Problem 1. Billet's Split Lens

Setup

If the field $U_L(x)$ is placed against a lens with focal length f and pupil function $P(x)$, the field $U_f(X)$ on the X-axis placed a distance f behind the lens is given by **Eq. (5-14) in Goodman** (modified to be 1D and I have reinserted the constant phase factor due to propagation over f in the z direction):

$$U_f(X) = \frac{\exp[jkf]\exp\left[j\frac{k}{2f}X^2\right]}{j\lambda f}\int_{-\infty}^{\infty} U_L(x)P(x)\exp\left[-j\frac{k}{f}xX\right]dx \qquad \textbf{Eq. 1}$$

Part (a)

Lens L (focal length f) is being illuminated by a spherical wave originating at location $z = -f$. Therefore, the field $U_L(x)$ incident on L is (using the paraxial approximation):

$$U_L(x) = E(x, z=0) = \left\{E_0\frac{\exp[jk(z+f)]}{j(z+f)}\exp\left[jk\frac{x^2}{2(z+f)}\right]\right\}\Bigg|_{z=0}$$

$$U_L(x) = E_0\frac{\exp[jkf]}{jf}\exp\left[jk\frac{x^2}{2f}\right]$$

The lens is partially obstructed by a block of width (W). Therefore, the field pattern $U_{output}(X)$ at the CCD detector can be viewed as follows. Let $U_\infty(X)$ be the field pattern that would result if the lens L was of infinite extent (with no obstructing block). Meanwhile, let $U_W(X)$ be the field pattern that would result if the lens L had finite extent W (again, with no obstructing block). $U_W(X)$ is essentially the field pattern blocked by the opaque block. Thus, by linearity, $U_{output}(X)$ will be $U_W(X)$ subtracted from $U_\infty(X)$:

$$U_{output}(X) = U_\infty(X) - U_W(X) \qquad \textbf{Eq. 2}$$

We can use **Eq. 1** to calculate $U_\infty(X)$ and $U_W(X)$ individually. $U_L(x)$ will be the same for each calculation. However, the pupil function $P(x)$ will differ as follows:

$P_\infty(x) = 1$ **Lens with infinite (∞) extent**

$P_W(x) = rect\left(\frac{x}{W}\right)$ **Lens with finite extent *W***

To calculate $U_\infty(X)$, we plug in $U_L(x)$ and $P_\infty(x)$ into **Eq. 1**:

$$U_\infty(X) = \frac{\exp[jkf]\exp\left[j\frac{k}{2f}X^2\right]}{j\lambda f}\int_{-\infty}^{\infty} E_0 \frac{\exp[jkf]}{jf}\exp\left[jk\frac{x^2}{2f}\right]\exp\left[-j\frac{k}{f}xX\right]dx$$

$$U_\infty(X) = \frac{-\exp[jk2f]\exp\left[j\frac{k}{2f}X^2\right]}{\lambda f^2}E_0 F\left\{\exp\left[jk\frac{x^2}{2f}\right]\right\}\bigg|_{f_X = X/\lambda f}$$

$$U_\infty(X) = \frac{-\exp[jk2f]\exp\left[j\frac{k}{2f}X^2\right]}{\lambda f^2}E_0 F\left\{\exp\left[j\pi\frac{1}{\lambda f}x^2\right]\right\}\bigg|_{f_X = X/\lambda f}$$

$$U_\infty(X) = \frac{-\exp[jk2f]\exp\left[j\frac{k}{2f}X^2\right]}{\lambda f^2}E_0 j\sqrt{\lambda f}\exp\left[-j\pi\lambda f f_X^2\right]_{f_X = X/\lambda f}$$

$$U_\infty(X) = \frac{\exp[jk2f]\exp\left[j\frac{k}{2f}X^2\right]}{jf\sqrt{\lambda f}}E_0 \exp\left[-j\frac{k}{2f}X^2\right]$$

$$U_\infty(X) = \frac{\exp[jk2f]}{jf\sqrt{\lambda f}}E_0$$

As expected for a spherical wave originating from the front focal point of a lens, $U_\infty(X)$ is a plane wave. To calculate $U_W(X)$, we plug in $U_L(x)$ and $P_W(x)$ into **Eq. 1**:

$$U_W(X) = \frac{\exp[jk2f]\exp\left[j\frac{k}{2f}X^2\right]}{j\lambda f}\int_{-\infty}^{\infty} E_0 \frac{\exp[jkf]}{jf}\exp\left[jk\frac{x^2}{2f}\right]rect\left(\frac{x}{W}\right)\exp\left[-j\frac{k}{f}xX\right]dx$$

$$U_W(X) = \frac{-\exp[jk2f]\exp\left[j\frac{k}{2f}X^2\right]}{\lambda f^2}E_0\int_{-\infty}^{\infty}\exp\left[jk\frac{x^2}{2f}\right]rect\left(\frac{x}{W}\right)\exp\left[-j\frac{k}{f}xX\right]dx$$

If the first exponential inside the integral is sufficiently constant (i.e., = 1) over the interval of the rect function (i.e., analogous to the Fraunhofer condition $f > 2W^2/\lambda$), then it can be neglected:

$$U_W(X) = \frac{-\exp[jk2f]\exp\left[j\frac{k}{2f}X^2\right]}{\lambda f^2}E_0\int_{-\infty}^{\infty} rect\left(\frac{x}{W}\right)\exp\left[-j\frac{k}{f}xX\right]dx$$

$$U_W(X) = \frac{-\exp[jk2f]\exp\left[j\frac{k}{2f}X^2\right]}{\lambda f^2} E_0 F\left\{rect\left(\frac{x}{W}\right)\right\}\Big|_{f_X = X/\lambda f}$$

$$U_W(X) = \frac{-\exp[jk2f]\exp\left[j\frac{k}{2f}X^2\right]}{\lambda f^2} E_0 \mathrm{W}\mathrm{sinc}\left(\frac{W}{\lambda f}X\right)$$

$$U_W(X) = \frac{-\exp[jk2f]\exp\left[j\frac{k}{2f}X^2\right]}{\lambda f^2} E_0 \mathrm{W}\frac{\sin\left(\pi\frac{W}{\lambda f}X\right)}{\pi\frac{W}{\lambda f}X}$$

$$U_W(X) = \frac{-\exp[jk2f]\exp\left[j\frac{k}{2f}X^2\right]}{f} E_0 \frac{\exp\left[j\pi\frac{W}{\lambda f}X\right] - j\exp\left[-j\pi\frac{W}{\lambda f}X\right]}{j2\pi X}$$

