Outline:

A. *Optical Invariant*
B. *Composite Lenses*
C. *Ray Vector and Ray Matrix*
D. *Location of Principal Planes for an Optical System*
E. *Aperture Stops, Pupils and Windows*

A. Optical Invariant

-*What happens to an arbitrary "axial" ray that originates from the axial intercept of the object, after passing through a series of lenses?*

If we make use of the relationship between launching angle and the imaging conditions, we have:

$$\theta_{in} = \frac{x_{in}}{s_o} \text{ and } \theta_{out} = -\frac{x_{in}}{s_i}$$

$$\frac{\theta_{in}}{\theta_{out}} = -\frac{s_i}{s_o} = \frac{h_i}{h_o}$$

Rearranging, we obtain:

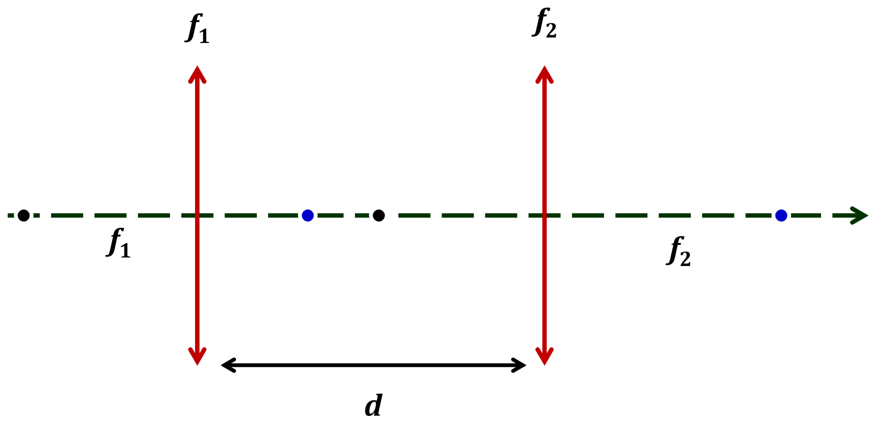
$$\theta_{in} h_o = \theta_{out} h_i$$

We see that the product of the image height and the angle with respect to the axis (**the components of the ray vector!**) remains a **constant**. Indeed a more general result, $n h_o sin\theta_{in} = n' h_i sin\theta_{out}$ is a constant (often referred as a *Lagrange invariant* in different textbooks) across any surface of the imaging system.

- The invariant may be used to deduce other quantities of the optical system, without the necessity of certain intermediate ray-tracing calculations.
- You may regard it as a precursor to *wave optics*: the angles are approximately proportional to lateral momentum of light, and the image height is equivalent to separation of two geometric points. For two points that are separated far apart, there is a limiting angle to transmit their information across the imaging system.

B. Composite Lenses

To elaborate the effect of lens in combinations, let's consider first two lenses separated by a distance d. We may apply the *thin lens* equation and cascade the imaging process by taking the image formed by lens 1 as the object for lens 2.



$$\frac{1}{s_{o1}} + \frac{1}{s_{i2}} = \left(\frac{1}{f_1} + \frac{1}{f_2}\right) - \frac{d}{(d - s_{i1})s_{i1}}$$

A few limiting cases:

a) Parallel beams from the left: s_{i2} is the back-focal length (BFL)

$$\frac{1}{\mathrm{BFL}} = \left(\frac{1}{f_1} + \frac{1}{f_2}\right) - \frac{d}{(d - f_1)f_1}$$

