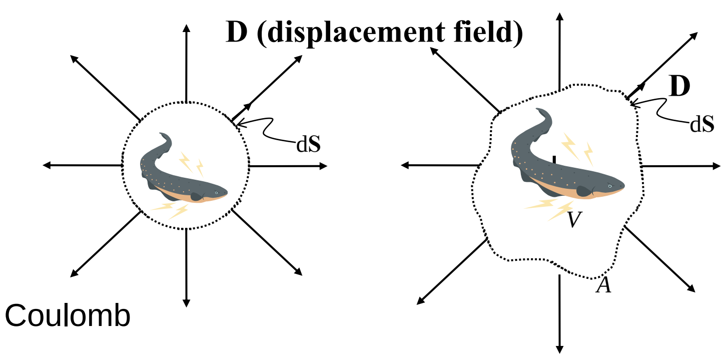
Outline:

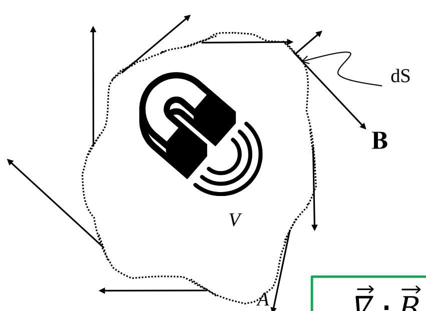
A. *Electromagnetism*
B. *Frequency Domain (Fourier transform)*
C. *EM waves in Cartesian coordinates*
D. *Energy Flow and Poynting Vector*
E. *Connection to geometrical optics*
F. *Eikonal Equations: Path of Light in an Inhomogeneous Medium*

A. Electromagnetism and Maxwell Equations, Differential Forms:

- In real space, time-dependent fields:



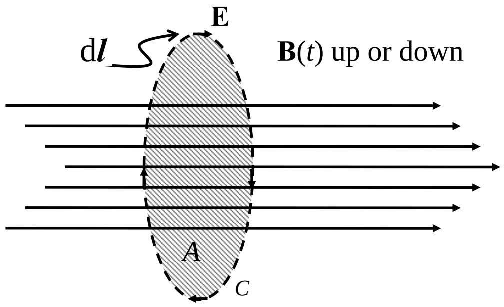
(Coulomb's Law, electric field) $$\vec{\nabla} \cdot \vec{D} = q$$ **(1)**



there are no magnetic charges

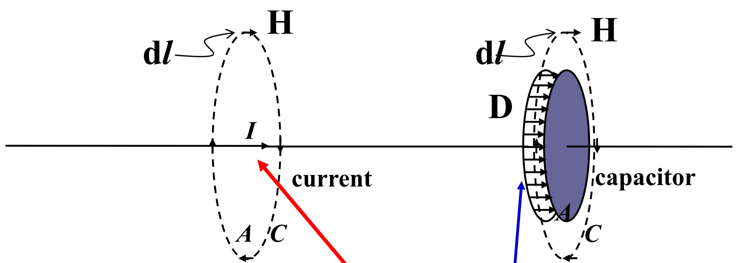
$$\vec{\nabla} \cdot \vec{B} = 0$$ **(2)**

(Gauss' Law, magnetic field)



$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (3)$$

(Faraday's Law)



$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad (4)$$

(Ampere-Maxwell's Law)

Note: J, q are sources of EM radiation and **E, D, H, B** are induced fields.

B. From time domain to frequency domain (Fourier Transform):

Continuous wave laser light field understudy are often mono-chromatic. These problems are mapped in the Maxwell equations by expanding complex time signals to a series of time harmonic components (often referred to as "**single**" wavelength light):

e.g. $\vec{E}(\vec{r}, t) = \int_{-\infty}^{\infty} \boldsymbol{E}(\vec{r}, \omega) exp(-i\omega t) d\omega$ **(5)**

Advantage:

$$\frac{\partial}{\partial t}\vec{E}(\vec{r}, t) = \int_{-\infty}^{\infty} \boldsymbol{E}(\vec{r}, \omega) \frac{\partial}{\partial t} exp(-i\omega t) d\omega = \int_{-\infty}^{\infty} \left[-\boldsymbol{i\omega E}(\vec{\boldsymbol{r}}, \boldsymbol{\omega})\right] exp(-i\omega t) d\omega \quad \textbf{(6)}$$