$$U_W(X) = \frac{\exp[jk2f]\exp\left[j\frac{k}{2f}X^2\right]}{j2\pi f} E_0 \frac{j\exp\left[-j\frac{k}{2f}WX\right] - \exp\left[j\frac{k}{2f}WX\right]}{X}$$

$$U_W(X) = \frac{j\exp[jk2f]}{j2\pi f} E_0 \frac{j\exp\left[j\frac{k}{2f}(X^2 - WX)\right] - \exp\left[j\frac{k}{2f}(X^2 + WX)\right]}{X}$$

$$U_W(X) = \frac{j\exp[jk2f]}{j2\pi f} E_0 \exp\left[-j\frac{k}{2f}\frac{W^2}{4}\right] \frac{j\exp\left[j\frac{k}{2f}\left(X^2 - WX + \frac{W^2}{4}\right)\right] - \exp\left[j\frac{k}{2f}\left(X^2 + WX + \frac{W^2}{4}\right)\right]}{X}$$

$$U_W(X) = \frac{\exp[jk2f]}{j2\pi f} E_0 \exp\left[-j\frac{k}{2f}\frac{W^2}{4}\right] \frac{j\exp\left[j\frac{k}{2f}\left(X - \frac{W}{2}\right)^2\right] - \exp\left[j\frac{k}{2f}\left(X + \frac{W}{2}\right)^2\right]}{X}$$

To find the field pattern $U_{output}(X)$ at the CCD detector, we plug $U_\infty(X)$ and $U_W(X)$ into **Eq. 2**:

$$U_{output}(X) = \frac{\exp[jk2f]}{j} E_0 \left\{ \frac{1}{f\sqrt{\lambda f}} + \frac{\exp\left[-j\frac{k}{2f}\frac{W^2}{4}\right]}{2\pi f} \frac{\exp\left[j\frac{k}{2f}\left(X + \frac{W}{2}\right)^2\right] - j\exp\left[j\frac{k}{2f}\left(X - \frac{W}{2}\right)^2\right]}{X} \right\}$$

Parts (b) and (c)

In the expression for $U_{output}(X)$, all three terms have the form of a spherical wave. Thus, $U_{output}(X)$ essentially represents the coherent superposition of three spherical waves as follows:

- **1st term**: a plane wave, equivalent to a diverging spherical wave originating at $(x, z) = (0, -\infty)$, i.e., on-axis at infinity (to the left)
- **2nd term**: a diverging spherical wave originating at $(x, z) = (-W/2, 0)$, i.e., the bottom edge of the opaque block, with a phase factor and a $1/X$ attenuation factor
- **3rd term**: a diverging spherical wave originating at $(x, z) = (+W/2, 0)$, i.e., the top edge of the opaque block, with a phase factor and a $1/X$ attenuation factor

For each pair of waves, bright and dark fringes will occur when the phase difference (δ) between the pair is a multiple m and (m+0.5) of 2π, respectively:

- **1st and 2nd waves (Plane & Spherical)**: bright fringes occur at X_m with increasing separation with m

$$2\pi m = \delta_{1-2} = \varphi_2 - \varphi_1 = \frac{k}{2f}\left[-\frac{W^2}{4} + \left(X + \frac{W}{2}\right)^2\right] \quad \rightarrow \quad X_m = \sqrt{2\lambda f m + \frac{W^2}{4}} - \frac{W}{2}$$

- **1st and 3rd waves (Plane & Spherical)**: bright fringes occur at X_m with increasing separation with m

$$2\pi m = \delta_{1-3} = \varphi_3 - \varphi_1 = \frac{k}{2f}\left[-\frac{W^2}{4} + \left(X - \frac{W}{2}\right)^2\right] \quad \rightarrow \quad X_m = \sqrt{2\lambda f m + \frac{W^2}{4}} + \frac{W}{2}$$

- **2nd and 3rd waves (Spherical & Spherical)**: bright fringes occur at X_m with constant separation with m (as in Young's double slit experiment with slit separation W)

$$2\pi m = \delta_{2-3} = \varphi_2 - \varphi_3 = \frac{k}{2f}\left[\left(X + \frac{W}{2}\right)^2 - \left(X - \frac{W}{2}\right)^2\right] \quad \rightarrow \quad X_m = \frac{\lambda f m}{W}$$

The total irradiance I on the CCD detector will be the aggregate of these fringe patterns, weighted by the irradiances (I_1, I_2, and I_3) of each wave. Note that I_2 and I_3 decrease with increasing X, due to the $1/X^2$ attenuation factor.

Parts (d)

If the illumination were moved off axis by a distance of x_0, then the field $U_L(x, y)$ incident on the lens L would become:

$$U_L(x) = E_0 \frac{\exp[jkf]}{jf} \exp\left[jk \frac{(x-x_0)^2}{2f} \right]$$

According to the **Shift Theorem**, $U_\infty(X)$ (from **Part a**) would become:

$$U_\infty(X) = \frac{\exp[jk2f]}{jf\sqrt{\lambda f}} E_0 \exp\left[-j\frac{k}{f} X x_0 \right]$$

$U_W(X)$ (from **Part a**) would become (plugging $P_W(x)$ and the new $U_L(x)$ into **Eq. 1**):

$$U_W(X) = \frac{\exp[jk2f]\exp\left[j\frac{k}{2f}X^2 \right]}{j\lambda f} \int_{-\infty}^{\infty} E_0 \frac{\exp[jkf]}{jf} \exp\left[jk\frac{(x-x_0)^2}{2f} \right] rect\left(\frac{x}{W} \right) \exp\left[-j\frac{k}{f} xX \right] dx$$

$$U_W(X) = \frac{-\exp[jk2f]\exp\left[j\frac{k}{2f}X^2 \right]}{\lambda f^2} E_0 \int_{-\infty}^{\infty} \exp\left[jk\frac{x^2}{2f} \right] \exp\left[-jk\frac{xx_0}{f} \right] \exp\left[jk\frac{x_0^2}{2f} \right] rect\left(\frac{x}{W} \right) \exp\left[-j\frac{k}{f} xX \right] dx$$

$$U_W(X) = \frac{-\exp[jk2f]\exp\left[j\frac{k}{2f}(X^2 + x_0^2) \right]}{\lambda f^2} E_0 \int_{-\infty}^{\infty} \exp\left[jk\frac{x^2}{2f} \right] rect\left(\frac{x}{W} \right) \exp\left[-j\frac{k}{f} x(X + x_0) \right] dx$$