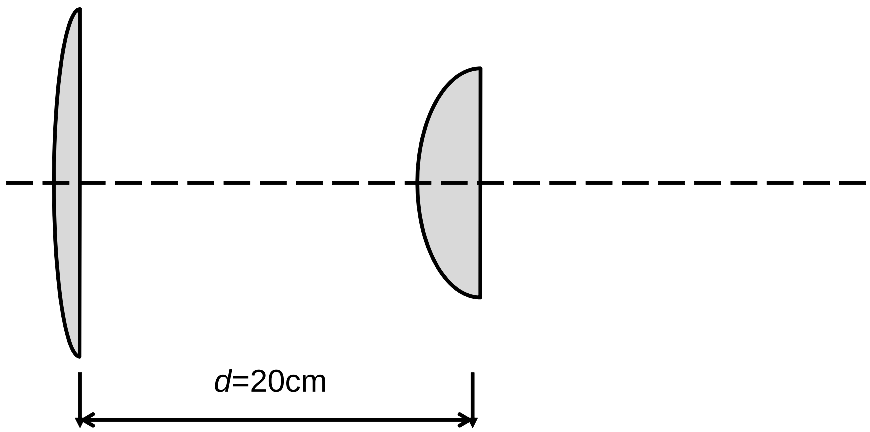
b) collimated beams to the right: s_{o1} is the front-focal length (FFL)

$$\frac{1}{\mathrm{FFL}} = \left(\frac{1}{f_1} + \frac{1}{f_2}\right) - \frac{d}{(d - f_2)f_2}$$

The composite lens does not have the same apparent focusing length in front and back end!

c) $d=f_1+f_2$: parallel beams illuminating the composite lens will remain parallel at the exit; the system is often called *afocal.* This is in fact the principle used in most **telescopes**, as the object is located at infinity and the function of the instrument is to send the image to the eye with a large angle of view. On the other hand, a point source located at the left focus of the first lens is imaged at the right focus of the second lens (the two are called *conjugate points*). This is often used as a **condenser** for illumination.

Practice Example: Huygens eyepiece



A **Huygens eyepiece** is designed with two plano-convex lenses separated by the average of the two focal length. Ideally, such eyepiece should produce a *virtual* image at *infinity* distance. Let f_1=30cm and f_2=10cm, so the spacing *d*=20cm, let's find these parameters:

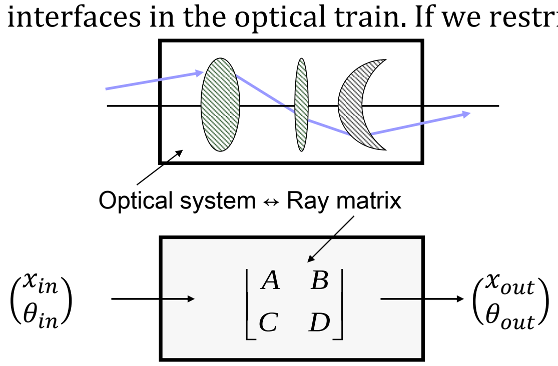
a) BFL and FFL,

b) the location of PPs,

c) the EFL.

C. Ray Vector and Ray Matrix

In principle, ray tracing can help us to analyze image formation in any given optical system as the rays refract or reflect at all interfaces in the optical train. If we restrict the analysis to paraxial rays only, then such process can be described in a matrix approach.



In the Feb 10 lecture, we defined a light ray by two co-ordinates:

a. its position, *x*

b. its slope, θ

These parameters define a **ray vector**, which will change with distance and as the ray propagates through optics.

Associated with the *input* ray vector $\begin{pmatrix} x_{in} \\ \theta_{in} \end{pmatrix}$ and *output* ray vector $\begin{pmatrix} x_{out} \\ \theta_{out} \end{pmatrix}$, we can express the effect of the optical elements in the general form of a 2x2 ray matrix:

$$\begin{pmatrix} x_{out} \\ \theta_{out} \end{pmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{pmatrix} x_{in} \\ \theta_{in} \end{pmatrix}$$

These matrices are often (uncreatively) called ***ABCD*** Matrices.
Since the displacements and angles are assumed to be small, we can think in terms of partial derivatives.

$$x_{out} = \left(\frac{\partial x_{out}}{\partial x_{in}} \right) x_{in} + \left(\frac{\partial x_{out}}{\partial \theta_{in}} \right) \theta_{in}$$

$$\theta_{out} = \left(\frac{\partial \theta_{out}}{\partial x_{in}} \right) x_{in} + \left(\frac{\partial \theta_{out}}{\partial \theta_{in}} \right) \theta_{in}$$

Therefore, we can connect the Matrix components with the functions of the imaging elements:

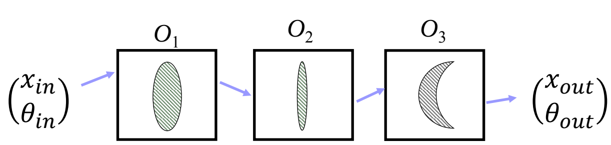
$A = \left(\frac{\partial x_{out}}{\partial x_{in}} \right)$: **spatial** magnification;

$D = \left(\frac{\partial \theta_{out}}{\partial \theta_{in}} \right)$: **angular** magnification;

$B = \left(\frac{\partial x_{out}}{\partial \theta_{in}} \right)$: mapping **angles(momentum)** to **position** (function of a prism);

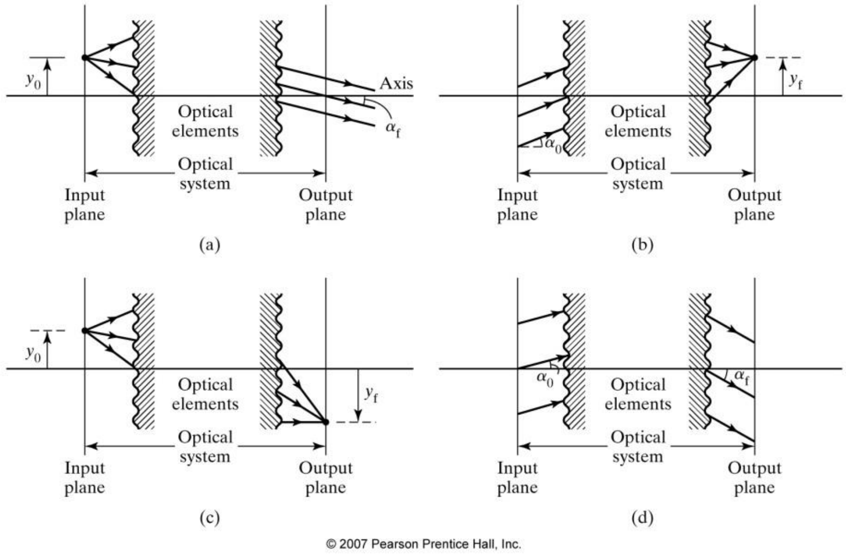
$C = \left(\frac{\partial \theta_{out}}{\partial x_{in}} \right)$: mapping **position** to **angles(momentum)** (also function of a prism).

For cascaded elements, we simply multiply ray matrices. (please notice the order of matrices starts from left to right on optical axis!!)



$$\begin{pmatrix} x_{out} \\ \theta_{out} \end{pmatrix} = [O_3][O_2][O_1] \begin{pmatrix} x_{in} \\ \theta_{in} \end{pmatrix}$$

Significance of the matrix elements: (Pedrotti Figure 18.9)



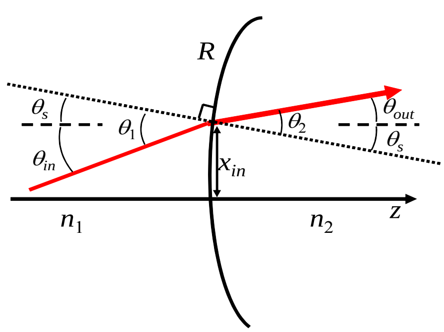
(a) If the input surface is at the front focal plane, the outgoing ray angles depend only on the incident height.

(b) Similarly, if the output surface is at the back focal plane, the outgoing ray heights depend only on the incoming angles.

(c) If the input and output plane are conjugate, then all incoming rays from constant height y_0 will converge at a constant height regardless of their angle.

(d) When the system is "afocal", the refracting angles of the outgoing beams are independent of the input positions.

