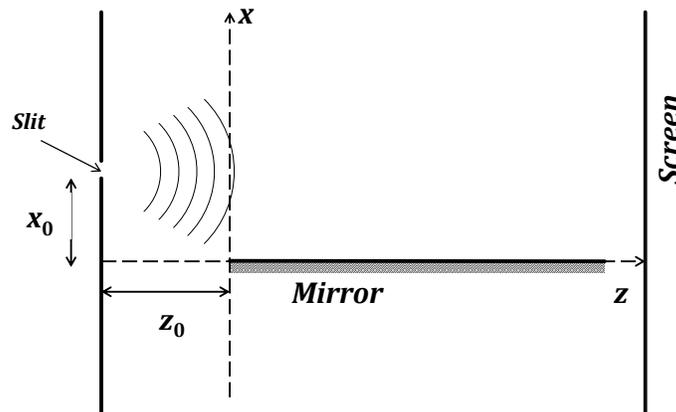


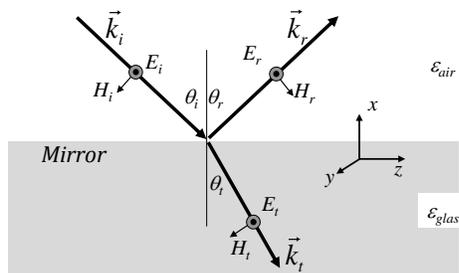
1. **A Lloyd Mirror Interferometer.** The Lloyd mirror is a wavefront-splitting interferometer. It consists of a flat glass mirror that reflects a portion of wavefront that comes from a narrow slit. Another portion of the wavefront proceeds directly to the screen. The interference of the two wavefronts form a set of bright and dark fringes that can be measured on the screen.



In our problem, let's assume the mirror is placed at the plane $x = 0$ and illuminated by a spherical wave originating from the slit at location $(x_0, -z_0)$ (where $x_0, z_0 > 0$). Using the paraxial approximation for a 1D spherical wave ($y=0$),

$$E(x, z) = E_0 \frac{\exp [ik(z + z_0)]}{i(z + z_0)} \exp \left[ik \frac{(x - x_0)^2}{2(z + z_0)} \right]$$

- (10%)** The reflected wavefront from the mirror can be considered as spherical wave radiation from a virtual source. Using your knowledge from geometric optics, determine the origin of the virtual source that radiates such spherical wave.
- (15%)** The source illuminating the slit has a wavelength of 500nm in air. If the slit is positioned at $x_0=1\text{mm}$ above the flat mirror, and the screen is placed 1 meter away from the slit, please estimate the spacing of the fringes on the screen.
- (15%)** In order to find the relative amplitude and phase of the virtual source with respect to the original spherical wave, we have to consider the Fresnel equations for reflected waves.



For simplicity, let's assume the E-field of source is polarized in the y-direction (S-polarization) and the slit is placed near the $x=0$ plane ($x_0 \ll z_0$, so $\theta_i = \theta_r \rightarrow 90^\circ$). Using these assumptions and based on the reflection coefficient for S-polarization at air-glass interface

$$r_S = \frac{E_{0r}}{E_{0i}} = \frac{k_{x,air} - k_{x,glass}}{k_{x,air} + k_{x,glass}}$$

$$\text{And } k_{x,air} = k \sqrt{n_{air}^2 - \sin^2 \theta_i}, \quad k_{x,glass} = k \sqrt{n_{glass}^2 - \sin^2 \theta_i} \quad (n_{air}=1, n_{glass}=1.5),$$

Please estimate the relative amplitude (i.e. $|r_S|$) and phase (i.e. $\arg(r_S)$) of the virtual source when θ_i approaches 90° .