Once again, if the first exponential inside the integral is sufficiently constant (i.e., = 1) over the interval of the rect function (i.e., analogous to the Fraunhofer condition $f > 2W^2/\lambda$), then it can be neglected as follows:

$$U_W(X) = \frac{-\exp[jk2f]\exp\left[j\frac{k}{2f}(X^2 + x_0^2) \right]}{\lambda f^2} E_0 F\left\{ rect\left(\frac{x}{W} \right) \right\} \Bigg|_{f_X = (X + x_0)/\lambda f}$$

$$U_W(X) = \frac{-\exp[jk2f]\exp\left[j\frac{k}{2f}(X^2 + x_0^2) \right]}{\lambda f^2} E_0 \mathrm{W} \mathrm{sinc}\left(\frac{W}{\lambda f}(X + x_0) \right)$$

$$U_W(X) = \frac{-\exp[jk2f]\exp\left[j\frac{k}{2f}(X^2 + x_0^2) \right]}{\lambda f^2} E_0 \mathrm{W} \frac{\sin\left(\pi\frac{W}{\lambda f}(X + x_0) \right)}{\pi\frac{W}{\lambda f}(X + x_0)}$$

$$U_W(X) = \frac{-\exp[jk2f]\exp\left[j\frac{k}{2f}(X^2+x_0^2)\right]}{f}E_0\frac{\exp\left[j\pi\frac{W}{\lambda f}(X+x_0)\right]-j\exp\left[-j\pi\frac{W}{\lambda f}(X+x_0)\right]}{j2\pi(X+x_0)}$$

$$U_W(X) = \frac{\exp[jk2f]\exp\left[j\frac{k}{2f}(X^2+x_0^2)\right]}{j2\pi f}E_0\frac{j\exp\left[-j\frac{k}{2f}W(X+x_0)\right]-\exp\left[j\frac{k}{2f}W(X+x_0)\right]}{X+x_0}$$

$$U_W(X) = \frac{\exp[jk2f]}{j2\pi f}E_0\exp\left[j\frac{k}{2f}x_0^2\right]\frac{j\exp\left[j\frac{k}{2f}(X^2-WX-Wx_0)\right]-\exp\left[j\frac{k}{2f}(X^2+WX+Wx_0)\right]}{X+x_0}$$

$$U_W(X) = \frac{\exp[jk2f]}{j2\pi f}E_0\exp\left[j\frac{k}{2f}\left(x_0^2-\frac{W^2}{4}\right)\right]\times$$

$$\frac{j\exp\left[-j\frac{k}{2f}Wx_0\right]\exp\left[j\frac{k}{2f}\left(X^2-WX+\frac{W^2}{4}\right)\right]-\exp\left[j\frac{k}{2f}Wx_0\right]\exp\left[j\frac{k}{2f}\left(X^2+WX+\frac{W^2}{4}\right)\right]}{X+x_0}$$

$$U_W(X) = \frac{\exp[jk2f]}{j2\pi f}E_0\exp\left[j\frac{k}{2f}\left(x_0^2-\frac{W^2}{4}\right)\right]\times$$

$$\frac{j\exp\left[-j\frac{k}{2f}Wx_0\right]\exp\left[j\frac{k}{2f}\left(X-\frac{W}{2}\right)^2\right]-\exp\left[j\frac{k}{2f}Wx_0\right]\exp\left[j\frac{k}{2f}\left(X+\frac{W}{2}\right)^2\right]}{X+x_0}$$

To find the field pattern $U_{output}(X)$ at the CCD detector, we plug $U_\infty(X)$ and $U_W(X)$ into **Eq. 2**:

$$U_{output}(X) = \frac{\exp[jk2f]}{j}E_0\times$$

$$\left\{\frac{\exp\left[-jk\frac{x_0}{f}X\right]}{f\sqrt{\lambda f}}+\exp\left[j\frac{k}{2f}\left(x_0^2-\frac{W^2}{4}\right)\right]\frac{\exp\left[j\frac{k}{2f}Wx_0\right]\exp\left[j\frac{k}{2f}\left(X+\frac{W}{2}\right)^2\right]-j\exp\left[-j\frac{k}{2f}Wx_0\right]\exp\left[j\frac{k}{2f}\left(X-\frac{W}{2}\right)^2\right]}{X+x_0}\right\}$$

The field $U_{output}(X)$ is still a coherent superposition of three spherical waves:

- **1st term**: a plane wave propagating at angle $-x_0/f$, equivalent to a diverging spherical wave originating off-axis at infinity (to the left)
- **2nd term**: a diverging spherical wave originating at $(x, z) = (-W/2, 0)$, i.e., the bottom edge of the opaque block, with a phase factor and a $1/(X+x_0)$ attenuation factor
- **3rd term**: a diverging spherical wave originating at $(x, z) = (+W/2, 0)$, i.e., the top edge of the opaque block, with a phase factor and a $1/(X+x_0)$ attenuation factor

Relative to **Part (a)**, the $U_{output}(X)$ here has changed as follows:

- **1st term**: this term is still a plane wave, but is now propagating at an angle of $-x_0/f$
- **2nd term**: this term is still a diverging spherical wave originating at $(x, z) = (-W/2, 0)$, but now has an additional phase factor of $\exp[jkWx_0/2f]$ and a $1/(X+x_0)$ attenuation factor (instead of just $1/X$)
- **3rd term**: this term is still a diverging spherical wave originating at $(x, z) = (+W/2, 0)$, but now has an additional phase factor of $\exp[-jkWx_0/2f]$ and a $1/(X+x_0)$ attenuation factor (instead of just $1/X$)

For each pair of waves, bright and dark fringes will occur when the phase difference (δ) between the pair is a multiple m and $(m+0.5)$ of 2π, respectively:

- **1st and 2nd waves**: bright fringes occur at X_m with increasing separation with m

$$2\pi m = \delta_{1-2} = \varphi_2 - \varphi_1 = \frac{k}{2f}\left[x_0^2 - \frac{W^2}{4} + \left(X + \frac{W}{2}\right)^2 + Wx_0 + 2Xx_0\right]$$

$$2\pi m = \frac{k}{2f}\left[-\frac{W^2}{4} + \left(X + x_0 + \frac{W}{2}\right)^2\right] \rightarrow X_m = \sqrt{2\lambda f m + \frac{W^2}{4}} - \frac{W}{2} - x_0$$

- **1st and 3rd waves**: bright fringes occur at X_m with increasing separation with m

$$2\pi m = \delta_{1-3} = \varphi_3 - \varphi_1 = \frac{k}{2f}\left[x_0^2 - \frac{W^2}{4} + \left(X - \frac{W}{2}\right)^2 - Wx_0 + 2Xx_0\right] \rightarrow X_m = \sqrt{2\lambda f m + \frac{W^2}{4}} + \frac{W}{2} - x_0$$

- **2nd and 3rd waves**: bright fringes occur at X_m with constant separation with m

$$2\pi m = \delta_{2-3} = \varphi_2 - \varphi_3 = \frac{k}{2f}\left[\left(X + \frac{W}{2}\right)^2 - \left(X - \frac{W}{2}\right)^2 + 2Wx_0\right] \rightarrow X_m = \frac{\lambda f m}{W} - x_0$$

Thus, relative to **Part (a)**, the bright (and dark) fringes on the CCD detector will be shifted by a distance of $-x_0$, i.e. in the opposite direction of the source shift. The attenuation on I_2 and I_3 will also be shifted by $-x_0$, due to the $1/(X+x_0)^2$ attenuation factor (instead of just $1/X^2$ in **Part a**).