Example 1: refraction matrix from a spherical interface (only changes θ but not x)



Right at the interface,

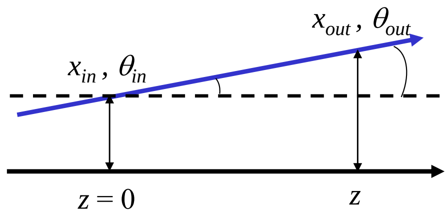
$$x_{in} = x_{out}$$

$$n_1(\theta_{in} + x_{in}/R) \approx n_2(\theta_{out} + x_{in}/R)$$

$$\theta_{out} \approx \left(\frac{n_1}{n_2}\right)\theta_{in} + \frac{\left[\left(\frac{n_1}{n_2}\right) - 1\right]}{R} x_{in}$$

So we can write the matrix:

Example 2: matrix of a ray propagating in a medium (changes *x* but not θ)

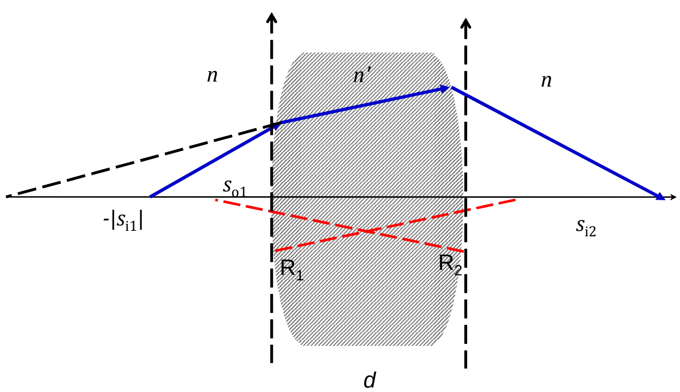


$$x_{out} = x_{in} + z\,\theta_{in}$$
$$\theta_{out} = \theta_{in}$$

Example 3: refraction matrix through a thin lens (combined refraction)

$$O_{thin\ lens} = O_{\substack{curved\\ interface\,2}}\; O_{\substack{curved\\ interface\,1}} = \begin{bmatrix} 1 & 0 \\ (n-1)/R_2 & n \end{bmatrix}\begin{bmatrix} 1 & 0 \\ (1/n-1)/R_1 & 1/n \end{bmatrix}$$

Example 4: Imaging matrix through a thick lens (combined refraction and translation)



From left to right:

- Translation O1:

$$\begin{bmatrix} 1 & s_{o1} \\ 0 & 1 \end{bmatrix}$$

- Refraction O2:

$$\begin{bmatrix} 1 & 0 \\ \dfrac{\left[\left(\frac{n}{n'}\right)-1\right]}{R_1} & \left(\frac{n}{n'}\right) \end{bmatrix}$$

- Translation O3:

$$\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$$

- Refraction O4:

$$\begin{bmatrix} 1 & 0 \\ \dfrac{\left[\left(\frac{n'}{n}\right)-1\right]}{-R_2} & \left(\frac{n'}{n}\right) \end{bmatrix}$$

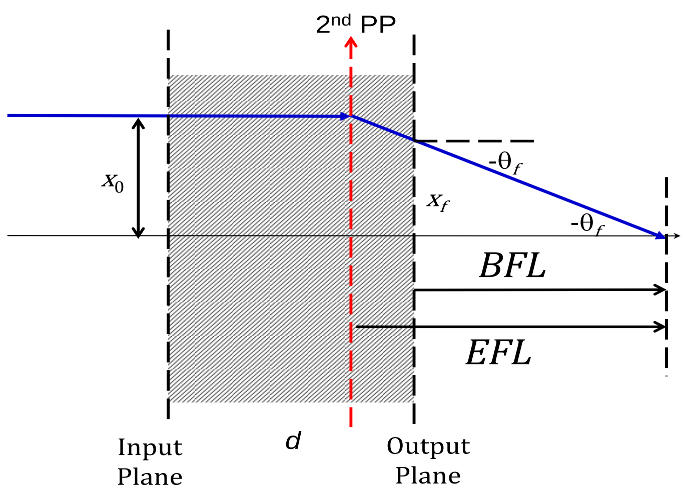
- Translation O5:

$$\begin{bmatrix} 1 & s_{i2} \\ 0 & 1 \end{bmatrix}$$

D. Location of Principal Planes for an Optical System

A ray matrix of the optical system (composite lenses and other elements) can give us a complete description of the rays passing through the overall optical train. In this session, we show that the focusing properties of the composite lens, such as the principal planes.

