Mathematical Preparation of Fourier Transform

- Fourier Transform in time domain

A time signal f(t) can be expressed as a series of frequency components F(ω):

$$f(\vec{r},t) = \int_{-\infty}^{\infty} F(\vec{r},\omega)\exp(-i\omega t)\,d\omega$$

$$F(\vec{r},\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\vec{r},t)\exp(+i\omega t)\,dt$$

The functions f(t) and F(ω) are referred to as Fourier Transform pairs.

- Fourier Transform in spatial domain

A spatially varying signal $f(x,y)$can be expressed as a series of spatial-frequency components $F(k_x,k_y)$:

$$f(x,y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(k_x,k_y)\exp(ik_x x)\exp(ik_y y)\,dk_x dk_y$$

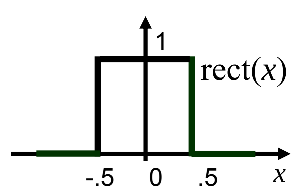
$$F(k_x,k_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\exp(-ik_x x)\exp(-ik_y y)\,dxdy$$

Accordingly, the functions $f(x,y)$and $F(k_x,k_y)$are referred to as spatial-Fourier Transform pairs.

- A few famous functions

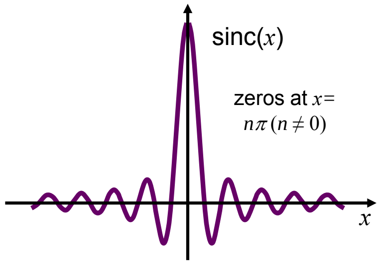
  - Rectangle function

$$rect(x) \equiv \begin{cases} 1, & |x| < \frac{1}{2} \\ \frac{1}{2}, & |x| = \frac{1}{2} \\ 0, & |x| > \frac{1}{2} \end{cases}$$



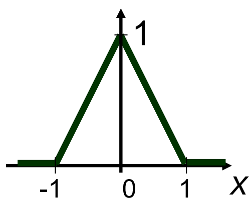
  - Sinc function

$$sinc(x) = \frac{\sin(\pi x)}{\pi x}$$



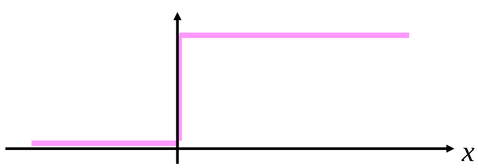
- Triangle Function

  $$\Lambda(x) \equiv \begin{cases} 1-|x|, & |x|<1 \\ 0, & |x| \geq 1 \end{cases}$$



- Step function

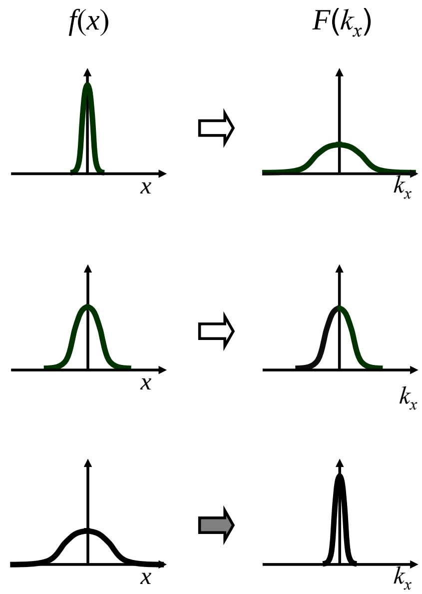
  $$H(x) \equiv \begin{cases} 1, & x>0 \\ \frac{1}{2}, & x=0 \\ 0, & x<0 \end{cases}$$



- Comb function

  $$comb(x) = \sum_{n=-\infty}^{\infty} \delta(x-n)$$

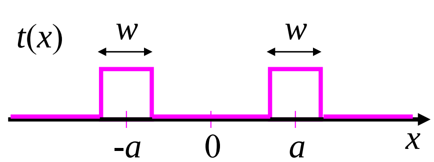
- Interesting properties of Fourier Transform:
  - Scale theorem $f(ax)$



  - Shift theorem $f(x-a)$,

    (e.g. double slit)

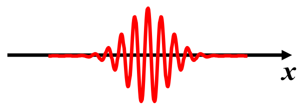
    $t(x) = \text{rect}[(x+a)/w] + \text{rect}[(x-a)/w]$

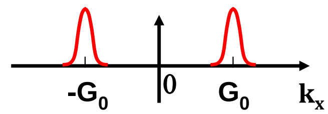


  - Complex Conjugate $f^*(x)$
  - Derivative $\frac{d}{dx}f(x)$

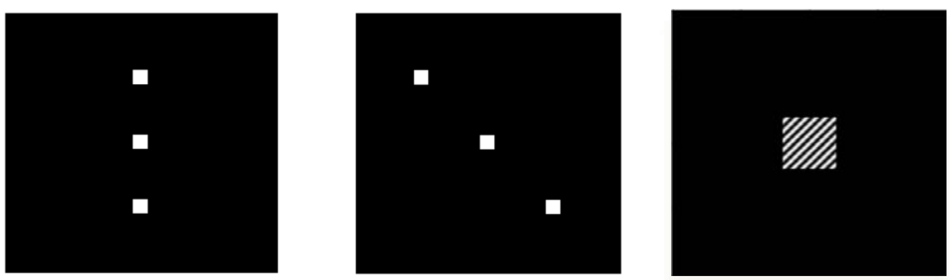
- Modulation $f(x)\cos(\frac{2\pi}{L}x)$

$$f(x)\cos(\frac{2\pi}{L}x) \qquad F(k_x)\otimes\big(\delta(k-G_0)+\delta(k+G_0)\big)$$





- Practice problem: can you apply these theorems in (x, y) plane?



Accordingly, the following functions $f(x,y)$and $F(k_x,k_y)$are referred to as spatial-Fourier Transform pairs.

Functions	*Fourier Transform Pairs*
$rect\left(\frac{x}{a}\right)$	$\lvert a\rvert sinc\left(\frac{ak_x}{2\pi}\right)$
$sinc\left(\frac{x}{a}\right)$	$\lvert a\rvert rect\left(\frac{ak_x}{2\pi}\right)$
$\Lambda\left(\frac{x}{a}\right)$	$\lvert a\rvert^2 sinc^2\left(\frac{ak_x}{2\pi}\right)$
$\mathrm{comb}\left(\frac{x}{a}\right)$	$\lvert a\rvert comb\left(\frac{ak_x}{2\pi}\right)$
$Gaussian\ exp\left(-\frac{x^2}{a^2}\right)$	$exp\left(-\frac{a^2}{4\pi}{k_x}^2\right)$
$Step\ function\ H(x)$	$\frac{1}{ik_x}+\frac{1}{2}\left(\delta(k_x)\right)$
$circ\left(\frac{\sqrt{x^2+y^2}}{a}\right)$	$\lvert a\rvert^2\frac{2\pi J_1\left(a\sqrt{{k_x}^2+{k_y}^2}\right)}{a\sqrt{{k_x}^2+{k_y}^2}}$