Appendix to Problem 1

My answers above were contingent on the assumption that $f > 2W^2/\lambda$, which allowed for easy calculation of $U_W(X)$, i.e., the field pattern block by the opaque block. To be more clear, I will now show what would happen if I did not make that assumption.

Part (a)

Without assuming $f > 2W^2/\lambda$, $U_L(X)$ does not change but $U_W(X)$ must be recalculated. Plugging $P_W(X)$ and the original $U_L(X)$ into **Eq. 1, t**he first exponential inside the integral cannot be neglected now:

$$U_W(X) = \frac{-\exp[jk2f]\exp\left[j\frac{k}{2f}X^2\right]}{\lambda f^2} E_0 \int_{-\infty}^{\infty} \exp\left[jk\frac{x^2}{2f}\right] rect\left(\frac{x}{W}\right) \exp\left[-j\frac{k}{f}xX\right] dx$$

$$U_W(X) = \frac{-\exp[jk2f]\exp\left[j\frac{k}{2f}X^2\right]}{\lambda f^2} E_0 F\left\{\exp\left[jk\frac{x^2}{2f}\right] rect\left(\frac{x}{W}\right)\right\}\Bigg|_{f_x = X/\lambda f}$$

$$U_W(X) = \frac{-\exp[jk2f]\exp\left[j\frac{k}{2f}X^2\right]}{\lambda f^2} E_0 F\left\{\exp\left[jk\frac{x^2}{2f}\right]\right\} \otimes F\left\{rect\left(\frac{x}{W}\right)\right\}\Bigg\|_{f_x = X/\lambda f}$$

$$U_W(X) = \frac{-\exp[jk2f]\exp\left[j\frac{k}{2f}X^2\right]}{\lambda f^2} E_0 j\sqrt{\lambda f}\exp\left[-j\pi\lambda f f_X^2\right] \otimes W\mathrm{sinc}(Wf_x)\Big\|_{f_x = X/\lambda f}$$

$$U_W(X) = \frac{\exp[jk2f]\exp\left[j\frac{k}{2f}X^2\right]}{jf\sqrt{\lambda f}} E_0 W\left\{\exp\left[-j\frac{k}{2f}X^2\right] \otimes \mathrm{sinc}\left(\frac{W}{\lambda f}X\right)\right\}$$

Note that we are left with a convolution of a converging spherical wave with a sinc, rather than a sinc by itself (as in my **Part (a)** previously). To find the new field pattern $U_{output}(X)$ at the CCD detector, we plug $U_\infty(X)$ and $U_W(X)$ into **Eq. 2**:

$$U_{output}(X) = \frac{\exp[jk2f]}{j} E_0 \left\{ \frac{1}{f\sqrt{\lambda f}} - \frac{\exp\left[j\frac{k}{2f}X^2\right]}{f\sqrt{\lambda f}} E_0 W\left\{\exp\left[-j\frac{k}{2f}X^2\right] \otimes \mathrm{sinc}\left(\frac{W}{\lambda f}X\right)\right\} \right\}$$

Part (b) and (c)

Without assuming $f > 2W^2/\lambda$, the expression for $U_{output}(X)$ can no longer be viewed as a superposition of three spherical waves. The first time is still a plane wave, i.e., a plane wave originating at infinity. However, the second term is now the product of a diverging spherical wave and the convolution of a converging spherical wave with a sinc. It is no longer fruitful to expand the sinc into complex exponentials in hope of creating two spherical waves (as in my **Part (a)** previously). Instead, $U_{output}(X)$ is the superposition of a plane wave (as before) and the Fresnel diffraction pattern of a rectangular aperture with plane wave incidence (not two spherical waves).

Part (d

Despite the changes to **Parts a, b, and c**, the answer here does not change. If the illumination were moved off axis by a distance of x_0, then $U_{output}(X)$ would still shift a distance of $-x_0$.

$U_\infty(X)$ can be written as:

$$U_\infty(X) = \frac{\exp[jk2f]\exp\left[-j\frac{k}{2f}(X^2+x_0^2)\right]}{jf\sqrt{\lambda f}} E_0 \exp\left[-j\frac{k}{2f}(X+x_0)^2\right]$$

$U_W(X)$ can be written as:

$$U_W(X) = \frac{\exp[jk2f]\exp\left[j\frac{k}{2f}(X^2+x_0^2)\right]}{jf\sqrt{\lambda f}} E_0 W\left\{\exp\left[-j\frac{k}{2f}(X+x_0)^2\right]\otimes \mathrm{sinc}\left(\frac{W}{\lambda f}X\right)\right\}$$

By observation, both of these terms are simply shifted versions (plus some phase change) of the $x_0 = 0$ case. Thus, the combined $U_{output}(X)$ will also be shifted.

Problem 2. 4F Imaging System

Setup

In a 4F imaging system, the primary fields of interest are:

- $U_{input}(x, y)$: the field pattern at the x-y input plane
- $U_{pupil}(X, Y)$: the field pattern incident on the X-Y pupil plane
- $U'_{pupil}(X, Y)$: the field pattern immediately behind the X-Y pupil plane
- $U_{output}(x', y')$: the field pattern at the x'-y' output plane

The field pattern $U_{input}(x, y)$ is simply the input image (to be specified in **Parts a**, **b**, and **c**). If field pattern $U_{input}(x, y)$ is placed a distance $d = f$ in front of L1 with focal length f and pupil function $P(X, Y) = 1$ (i.e., neglecting the finite extent of L1), the field pattern $U_{pupil}(X, Y)$ on the X-Y pupil plane placed a distance f behind L1 is given by **Eq. (5-19) in Goodman** (Note: I have reinserted the constant phase factor due to propagation over $2f$ in the z direction):

$$U_{pupil}(X,Y) = \frac{e^{jk2f}}{j\lambda f}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{U_{input}(x,y)\right\}\exp\left[-j\frac{k}{f}(xX+xY)\right]dxdy$$

$$U_{pupil}(X,Y) = \frac{e^{jk2f}}{j\lambda f}F\left\{U_{input}(x,y)\right\}\Big|_{\substack{f_x = X/\lambda f \\ f_y = Y/\lambda f}}$$

$$U_{pupil}(X,Y) = \frac{e^{jk2f}}{j\lambda f}\hat{U}_{input}\left(\frac{X}{\lambda f},\frac{Y}{\lambda f}\right)$$