In order to facilitate our analysis, we choose the *input plane* to be the front surface of the lens arrays, and the *output plane* to be the back surface of the lenses.



(Adapted from Pedrotti **Figure 18-12**)

Let's start with the process of focusing at back focus first. In this case, an incoming parallel ray$\begin{pmatrix} x_0 \\ 0 \end{pmatrix}$ is refracted from the 2nd principal plane (PP) so it passes through the back focal point (BF). At the output plane, the ray vector of the refracted ray reads$\begin{pmatrix} x_f \\ -\theta_f \end{pmatrix}$.

$$\begin{pmatrix} x_f \\ -\theta_f \end{pmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{pmatrix} x_0 \\ 0 \end{pmatrix}$$

This gives $x_f = Ax_0$ and $-\theta_f = Cx_0$.

Using the small angle approximation, we can connect the ratio of beam height x_0 and the effective focal length (EFL) by the steering angle θ_f:

$$\theta_f = x_0/EFL$$

Thus

$$EFL = -1/C.$$

Also from the similar triangles,

$$x_f/x_0 = BFL/EFL.$$

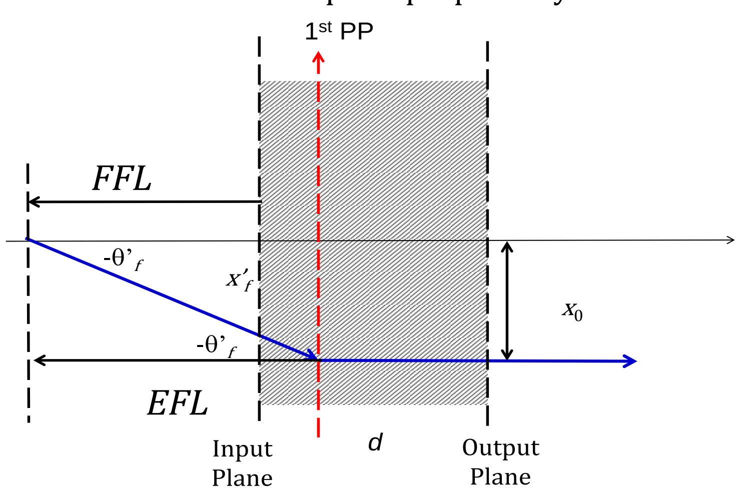
We can find BFL:

$$BFL = -A/C.$$

Thus the 2nd PP is located at a distance from the output plane given by:

$$BFL - EFL = -(A-1)/C.$$

Likewise, we can find FFL and the first principal plane by the matrix components.



$$\begin{pmatrix} x'_0 \\ 0 \end{pmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{pmatrix} -x'_f \\ -\theta'_f \end{pmatrix}$$

You could consider this as an inverse problem of the previous example, or solve the relationship:

$$x'_0 = -Ax'_f - B\theta'_f$$

$$0 = -Cx'_f - D\theta'_f$$

$$\theta'_f = x'_0/EFL \; and \; \theta'_f = x'_f/FFL$$

So how is the ray matrix experimentally determined by ray tracing?

Generally, for a given (2D) optical system with unknown details, one way to determine the transfer matrix is to take measurement of two arbitrary input and output rays. To elaborate that idea, we can treat a pair of the input ray vectors as a 2x2 matrix:

$$\begin{pmatrix} x_{out}^1 & x_{out}^2 \\ \theta_{out}^1 & \theta_{out}^2 \end{pmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{pmatrix} x_{in}^1 & x_{in}^2 \\ \theta_{in}^1 & \theta_{in}^2 \end{pmatrix}$$