***d) (10%)** Using the result you obtained from c), show that the fringe pattern is dark at $x=0$ on the screen.

Solution:

a) (4 pts) By using the law of reflection to draw a few rays leaving the slit, it can be readily that there is a virtual point source located at:

$$(x, z) = (-x_0, -z_0) \quad [1-1]$$

b) (6 pts) The observed fringe pattern will be described by determining the intensity at the observation screen. This can be done by finding the total field at the screen by superimposing the waves from both the actual and virtual 1D point sources. Using the equation provided for a 1D paraxial spherical wave, the wave leaving the actual point source is:

$$E_{actual}(x, z) = \frac{E_0}{i(z + z_0)} e^{ik(z+z_0)} e^{ik \frac{(x-x_0)^2}{2(z+z_0)}} \quad [1-2]$$

Likewise, the virtual point source originates from the location found in part (a):

$$E_{virtual}(x, z) = \frac{AE_0}{i(z + z_0)} e^{ik(z+z_0)} e^{ik \frac{(x+x_0)^2}{2(z+z_0)}} \quad [1-3]$$

(A is reflection coefficient to be determined by part (c), we will show $A=-1$ in this case) Adding Equations [1-2] and [1-3] yields the field at the observation screen, located at a location $(z + z_0)$ away from the slits:

$$E_{out}(x, z) = E_{actual}(x, z) + E_{virtual}(x, z) \quad [1-4]$$

$$E_{out}(x, z) = \frac{E_0}{i(z + z_0)} e^{ik(z+z_0)} \left[e^{ik \frac{(x-x_0)^2}{2(z+z_0)}} + A e^{ik \frac{(x+x_0)^2}{2(z+z_0)}} \right]$$

Equation [1-4] can be simplified by expanding the quadratics in the exponentials of the bracketed term.

$$E_{out}(x, z) = \frac{E_0}{i(z + z_0)} e^{ik(z+z_0)} \left[e^{ik \frac{x^2 - 2xx_0 + x_0^2}{2(z+z_0)}} + A e^{ik \frac{x^2 + 2xx_0 + x_0^2}{2(z+z_0)}} \right] \quad [1-5]$$

$$E_{out}(x, z) = \frac{E_0}{i(z + z_0)} e^{ik(z+z_0)} e^{ik \frac{(x^2 + x_0^2)}{2(z+z_0)}} \left[e^{-ik \frac{xx_0}{z+z_0}} + A e^{ik \frac{xx_0}{z+z_0}} \right] \quad [1-6]$$

$$E_{out}(x, z) = \frac{E_0}{i(z + z_0)} e^{ik(z+z_0)} e^{ik\frac{(x^2+x_0^2)}{2(z+z_0)}} \left[A \left(e^{-ik\frac{xx_0}{z+z_0}} + e^{ik\frac{xx_0}{z+z_0}} \right) + (1 - A) \left(e^{-ik\frac{xx_0}{z+z_0}} \right) \right] \quad [1-7]$$

For fringe spacing we can focus on Equation [1-7], which is further reduced by using the formula: $2\cos x = e^{ix} + e^{-ix}$:

$$E_{out}(x, z) = \frac{E_0}{i(z + z_0)} e^{ik(z+z_0)} e^{ik\frac{(x^2+x_0^2)}{2(z+z_0)}} \left[2A \cos \left(\frac{2\pi x}{\lambda(z + z_0)/x_0} \right) + (1 - A) \left(e^{-ik\frac{xx_0}{z+z_0}} \right) \right] \quad [1-8]$$

It can be seen from equation [1-8] that the field will be a maximum or minimum whenever the cosine term is $n\pi$ (where $n = 0, \pm 1, \pm 2, \dots$) For simplicity, take $A = 1$ to determine the fringe spacing. Now that the field is known, the intensity can be found:

$$\begin{aligned} I_{out}(x, z) &\propto E_{out}(x, z)E_{out}(x, z)^* = \\ &= \frac{E_0}{i(z + z_0)} e^{ik(z+z_0)} e^{ik\frac{(x^2+x_0^2)}{2(z+z_0)}} \left[2 \cos \left(\frac{2\pi x}{\lambda(z + z_0)/x_0} \right) \right] \\ &\cdot \frac{E_0}{-i(z + z_0)} e^{-ik(z+z_0)} e^{-ik\frac{(x^2+x_0^2)}{2(z+z_0)}} \left[2 \cos \left(\frac{2\pi x}{\lambda(z + z_0)/x_0} \right) \right] \end{aligned} \quad [1-9]$$

$$I_{out}(x, z) \propto 4 \left(\frac{E_0}{z + z_0} \right)^2 \cos^2 \left(\frac{2\pi x}{\lambda(z + z_0)/x_0} \right) \quad [1-10]$$