Thus, $U_{pupil}(X, Y)$ is simply the Fourier transform of the input field. Meanwhile, the pupil plane has a mask with amplitude transmission function $t(X, Y)$. If $U_{pupil}(X, Y)$ is incident on the pupil plane, then the field pattern $U'_{pupil}(X, Y)$ immediately behind the pupil plane is the product of $U_{pupil}(X, Y)$ and $t(X, Y)$:

$$U'_{pupil}(X,Y) = U_{pupil}(X,Y)t(X,Y)$$

If field pattern $U'_{pupil}(X, Y)$ is placed a distance $d = f$ in front of L2 with focal length f and pupil function $P(X, Y) = 1$ (i.e., neglecting the finite extent of L2), the field pattern $U_{output}(x', y')$ on the x'-y' outuput plane placed a distance f behind L2 is given **Eq. (5-19) in Goodman** (Note: I have reinserted the constant phase factor due to propagation over $2f$ in the z direction):

$$U_{output}(x',y') = \frac{e^{jk2f}}{j\lambda f}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}U'_{pupil}(X,Y)\exp\left[-j\frac{k}{f}(x'X+y'Y)\right]dXdY$$

$$U_{output}(x',y') = \frac{e^{jk2f}}{j\lambda f}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U_{pupil}(X,Y)t(X,Y)\exp\left[-j\frac{k}{f}(x'X+y'Y)\right]dXdY$$

$$U_{output}(x',y') = \frac{e^{jk4f}}{j\lambda^2 f^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{U}_{input}\left(\frac{X}{\lambda f},\frac{Y}{\lambda f}\right)t(X,Y)\exp\left[-j\frac{k}{f}(x'X+y'Y)\right]dXdY$$

$$U_{output}(x',y') = \frac{e^{jk4f}}{j\lambda^2 f^2}F\left\{\hat{U}_{input}\left(\frac{X}{\lambda f},\frac{Y}{\lambda f}\right)t(X,Y)\right\}\Bigg|_{\substack{f_X=x'/\lambda f\\ f_Y=y'/\lambda f}} \qquad \textbf{Eq. 1}$$

$$U_{output}(x',y') = \frac{e^{jk4f}}{j\lambda^2 f^2}F\left\{\hat{U}_{input}\left(\frac{X}{\lambda f},\frac{Y}{\lambda f}\right)\right\}\otimes F\{t(X,Y)\}\Bigg|_{\substack{f_X=x'/\lambda f\\ f_Y=y'/\lambda f}}$$

$$U_{output}(x',y') = \frac{e^{jk4f}}{j\lambda^2 f^2}\lambda^2 f^2 U_{input}(-\lambda f f_X,-\lambda f f_Y)\otimes\hat{t}(f_X,f_Y)\Big|_{\substack{f_X=x'/\lambda f\\ f_Y=y'/\lambda f}}$$

$$U_{output}(x',y') = \frac{e^{jk4f}}{j}U_{input}(-x',-y')\otimes\hat{t}\left(\frac{x'}{\lambda f},\frac{y'}{\lambda f}\right) \qquad \textbf{Eq. 2}$$

$$U_{output}(x',y') = \frac{e^{jk4f}}{j}U_{input}(-x',-y')\otimes PSF(x',y')$$

Thus, the output field $U_{output}(x', y')$ can be calculated in two equivalent ways (whichever is more convenient for the given input field):

- **Eq. 1**: $U_{output}(x', y')$ is the Fourier transform of the product of the Fourier transform $\hat{U}_{input}(X/\lambda f, Y/\lambda f)$ of the input image and the amplitude transmission function $t(X, Y)$ of the mask.
- **Eq. 2**: $U_{output}(x', y')$ is the convolution of the inverted input image $U_{input}(-x', -y')$ and the Fourier transform $\hat{t}(x'/\lambda f, y'/\lambda f)$ of the mask's amplitude transmission function ((i.e., the point spread function $PSF(x', y')$).

In either case, the intensity pattern can then be calculated as follows:

$$I_{output}(x',y') = \left|U_{output}(x',y')\right|^2 \qquad \textbf{Eq. 3}$$

Part (a)

In this case, the output field $U_{output}(x', y')$ is most easily found using **Eq. 1**. If the input plane is illuminated by a coherent plane wave, the input field $U_{input}(x, y)$ is a constant and its Fourier transform $\hat{U}_{input}(f_X, f_Y)$ is a δ function:

$$U_{input}(x, y) = 1 \qquad \rightarrow \qquad \hat{U}_{input}(f_X, f_Y) = F\{1\} = \delta(f_x, f_y)$$

Meanwhile, the pupil plane's mask is a rectangular grid of squares, specifically a a x b (Y-width by X-height) rectangle of (Λ – d) x (Λ – d) squares repeating every Λ in each direction. Therefore, $t(X, Y)$ is the convolution of the square with a comb function bounded by a rectangle:

$$t(X,Y) = rect\left(\frac{X}{\Lambda - d}\right) rect\left(\frac{Y}{\Lambda - d}\right) \otimes comb\left(\frac{X}{\Lambda}\right) comb\left(\frac{Y}{\Lambda}\right) rect\left(\frac{X}{b}\right) rect\left(\frac{Y}{a}\right)$$

Note the grid has a transparent square at the center, such that $t(0, 0) = 1$.