Therefore

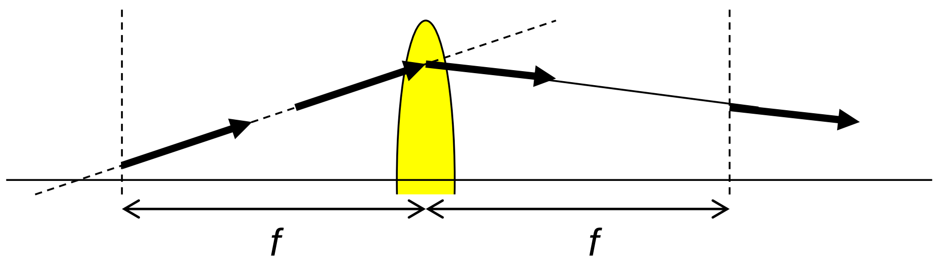
$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{pmatrix} x_{out}^1 & x_{out}^2 \\ \theta_{out}^1 & \theta_{out}^2 \end{pmatrix} \begin{pmatrix} x_{in}^1 & x_{in}^2 \\ \theta_{in}^1 & \theta_{in}^2 \end{pmatrix}^{-1}$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \frac{1}{(x_{in}^1\theta_{in}^2 - x_{in}^2\theta_{in}^1)} \begin{pmatrix} x_{out}^1\theta_{in}^2 - x_{out}^2\theta_{in}^1 & x_{out}^2 x_{in}^1 - x_{out}^1 x_{in}^2 \\ \theta_{out}^1\theta_{in}^2 - \theta_{out}^2\theta_{in}^1 & \theta_{out}^2 x_{in}^1 - \theta_{out}^1 x_{in}^2 \end{pmatrix}$$

As a special case you may select the two rays to be marginal and chief rays as defined in the following section.

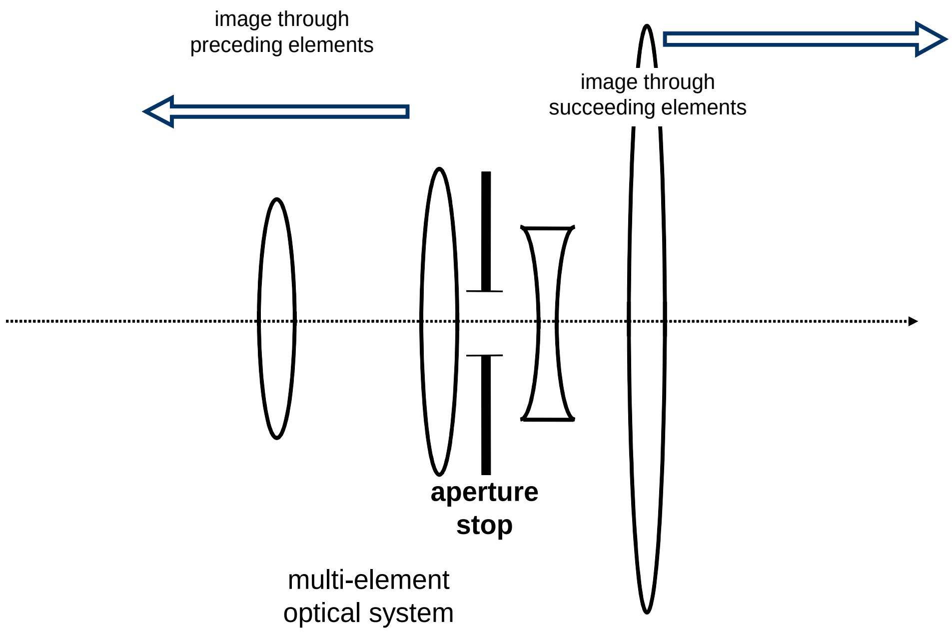
Practice Example: Rays Going Through 2F/4F Lens system

Please determine the ray transfer matrix of the following lens elements, with their input and output planes located at the front and back focal point of the corresponding lens.