Using the identity $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$, Equation [1-10] becomes:

$$I_{out}(x, z) \propto 2 \left(\frac{E_0}{z + z_0} \right)^2 \left[1 + \cos \left(\frac{2\pi x}{\lambda(z + z_0)/2x_0} \right) \right] \quad [1-11]$$

The denominator of the cosine term in Equation [1-11] is the spatial wavelength of the interference pattern. Therefore, the spacing between the fringes at the observation screen is:

$\Lambda = \frac{\lambda(z + z_0)}{2x_0} = \frac{(500 \cdot 10^{-9}m)(1m)}{2(1 \cdot 10^{-3}m)} = 2.5 \cdot 10^{-4}m = 0.25mm$	[1-12]
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Alternatively, if we draw two rays from the slit and its virtual image, and let them intersect on the screen at the point (x', z) , we come up with the following path differences:

$$\Delta OPL = \sqrt{(x' - x_0)^2 + (z + z_0)^2} - \sqrt{(x' + x_0)^2 + (z + z_0)^2} \quad [1-13]$$

Under paraxial limit, $z+z_0 \gg x, x_0$, so we can use the approximation: $\sqrt{1 + b^2} \approx 1 + \frac{b^2}{2}$

$$\Delta OPL = (z + z_0) \left[1 + \frac{(x' - x_0)^2}{2(z + z_0)^2} \right] - (z + z_0) \left[1 + \frac{(x' + x_0)^2}{2(z + z_0)^2} \right] \quad [1-14]$$

$$\Delta OPL = \frac{(z + z_0)}{2(z + z_0)^2} [(x' - x_0)^2 - (x' + x_0)^2] \quad [1-15]$$

$$\Delta OPL = \frac{-2x'x_0}{z+z_0} \quad [1-16]$$

In addition, the reflection may introduce a phase shift (as shown in part (c)). Nevertheless, if the difference in the path lengths are integer values of the wavelength, $\Delta OPL = m\lambda$, then constructive interference will occur at that point on the screen:

$$m\lambda = \frac{-2x'x_0}{z+z_0}, \quad [1-17]$$

Therefore, the distance between two bright spots is given by

$$\lambda = \frac{-2\Delta x'x_0}{z+z_0}, \rightarrow \Delta x' = \frac{\lambda(z+z_0)}{2x_0} = 0.25mm \quad [1-18]$$

- c) (6 pts) The reflection coefficient can be determined using the equation provided. First consider the wave numbers at $\theta = 90^\circ$ and $n_{air} = 1, n_{glass} = 1.5$:

$$k_{x,air} = k \sqrt{n_{air}^2 - \sin^2\theta_i} = k\sqrt{1-1} = 0 \quad [1-19]$$

and,

$$k_{x,glass} = k \sqrt{n_{glass}^2 - \sin^2\theta_i} = k \sqrt{\left(\frac{3}{2}\right)^2 - 1} = k \sqrt{\frac{5}{4}} \quad [1-20]$$

Substituting Equations [1-19] and [1-20] into the definition of the reflection coefficient provided:

$$r_s = \frac{k_{x,air} - k_{x,glass}}{k_{x,air} + k_{x,glass}} = \frac{0 - k\sqrt{\frac{5}{4}}}{k\sqrt{\frac{5}{4}} - 0} = -1 \quad [1-21]$$

The magnitude of the reflection coefficient is therefore:

$ r_s = 1$	[1-22]
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We can make use of the Euler equation to find the phase:

$r_s = r_s e^{i\phi} = r_s (\cos\phi + i\sin\phi) = -1 \rightarrow \phi = \pi$	[1-23]
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- d) (4 pts) Equation [1-22] informs us that the wavefront reflecting off of the glass mirror actually incurs a phase shift of 180° with respect to the wavefront incident on the glass. Therefore, we must account for this shift by adding a phase delay into the equation for the field emanating from the virtual point source. Equation [1-3] should be rewritten as:

$$E_{virtual}(x, z) = \frac{E_0}{i(z+z_0)} e^{ik(z+z_0)} e^{ik\frac{(x+x_0)^2}{2(z+z_0)}} e^{i\pi} = -\frac{E_0}{i(z+z_0)} e^{ik(z+z_0)} e^{ik\frac{(x+x_0)^2}{2(z+z_0)}} \quad [1-24]$$

Therefore the field at the observation screen is instead:

$$E_{out}(x, z) = \frac{E_0}{i(z+z_0)} e^{ik(z+z_0)} \left[e^{ik\frac{x^2-2xx_0+x_0^2}{2(z+z_0)}} - e^{ik\frac{x^2+2xx_0+x_0^2}{2(z+z_0)}} \right] \quad [1-25]$$

$$E_{out}(x, z) = \frac{E_0}{i(z+z_0)} e^{ik(z+z_0)} e^{ik\frac{(x^2+x_0^2)}{2(z+z_0)}} \left[e^{-ik\frac{xx_0}{z+z_0}} - e^{ik\frac{xx_0}{z+z_0}} \right] \quad [1-26]$$

The term in the brackets of Equation [1-26] can be written as a sine using the relation $\sin x = \frac{1}{2i}e^{ix} - \frac{1}{2i}e^{-ix}$:

$$E_{out}(x, z) = \frac{E_0}{(z + z_0)} e^{ik(z+z_0)} e^{ik\frac{(x^2+x_0^2)}{2(z+z_0)}} \left[-2 \sin\left(\frac{2\pi x}{\lambda(z+z_0)/x_0}\right) \right] \quad [1-27]$$

Therefore, the intensity at the observation screen is actually:

$$\begin{aligned} I_{out}(x, z) &\propto E_{out}(x, z)E_{out}(x, z)^* = \\ &= \frac{E_0}{(z + z_0)} e^{ik(z+z_0)} e^{ik\frac{(x^2+x_0^2)}{2(z+z_0)}} \left[-2 \sin\left(\frac{2\pi x}{\lambda(z+z_0)/x_0}\right) \right] \\ &\cdot \frac{E_0}{(z + z_0)} e^{-ik(z+z_0)} e^{-ik\frac{(x^2+x_0^2)}{2(z+z_0)}} \left[-2 \sin\left(\frac{2\pi x}{\lambda(z+z_0)/x_0}\right) \right] \end{aligned} \quad [1-28]$$

$$I_{out}(x, z) \propto 4 \left(\frac{E_0}{z + z_0}\right)^2 \sin^2\left(\frac{2\pi x}{\lambda(z+z_0)/x_0}\right) \quad [1-29]$$

From Equation [1-29] it is obvious that the intensity of the pattern at $x = 0$ is zero. Therefore, as expected, there will be a dark spot at the origin of the observation screen.

$I_{out}(0, z) \propto 4 \left(\frac{E_0}{z + z_0}\right)^2 \sin^2(0) = 0$	[1-30]
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2. Optical Fourier transforms.

a) (20%) Calculate the Fourier transform of the function:

$$g(x) = \frac{1}{2} \exp(-i\pi\alpha^2 x^2) \left[1 + \cos\left(\frac{2\pi x}{\Lambda}\right) \right]$$

b) (30%) Assuming $\alpha = 10\text{mm}^{-1}$, $\Lambda = 10\mu\text{m}$, design an optical system which at its output plane creates an exact replica of the Fourier transform of the previous question.

Additional constraints on your design are:

1. Your system should be designed for spatially and temporally coherent illumination consisting of a plane wave at wavelength $\lambda = 500\text{nm}$, incident on-axis.
2. You may only use "standard" optical elements such as lenses, gratings, prisms, and free-space propagation.
3. The spatial frequency component $u = 50\text{mm}^{-1}$ should be mapped at distance of 3 cm away from the optical axis at the output plane.