Plugging $\hat{U}_{input}(f_X, f_Y)$ and $t(X, Y)$ into **Eq. 1**

$$U_{output}(x', y') = \frac{e^{jk4f}}{j\lambda^2 f^2} F\left\{\hat{U}_{input}\left(\frac{X}{\lambda f}, \frac{Y}{\lambda f}\right) t(X,Y)\right\}\Bigg|_{\substack{f_X = x'/\lambda f \\ f_Y = y'/\lambda f}}$$

$$U_{output}(x', y') = \frac{e^{jk4f}}{j\lambda^2 f^2} F\left\{\delta\left(\frac{X}{\lambda f}, \frac{Y}{\lambda f}\right) t(X,Y)\right\}\Bigg|_{\substack{f_X = x'/\lambda f \\ f_Y = y'/\lambda f}}$$

$$U_{output}(x', y') = \frac{e^{jk4f}}{j\lambda^2 f^2} F\left\{\delta\left(\frac{X}{\lambda f}, \frac{Y}{\lambda f}\right) t(0,0)\right\}\Bigg|_{\substack{f_X = x'/\lambda f \\ f_Y = y'/\lambda f}}$$

$$U_{output}(x', y') = \frac{e^{jk4f}}{j\lambda^2 f^2} F\left\{\delta\left(\frac{X}{\lambda f}, \frac{Y}{\lambda f}\right)\right\}\Bigg|_{\substack{f_X = x'/\lambda f \\ f_Y = y'/\lambda f}} \qquad \text{since } t(0,0) = 1$$

$$U_{output}(x', y') = \frac{e^{jk4f}}{j\lambda^2 f^2} \lambda^2 f^2 \Big|_{\substack{f_X = x'/\lambda f \\ f_Y = y'/\lambda f}}$$

$$U_{output}(x', y') = \frac{e^{jk4f}}{j\lambda^2 f^2} \lambda^2 f^2$$

$$U_{output}(x', y') = \frac{e^{jk4f}}{j}$$

By **Eq. 3**, the output intensity pattern $I_{output}(x', y')$ is:

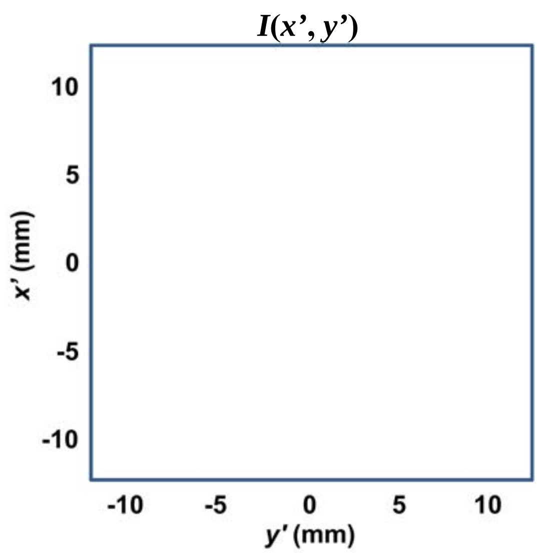
$$I_{output}(x', y') = \left|U_{output}(x', y')\right|^2 = 1$$

Putting the results together:

$$U_{output}(x', y') = \frac{e^{jk4f}}{j} \qquad\qquad I_{output}(x', y') = 1$$

Thus, the output field $U_{output}(x', y')$ is a plane wave (with some phase delay) and the output intensity pattern $I_{output}(x', y')$ is a constant. This result is easily explained. The input field is a plane wave, i.e., with only a DC component. Because the mask is transparent at the point (X, Y) = (0, 0) (via the center square), the DC component of the input is passed by the mask. Therefore, the entire input is preserved and a plane wave appears again at the output.

Below, I have plotted the intensity pattern (generated in Matlab), which is simply a constant background.



Part (b)

In this case, the output field $U_{output}(x', y')$ is most easily found using **Eq. 2**. If a coherent point source is placed at the origin of the input plane, the input field $U_{input}(x, y)$ is a δ function.

$$U_{input}(x, y) = \delta(x, y)$$

The Fourier transform $\hat{t}(f_X, f_Y)$ of the mask $t(X, Y)$ can be calculated as follows:

$$\hat{t}(f_X, f_Y) = F\{t(X,Y)\}$$

$$\hat{t}(f_X, f_Y) = F\left\{rect\left(\frac{X}{\Lambda - d}\right)rect\left(\frac{Y}{\Lambda - d}\right) \otimes comb\left(\frac{X}{\Lambda}\right)comb\left(\frac{Y}{\Lambda}\right)rect\left(\frac{X}{b}\right)rect\left(\frac{Y}{a}\right)\right\}$$

$$\hat{t}(f_X, f_Y) = F\left\{rect\left(\frac{X}{\Lambda - d}\right)rect\left(\frac{Y}{\Lambda - d}\right)\right\}\left[F\left\{comb\left(\frac{X}{\Lambda}\right)comb\left(\frac{Y}{\Lambda}\right)\right\} \otimes F\left\{rect\left(\frac{X}{b}\right)rect\left(\frac{Y}{a}\right)\right\}\right]$$

$$\hat{t}(f_X, f_Y) = (\Lambda - d)^2 \mathrm{sinc}[(\Lambda - d)f_X]\mathrm{sinc}[(\Lambda - d)f_Y] \cdot \left[\Lambda^2 comb(\Lambda f_X)comb(\Lambda f_Y) \otimes ab\,\mathrm{sinc}(bf_X)\mathrm{sinc}(af_Y)\right]$$

$$\hat{t}(f_X, f_Y) = ab\Lambda^2(\Lambda - d)^2 \mathrm{sinc}[(\Lambda - d)f_X]\mathrm{sinc}[(\Lambda - d)f_Y] \cdot \left[\left(\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(\Lambda f_X - m)(\Lambda f_Y - n)\right) \otimes \mathrm{sinc}(bf_X)\mathrm{sinc}(af_Y)\right]$$

$$\hat{t}(f_X, f_Y) = ab\Lambda^2(\Lambda - d)^2 \mathrm{sinc}[(\Lambda - d)f_X]\mathrm{sinc}[(\Lambda - d)f_Y] \cdot \left[\left(\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta\left(f_X - \frac{m}{\Lambda}\right)\delta\left(f_Y - \frac{n}{\Lambda}\right)\right) \otimes \mathrm{sinc}(bf_X)\mathrm{sinc}(af_Y)\right]$$

$$\hat{t}(f_X, f_Y) = ab\Lambda^2(\Lambda - d)^2 \mathrm{sinc}[(\Lambda - d)f_X]\mathrm{sinc}[(\Lambda - d)f_Y] \cdot \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\mathrm{sinc}\left[b\left(f_X - \frac{m}{\Lambda}\right)\right]\mathrm{sinc}\left[a\left(f_Y - \frac{n}{\Lambda}\right)\right]$$

$$\hat{t}(f_X, f_Y) = ab\Lambda^2(\Lambda - d)^2 \mathrm{sinc}[(\Lambda - d)f_X]\mathrm{sinc}[(\Lambda - d)f_Y] \cdot \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\mathrm{sinc}\left[b\left(f_X - \frac{m}{\Lambda}\right)\right]\mathrm{sinc}\left[a\left(f_Y - \frac{n}{\Lambda}\right)\right]$$

$$\hat{t}(f_X, f_Y) = \mathrm{sinc}[(\Lambda - d)f_X]\mathrm{sinc}[(\Lambda - d)f_Y] \cdot \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\mathrm{sinc}\left[b\left(f_X - \frac{m}{\Lambda}\right)\right]\mathrm{sinc}\left[a\left(f_Y - \frac{n}{\Lambda}\right)\right]$$

where the constant factor $ab\Lambda^2(\Lambda - d)^2$ has been dropped for simplicity in the last step above. Thus, $\hat{t}(f_X, f_Y)$ is a grid of skinny 2D sinc functions modulated by a wider 2D sinc function envelope.