E. Aperture Stops, Pupils and Windows

- The Aperture Stops and **Numerical Aperture**



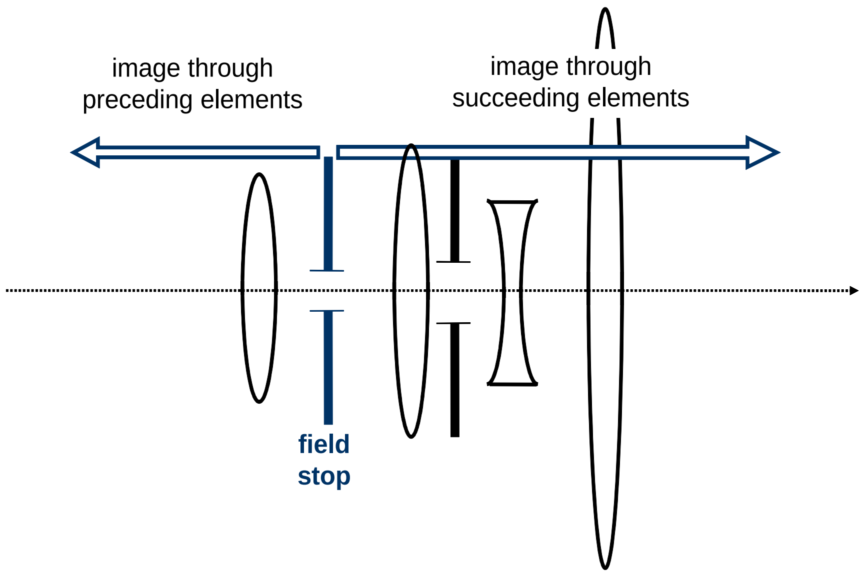
- **Numerical Aperture(NA)**:
  - limits the optical *flux* that is admitted through the system;
  - also defines the *resolution* (or *resolving power*) of the optical system

- The concept of ***marginal rays*** and ***chief rays***
  - **Marginal ray:** the ray that passes through the edge of the aperture.
  - **Chief ray** (also called *principal rays*): the ray from an object point that passes through the axial point of the aperture stop (also appears as emitting from the axis of exit pupil).

Together, the C.R. and M.R. define the angular acceptance of spherical ray bundles originating from an off-axis object.

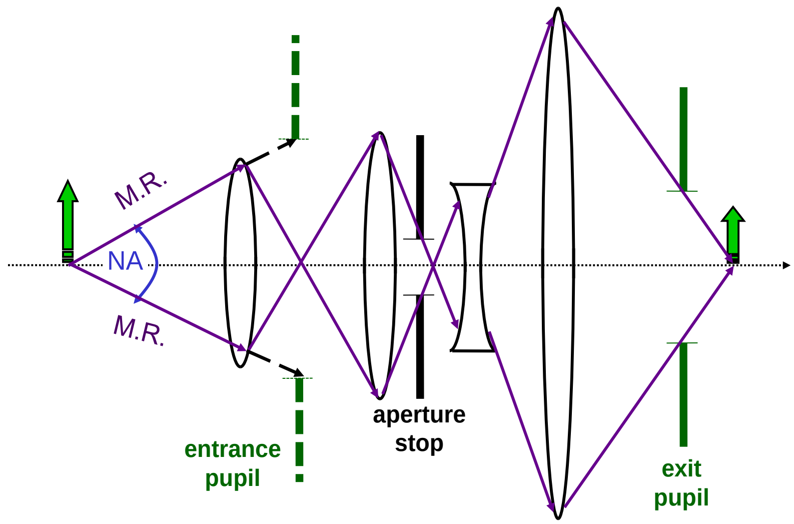
- The **entrance** and **exit pupils**

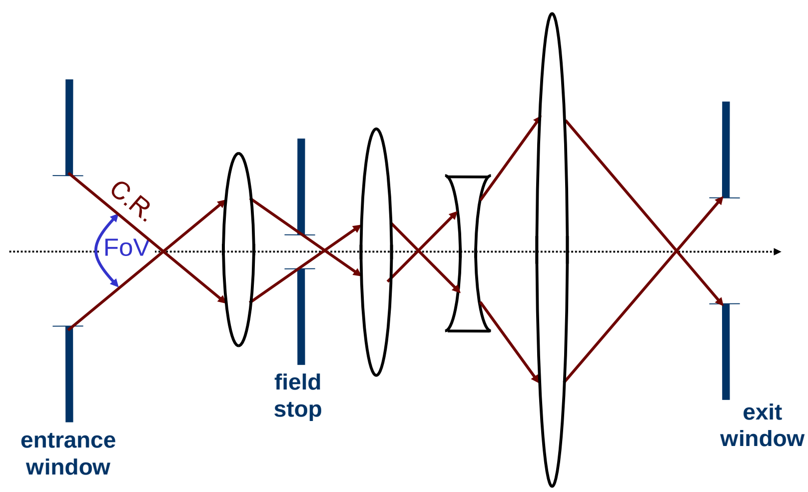
- The field stop and corresponding windows