Solution:

a) (8 pts) The first step is to rewrite the cosine term of the expression provided as a set of exponentials:

$$g(x) = \frac{1}{2} e^{-i\pi\alpha^2 x^2} \left[1 + \cos\left(\frac{2\pi x}{\Lambda}\right) \right] = e^{-i\pi\alpha^2 x^2} \left[\frac{1}{2} + \frac{1}{4} e^{i2\pi x/\Lambda} + \frac{1}{4} e^{-i2\pi x/\Lambda} \right] \quad [2-1]$$

Now the convolution theorem can be used:

$$G(u) = \mathfrak{F}\{e^{-i\pi\alpha^2 x^2}\} \otimes \mathfrak{F}\left\{\frac{1}{2} + \frac{1}{4} e^{i2\pi x/\Lambda} + \frac{1}{4} e^{-i2\pi x/\Lambda}\right\} \quad [2-2]$$

$$G(u) = \sqrt{\frac{i}{-\alpha^2}} e^{\frac{i\pi u^2}{\alpha^2}} \otimes \left[\frac{1}{2} \delta(u) + \frac{1}{4} \delta\left(u - \frac{1}{\Lambda}\right) + \frac{1}{4} \delta\left(u + \frac{1}{\Lambda}\right) \right] \quad [2-3]$$

Each delta function can be convolved with the exponential in a straightforward manner:

$$G(u) = \frac{1}{\alpha\sqrt{i}} \left[\frac{1}{2} e^{\frac{i\pi u^2}{\alpha^2}} + \frac{1}{4} e^{\frac{i\pi(u-\frac{1}{\Lambda})^2}{\alpha^2}} + \frac{1}{4} e^{\frac{i\pi(u+\frac{1}{\Lambda})^2}{\alpha^2}} \right] \quad [2-4]$$

Equation [2-4] can be further simplified:

$$G(u) = \frac{1}{2\alpha\sqrt{i}} \left[e^{\frac{i\pi u^2}{\alpha^2}} + \frac{1}{2} e^{\frac{i\pi(u^2 - 2\frac{u}{\Lambda} + \frac{1}{\Lambda^2})}{\alpha^2}} + \frac{1}{2} e^{\frac{i\pi(u^2 + 2\frac{u}{\Lambda} + \frac{1}{\Lambda^2})}{\alpha^2}} \right] \quad [2-5]$$

$$G(u) = \frac{1}{2\alpha\sqrt{i}} e^{\frac{i\pi u^2}{\alpha^2}} \left[1 + \frac{1}{2} e^{\frac{i\pi(\frac{1}{\Lambda^2})}{\alpha^2}} \left(e^{\frac{-i\pi 2u}{\Lambda\alpha^2}} + e^{\frac{i\pi 2u}{\Lambda\alpha^2}} \right) \right] \quad [2-6]$$

$G(u) = \frac{1}{2\alpha\sqrt{i}} e^{\frac{i\pi u^2}{\alpha^2}} \left[1 + e^{\frac{i\pi(\frac{1}{\Lambda^2})}{\alpha^2}} \cos\left(\frac{2\pi u}{\Lambda\alpha^2}\right) \right]$	[2-7]
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Note that $\sqrt{i} = e^{i\frac{\pi}{4}}$, therefore another way of writing Equation [2-7] is:

$G(u) = \frac{1}{2\alpha} e^{i\left(\frac{\pi u^2}{\alpha^2} + \frac{\pi}{4}\right)} \left[1 + e^{\frac{i\pi(\frac{1}{\Lambda^2})}{\alpha^2}} \cos\left(\frac{2\pi u}{\Lambda\alpha^2}\right) \right]$	[2-8]
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- b)** (12 pts) There are always multiple ways to achieve the same output field. You can achieve a Fourier transform by using a lens or by placing a screen far enough away from an input field such that the Fraunhofer diffraction pattern is produced, therefore, we must first be able to produce the provided field at the input plane:

$$g_{in}(x) = \frac{1}{2} \left[1 + \cos\left(\frac{2\pi x}{\Lambda}\right) \right] \{e^{-i\pi\alpha^2 x^2}\} \quad [2-9]$$