So, we can replace all time derivatives $\frac{\partial}{\partial t}$ by $-i\omega$ in frequency domain:

(Faraday's Law:) $$\vec{\nabla} \times \vec{E} = +i\omega\vec{B} \quad (7)$$

(Ampere-Maxwell's Law:) $$\vec{\nabla} \times \vec{H} = \vec{J} - i\omega\vec{D} \quad (8)$$

The other two equations remain unaltered.

There are total of 12 unknowns (***E, H, D, B***) but so far we only obtained 8 equations from the Maxwell equations (2 vector form x3 + 2 scalar forms) so more information needed to understand the complete wave behavior!

Generally we may start to construct the response of a material by applying a excitation field E or H in vacuum. Therefore it is more typical to consider the E, H field as input and D, B fields as output. In common optical materials, we may enjoy the following simplification of local (i.e. independent of neighbors) and linear relationship:

$$\vec{\boldsymbol{D}}(\omega) = \varepsilon(\omega)\varepsilon_0\vec{\boldsymbol{E}}(\omega) \quad \textbf{(9)}$$

$$\vec{\boldsymbol{B}}(\omega) = \mu(\omega)\mu_0\vec{\boldsymbol{H}}(\omega) \quad \textbf{(10)}$$

The so called (electric) permittivity $\varepsilon(\omega)$ and (magnetic) permeability $\mu(\omega)$are unitless parameters that depend on the frequency of the input field. In the case of anisotropic medium, both $\varepsilon(\omega)$ and $\mu(\omega)$become 3x3 dimension tensor.

Now we have 6 more equations from material response, we can include them together with Maxwell equations to obtain a complete solution of optical fields with proper boundary condition.

C. Maxwell's Equations in Cartesian Coordinates:

To solve Maxwell equations in Cartesian coordinates, we need to practice on the vector operators accordingly. Most unfamiliar one is probably the curl of a vector $\vec{A}$. In Cartesian coordinates it is often written as a matrix determinant:

$$\boldsymbol{\nabla} \times \vec{\boldsymbol{A}} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_{\boldsymbol{x}} & A_{\boldsymbol{y}} & A_{\boldsymbol{z}} \end{vmatrix} \quad \textbf{(11)}$$

In this fashion, we may write the Faraday's law in frequency domain, $\vec{\nabla} \times \vec{E} = +i\omega\vec{B}$, with 3 components in Cartesian coordinates:

$$\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) = i\omega B_x \quad (12)$$

$$\left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right) = i\omega B_y \quad (13)$$

$$\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) = i\omega B_z \quad (14)$$

Likewise, we arrive at the rest of Maxwell's equations:

$$\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) = -i\omega D_x \quad (15)$$

$$\left(\frac{\partial H_z}{\partial x} - \frac{\partial H_x}{\partial z}\right) = -i\omega D_y \quad (16)$$

$$\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) = -i\omega D_z \quad (17)$$

together with

$$\frac{\partial}{\partial x}D_x + \frac{\partial}{\partial y}D_y + \frac{\partial}{\partial z}D_z = 0 \quad (18)$$

And

$$\frac{\partial}{\partial x}B_x + \frac{\partial}{\partial y}B_y + \frac{\partial}{\partial z}B_z = 0 \quad (19)$$

- **Example: Plane EM wave in 1-D homogeneous medium (e.g. an expanded laser beam in +z direction, $\boldsymbol{\frac{\partial}{\partial x} = 0, \frac{\partial}{\partial y} = 0}$)**

We can now further simplify from the above equations in Cartesian coordinates:

$$\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) = i\omega B_x \quad (20)$$

$$\left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right) = i\omega B_y \quad (21)$$

And

$$\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) = -i\omega D_x \quad (22)$$

$$\left(\frac{\partial H_z}{\partial x} - \frac{\partial H_x}{\partial z}\right) = -i\omega D_y \quad (23)$$

From the above we found only 4 non-trivial equations. In the isotropic case, they can be further divided into 2 independent sub-groups (two Polarizations!):