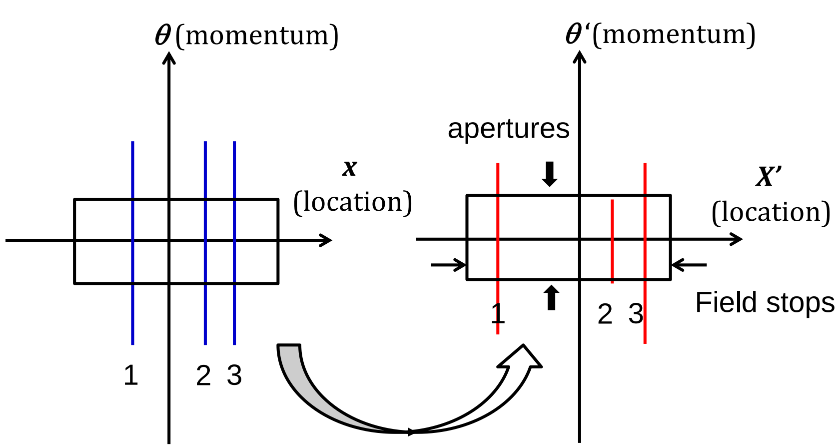
  - **Field stop**:
    - Limits the angular acceptance of ***Chief Rays***
    - Defines the **Field of View**
    - Proper FS should be at intermediate image plane
  - ***Entrance & Exit* Windows**

- ***Effect of Aperture and field stops***

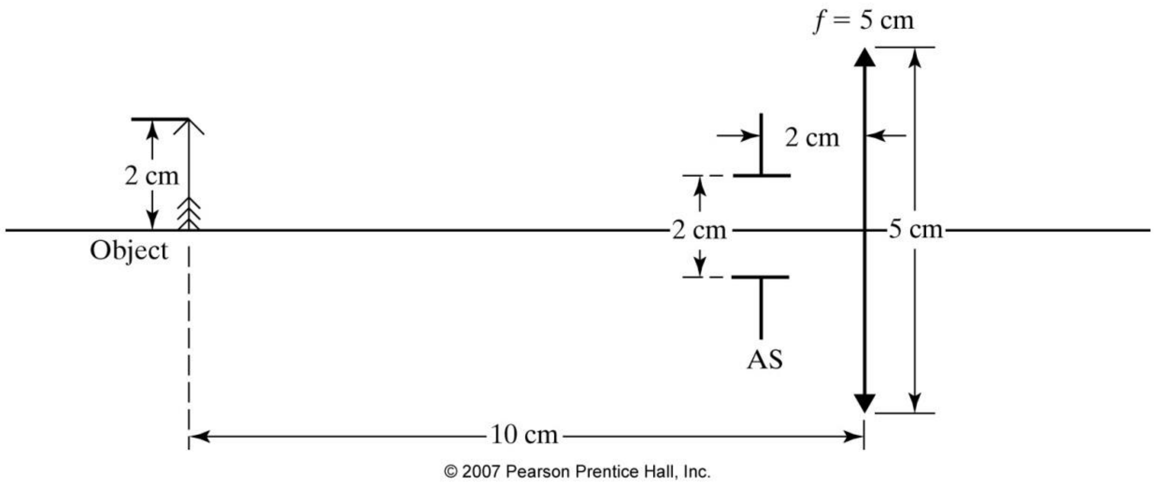




Effect of Apertures and stops



Practice Example: Single lens camera:



- Please determine the position and size of the image.
- Please determine the entrance and exit pupils.
- Please sketch the chief ray and marginal rays from the top of the object to the image.