Plugging $U_{input}(x, y)$ and $\hat{t}(f_X, f_Y)$ into **Eq. 2**:

$$U_{output}(x', y') = \frac{e^{jk4f}}{j}U_{input}(-x', -y') \otimes \hat{t}\left(\frac{x'}{\lambda f}, \frac{y'}{\lambda f}\right)$$

$$U_{output}(x', y') = \frac{e^{jk4f}}{j}\delta(-x', -y') \otimes \hat{t}\left(\frac{x'}{\lambda f}, \frac{y'}{\lambda f}\right)$$

$$U_{output}(x',y') = \frac{e^{jk4f}}{j}\hat{t}\left(\frac{x'}{\lambda f},\frac{y'}{\lambda f}\right)$$

$$U_{output}(x',y') = \frac{e^{jk4f}}{j}\text{sinc}\left[\frac{(\Lambda-d)}{\lambda f}x'\right]\text{sinc}\left[\frac{(\Lambda-d)}{\lambda f}y'\right]\cdot\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\text{sinc}\left[b\left(\frac{x'}{\lambda f}-\frac{m}{\Lambda}\right)\right]\text{sinc}\left[a\left(\frac{y'}{\lambda f}-\frac{n}{\Lambda}\right)\right]$$

$$U_{output}(x',y') = \frac{e^{jk4f}}{j}\text{sinc}\left[\frac{(\Lambda-d)}{\lambda f}x'\right]\text{sinc}\left[\frac{(\Lambda-d)}{\lambda f}y'\right]\cdot\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\text{sinc}\left[\frac{b}{\lambda f}\left(x'-\frac{\lambda f}{\Lambda}m\right)\right]\text{sinc}\left[\frac{b}{\lambda f}\left(y'-\frac{\lambda f}{\Lambda}n\right)\right]$$

Plugging in $\Lambda = 10$ μm, $d = 2$ μm, $a = 5$mm, $b = 3$ mm, $\lambda = 0.5$ μm, and $f = 10$ cm:

$$U_{output}(x',y') = \frac{e^{jk4f}}{j}\text{sinc}\left[\frac{8\mu\text{m}}{0.5\mu\text{m}\cdot10\text{cm}}x'\right]\text{sinc}\left[\frac{8\mu\text{m}}{0.5\mu\text{m}\cdot10\text{cm}}y'\right]\cdot$$
$$\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\text{sinc}\left[\frac{3\text{mm}}{0.5\mu\text{m}\cdot10\text{cm}}\left(x'-\frac{0.5\mu\text{m}\cdot10\text{cm}}{10\mu\text{m}}m\right)\right]\text{sinc}\left[\frac{5\text{mm}}{0.5\mu\text{m}\cdot10\text{cm}}\left(y'-\frac{0.5\mu\text{m}\cdot10\text{cm}}{10\mu\text{m}}n\right)\right]$$

$$U_{output}(x',y') = \frac{e^{jk4f}}{j}\text{sinc}\left[\frac{8\mu\text{m}}{0.5\mu\text{m}\cdot100\text{mm}}x'\right]\text{sinc}\left[\frac{8\mu\text{m}}{0.5\mu\text{m}\cdot100\text{mm}}y'\right]\cdot$$
$$\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\text{sinc}\left[\frac{3000\mu\text{m}}{0.5\mu\text{m}\cdot100\text{mm}}\left(x'-\frac{0.5\mu\text{m}\cdot100\text{mm}}{10\mu\text{m}}m\right)\right]\text{sinc}\left[\frac{5000\mu\text{m}}{0.5\mu\text{m}\cdot100\text{mm}}\left(y'-\frac{0.5\mu\text{m}\cdot100\text{mm}}{10\mu\text{m}}n\right)\right]$$

$$U_{output}(x',y') = \frac{e^{jk4f}}{j}\text{sinc}\left[0.16\text{mm}^{-1}\cdot x'\right]\text{sinc}\left[0.16\text{mm}^{-1}\cdot y'\right]\cdot$$
$$\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\text{sinc}\left[60\text{mm}^{-1}\cdot(x'-5\text{mm}\cdot m)\right]\text{sinc}\left[100\text{mm}^{-1}\cdot(y'-5\text{mm}\cdot n)\right]$$

Thus, the output field $U_{output}(x', y')$ is a grid of skinny 2D sinc functions separated by 5 mm in each direction and modulated by a wider 2D sinc function envelope. Each skinny sinc function has a main lobe with a width of $\Delta x' = 2 / 60$ mm^{-1} = 0.033 mm in the (vertical) x'-direction and $\Delta y' = 2 / 100$mm^{-1} = 0.02 mm in the (horizontal) y'-direction. The wider sinc function envelope has a main lobe with a width of $\Delta x' = \Delta y' = 2 / 0.16$ mm^{-1} = 12.5 mm.

The output intensity pattern $I_{output}(x', y')$ can be calculated by **Eq. 3**. Since each sinc function is reasonably far from all neighboring sinc functions relative to their widths (5 mm > 0.033 mm), let's assume for simplicity that there is negligible overlap among the sinc functions in the intensity calculation:

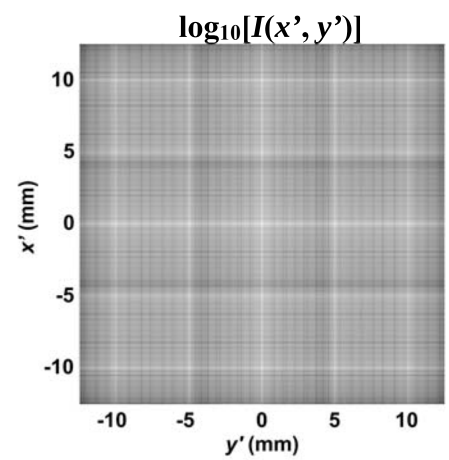
$$I_{output}(x',y') \approx \text{sinc}^2\left[0.16\text{mm}^{-1}\cdot x'\right]\text{sinc}^2\left[0.16\text{mm}^{-1}\cdot y'\right]\cdot$$
$$\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\text{sinc}^2\left[60\text{mm}^{-1}\cdot(x'-5\text{mm}\cdot m)\right]\text{sinc}^2\left[100\text{mm}^{-1}\cdot(y'-5\text{mm}\cdot n)\right]$$