(E_x, H_y only)

$$i\omega\mu\mu_0 H_y = -\frac{\partial}{\partial z}E_x \tag{24}$$

and $$i\omega\varepsilon\varepsilon_0 E_z = \frac{\partial}{\partial z}H_y \tag{25}$$

Or

(E_y, H_x only)

$$i\omega\mu\mu_0 H_x = -\frac{\partial}{\partial z}E_y \tag{26}$$

And $$i\omega\varepsilon\varepsilon_0 E_y = \frac{\partial}{\partial z}H_x \tag{27}$$

Observations:

- We see that wave propagation in such medium is purely transverse, i.e. only components of E, H field that are orthogonal to propagation direction (+z) survived in the wave field.
- Taking the derivative $\frac{\boldsymbol{\partial}}{\boldsymbol{\partial z}}$ again on any of these equations, we obtain wave equation such as:

$$\frac{\partial^2}{\partial z^2}E_x = -i\omega\mu\mu_0\frac{\partial}{\partial z}H_y = -i\omega\mu\mu_0(i\omega\varepsilon\varepsilon_0)E_x \tag{28}$$

$$\frac{\partial^2}{\partial z^2}E_x = (\omega^2\mu_0\varepsilon_0)\varepsilon\mu E_x = \left(\frac{\varepsilon\mu\omega^2}{c_0^2}\right)E_x \tag{29}$$

since the speed of light c_0 in vacuum satisfy $\left(\ c_0^2 = \frac{1}{\varepsilon_0\mu_0}\right)$

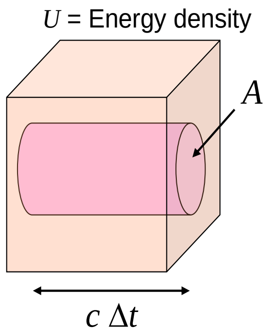
Therefore the index of refraction is found as:

$$n(\omega) = \sqrt{\varepsilon(\omega)\mu(\omega)} \tag{30}$$

D. The Poynting Vector

The Poynting vector, $\vec{\boldsymbol{S}} = \varepsilon_0 c_0^2 \vec{E} \times \vec{B}$, is used to power per unit area in the direction of propagation.

- Justification:



Energy passing through area A in time Δt:

So the energy per unit time per unit area:

- The **Irradiance** (often called the Intensity)
  Visible light wave oscillates in 10^{14}-10^{15}Hz. Since we don't have a detector that responds in such a high speed yet, for convenience we take the average power per unit area, as the irradiance.

  Substituting a sinusoidal light wave into the expression for the Poynting vector yields

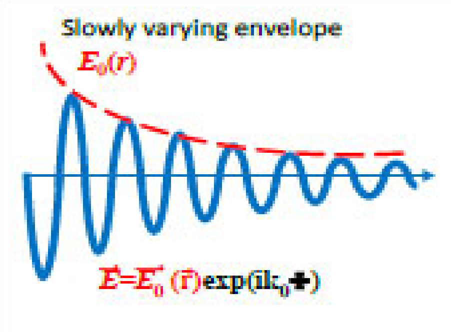
$$\langle \vec{\boldsymbol{S}} \rangle = \varepsilon_0 c_0^2 \langle \vec{E} \times \vec{B} \rangle = \frac{1}{2} \varepsilon_0 c_0^2 E_0 B_0 \quad (31)$$

E. High Frequency Limit, connection to Geometric Optics:

How can we obtain Geometric optics picture such as ray tracing from wave equations? Now let's go back to real space and frequency domain (in an isotropic medium but with spatially varying permittivity $\varepsilon(x, z)$, for example).

$$\frac{\partial^2}{\partial x^2} E_x + \frac{\partial^2}{\partial z^2} E_x + \varepsilon(x, z) \left(\frac{\omega^2}{c_0^2}\right) E_x = 0 \quad (32)$$

Now we decompose the field $\mathbf{E}(\mathbf{r}, \omega)$ into two forms: a fast oscillating component $\exp(ik_0\Phi)$, $k_0 = \omega/c_0$ and a slowly varying envelope $\mathbf{E}_0(\mathbf{r})$ as illustrated in the textbox.