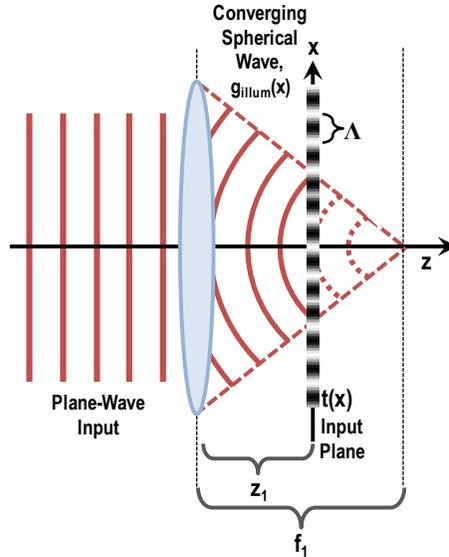
Since the input field described by Equation [2-2] should be the field immediately after the input transparency, it should be the product of an illumination field incident on the input plane and the transmission function of the object. By inspection of Equation [2-2], we can recognize that the term in the braces has the form of a *converging* one dimensional spherical wave and the term in front of it corresponds to the transmission function of a sinusoidal grating of perfect contrast:

$$g_{in}(x) = g_{illum}(x)t(x) = \{e^{-i\pi\alpha^2 x^2}\} \left\{ \frac{1}{2} \left[1 + \cos\left(\frac{2\pi x}{\Lambda}\right) \right] \right\} \quad [2-10]$$

Since our system must use an on-axis plane wave as an input, we need to be able to convert it into a converging spherical wave. This can be easily done with a one-dimensional lens (i.e. a cylindrical lens) that is located less than one focal length in front of the sinusoidal grating. However, care must be taken to relate the variable α to physical parameters in our system. The general form of a paraxial spherical wave originating at (x_0, z_0) is:

$$g_{spherical}(x, z) = \frac{E_0}{i(z - z_0)} e^{ik(z - z_0)} e^{ik\frac{(x - x_0)^2}{2(z - z_0)}} \quad [2-11]$$

We can imagine that if the spherical wave is centered *after* the sinusoidal grating, the illuminating wavefront is a converging spherical wave when it strikes the grating at the input plane.



Therefore we can write the field for the converging spherical wave that corresponds to the above sketch as:

$$g_{spherical}(x, z) = \frac{E_0}{i(z - (f_1 - z_1))} e^{ik(z - (f_1 - z_1))} e^{ik \frac{x^2}{2(z - (f_1 - z_1))}} \quad [2-12]$$

Therefore the field at the input plane where $z = 0$ is:

$$g_{spherical}(x, 0) = \frac{iE_0}{(f_1 - z_1)} e^{-\frac{i2\pi}{\lambda}(f_1 - z_1)} e^{\frac{-i\pi x^2}{\lambda(f_1 - z_1)}} \quad [2-13]$$

Note that equation [2-13] has the functional form we are interested in for the illuminating wavefront. In order to make $g_{illum}(x) = g_{spherical}(x, z = 0)$ we must make sure that the amplitudes are equal and solve for α in terms of physical quantities, which can be done by comparing Equation [2-13] with [2-10]:

$$g_{illum}(x) = g_{spherical}(x, 0) \quad [2-14]$$

$$e^{-i\pi\alpha^2 x^2} = \frac{iE_0}{(f_1 - z_1)} e^{-\frac{i2\pi}{\lambda}(f_1 - z_1)} e^{\frac{-i\pi x^2}{\lambda(f_1 - z_1)}} \quad [2-14]$$

In order to make the amplitudes equal:

$\frac{iE_0}{(f_1 - z_1)} e^{-\frac{i2\pi}{\lambda}(f_1 - z_1)} = 1 \rightarrow E_0 = -i(f_1 - z_1) e^{\frac{i2\pi}{\lambda}(f_1 - z_1)}$	[2-15]
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Also comparing the quadratic exponentials:

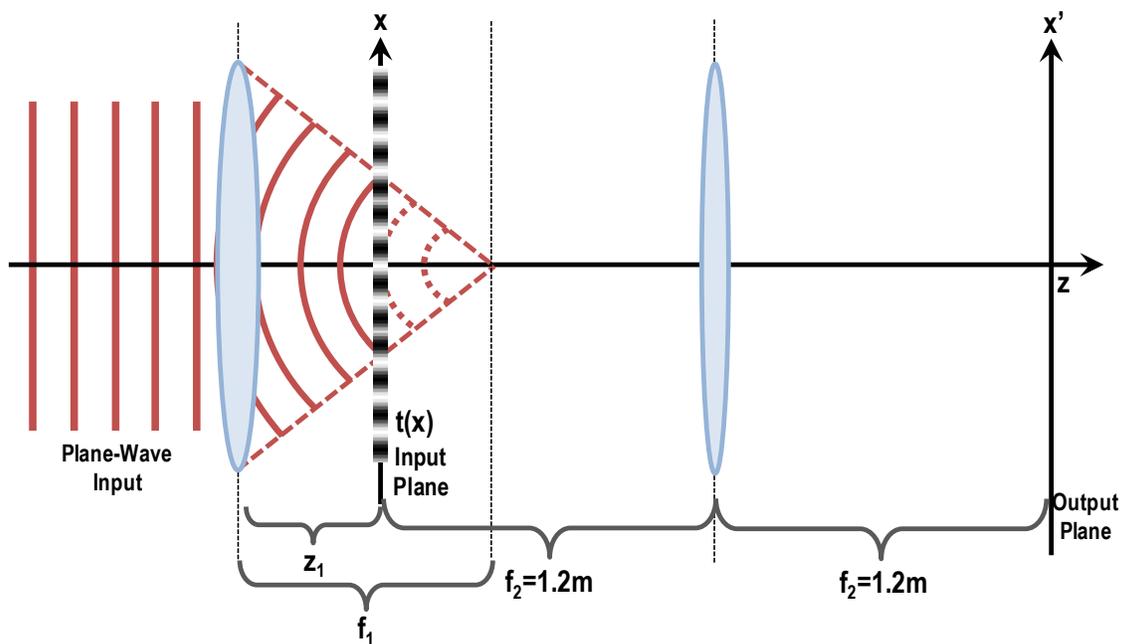
$-i\pi\alpha^2 x^2 = \frac{-i\pi x^2}{\lambda(f_1 - z_1)} \rightarrow \alpha^2 = \frac{1}{\lambda(f_1 - z_1)}$	[2-16]
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(Note, you could also set $z_i=0$ and choose a lens with $f_1 = \frac{1}{\lambda\alpha^2}$)

Now that the input field has been created, we must decide whether to perform the Fourier transform with a lens or through far-field propagation. In order to have a spatial frequency of $u=50\text{mm}^{-1}$ mapped to a spatial location of $x' = 3\text{cm}$ at the output plane, we must have either a lens with the appropriate focal length or place the output screen at a particular distance from the input.

Far-field Diffraction: $u = \frac{x'}{\lambda z} \rightarrow z = \frac{x'}{\lambda u} = \frac{(3 \cdot 10^{-2}m)}{(500 \cdot 10^{-9}m) \left(50 \frac{1}{mm}\right) \left(\frac{1000mm}{1m}\right)} = 1.2m$	[2-17]
Lens Transform: $u = \frac{x'}{\lambda f} \rightarrow f = \frac{x'}{\lambda u} = 1.2m$	

Given the design choice, it is safer to use a lens since 1.2 meters might not be far enough for the far-field diffraction pattern to be fully developed. Therefore, we will use a lens with a focal length of 1.2 meters and place it 1.2 meters behind the input plane and place the observation (output) screen 1.2 meters behind the lens.



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