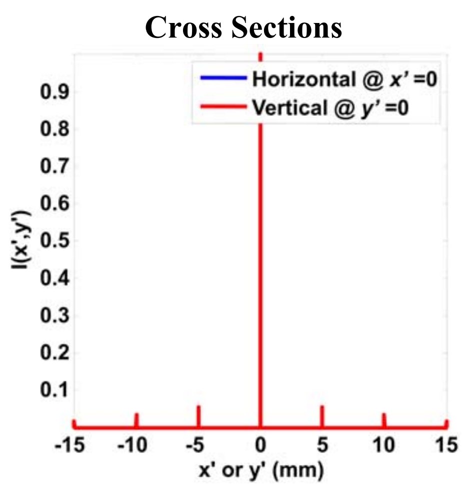
$$U_{output}(x',y') = \frac{e^{jk4f}}{j}\,\mathrm{sinc}[0.16x']\,\mathrm{sinc}[0.16y']\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\mathrm{sinc}[60(x'-5m)]\,\mathrm{sinc}[100(y'-5n)]$$

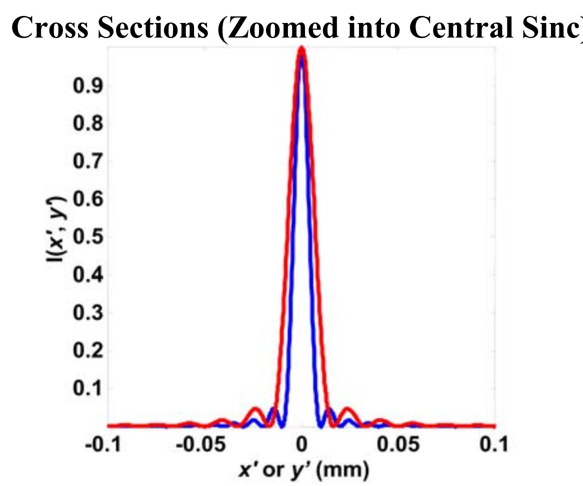
$$I_{output}(x',y') \approx \mathrm{sinc}^2[0.16x']\,\mathrm{sinc}^2[0.16y']\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\mathrm{sinc}^2[60(x'-5m)]\,\mathrm{sinc}^2[100(y'-5n)]$$

where *x'* and *y'* are in units of millimeters (mm). Because the input field was a point source, the output field $U_{output}(x', y')$ and intensity $I_{output}(x', y')$ here are equivalent to the coherent and incoherent PSF(*x'*, *y'*), respectively, of this 4F imaging system.

Below, I have plotted the intensity pattern (generated in Matlab) on a $\log_{10}$ scale, which makes it easy to see the grid of sinc functions repeating every 5 mm in each direction. For more clarity, the bottom figures show the horizontal (@ *x'* = 0) and vertical (@ *y'* = 0) cross sections of the intensity pattern. In the "zoomed out" figure, one can easily see the amplitude of the skinny sinc functions decreasing outward from the center, due to the wider sinc envelope. In the "zoomed in" figure, the vertical cross section is broader than the horizontal cross section because the mask is wider than it is tall, i.e., a > b.







Part (c)

In this case, the output field $U_{output}(x', y')$ is most easily found using **Eq. 2**. If the transparent stamp of Tim the beaver is illuminated by a coherent plane wave, the input field $U_{input}(x, y)$ is a equivalent to the (amplitude transmission of the) stamp itself:

$$U_{input}(x, y) = Tim(x, y)$$

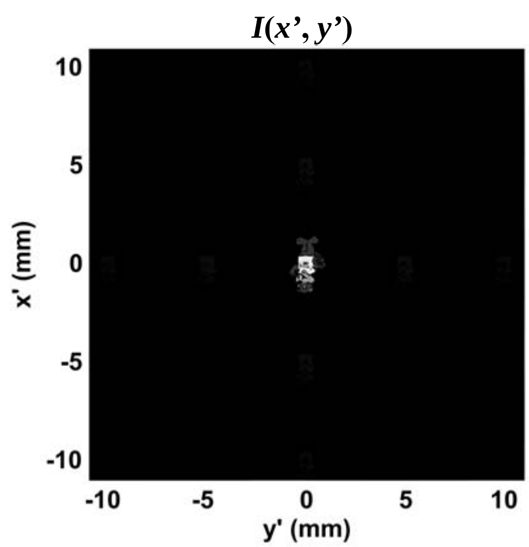
Plugging $U_{input}(x, y)$ and $\hat{t}(f_X, f_Y)$ (calculated in **Part b**) into **Eq. 2**:

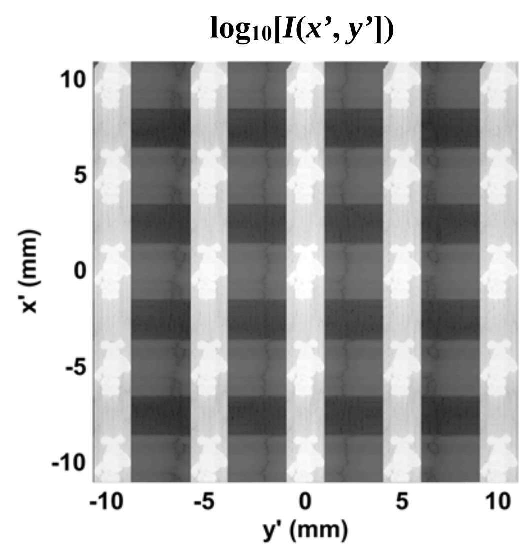
$$U_{output}(x', y') = \frac{e^{jk4f}}{j} Tim(-x', -y') \otimes \left[\mathrm{sinc}[0.16x']\mathrm{sinc}[0.16y'] \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \mathrm{sinc}[60(x'-5m)]\mathrm{sinc}[100(y'-5n)] \right]$$

$$I_{output}(x', y') = \left| U_{output}(x', y') \right|^2$$

where x' and y' are in units of millimeters (mm). The output field $U_{output}(x', y')$ is the convolution of the Tim stamp Tim(x', y') and the coherent PSF. The result is a grid of Tim stamps repeating every 5 mm in each direction and modulated by a sinc function envelope. Each Tim stamp is blurry due to convolution with a low-pass filter, i.e., the skinny sinc function repeated in each direction in the PSF.

Below, I have plotted the intensity pattern (generated in Matlab). The left figure is plotted on a linear scale. On this scale, only the central Tim is clearly visible, due to the sinc envelope making the neighboring Tims considerably darker. For more clarity, the right figure is plotted on a $\log_{10}$ scale. On this scale, the grid of Tims, repeating every 5 mm in each direction, is clearly visible.





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