Example of decomposition of E field into the product of slowly varying envelope and a fast oscillating phase $\exp(ik_0\Phi)$

With this tentative solution, we can rewrite the wave equation:

$$\frac{\partial}{\partial x}E_x = \left(\frac{\partial}{\partial x}E_0(x,z)\right)\exp\left(ik\Phi(x,z)\right) + E_0(x,z)\left[\frac{\partial}{\partial x}\exp\left(ik\Phi(x,z)\right)\right] \quad (33)$$

$$\frac{\partial}{\partial x}E_x = \left(\frac{\partial}{\partial x}E_0(x,z)\right)\exp\left(ik\Phi(x,z)\right) + E_0(x,z)\left[ik\frac{\partial\Phi(x,z)}{\partial x}\right]\exp\left(ik\Phi(x,z)\right) \quad (34)$$

$$\frac{\partial}{\partial x}E_x = \left[\frac{\partial}{\partial x}E_0(x,z) + ikE_0(x,z)\frac{\partial\Phi(x,z)}{\partial x}\right]\exp\left(ik\Phi(x,z)\right) \quad (35)$$

$$\frac{\partial^2}{\partial x^2}E_x = \left[\frac{\partial^2}{\partial x^2}E_0(x,z) + ikE_0(x,z)\frac{\partial^2}{\partial x^2}\Phi(x,z) + 2ik\frac{\partial}{\partial x}E_0(x,z)\frac{\partial}{\partial x}\Phi(x,z) - k^2E_0(x,z)\left(\frac{\partial}{\partial x}\Phi(x,z)\right)^2\right]\exp\left(ik\Phi(x,z)\right) \quad (36)$$

Similarly you can find the derivative along the z direction. So our wave equation becomes:

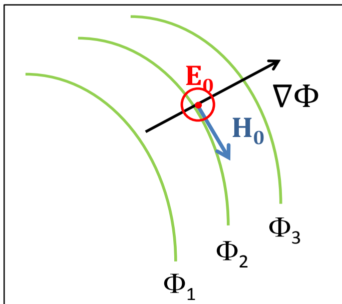
$$k^2\left[\varepsilon(x,z) - \left(\frac{\partial}{\partial x}\Phi(x,z)\right)^2 - \left(\frac{\partial}{\partial z}\Phi(x,z)\right)^2\right]E_0(x,z)$$

$$+\left[\frac{\partial^2}{\partial x^2}E_0(x,z) + \frac{\partial^2}{\partial z^2}E_0(x,z)\right] + 2ik\left[\frac{\partial}{\partial x}E_0(x,z)\frac{\partial}{\partial x}\Phi(x,z) + \frac{\partial}{\partial z}E_0(x,z)\frac{\partial}{\partial z}\Phi(x,z)\right] + ikE_0(x,z)\left[\frac{\partial^2}{\partial x^2}\Phi(x,z) + \frac{\partial^2}{\partial z^2}\Phi(x,z)\right] = 0 \quad (37)$$

Furthermore, if the envelope of field varies slowly with wavelength (e.g. $\frac{1}{k}\frac{\partial}{\partial x}E_0 \ll 1, \frac{1}{k}\frac{\partial}{\partial z}E_0 \ll 1$) then only the first term is important.

$$\left(\frac{\partial\Phi}{\partial x}\right)^2 + \left(\frac{\partial\Phi}{\partial z}\right)^2 = \varepsilon(x,z) = n^2(x,z) \quad (38)$$

This is the well-known Eikonal equation, Φ being the *eikonal (derived from a Greek word, meaning image*).



Geometrical relationship of E, H, and ∇Φ

F. Path of Light in an Inhomogeneous Medium

A. Example 1: 1D problems (Gradient index waveguides, Mirage Effects)

The best known example of this kind is probably the Mirage effect in desert or near a seashore, and we heard of the explanation such as the refractive index increases with density (and hence decreases with temperature at a given altitude). With the picture in mind, now can we predict more accurately the ray path and image forming processes?

Image of Mirage effect removed due to copyright restrictions.

Starting from the Eikonal equation and we assume $n^2(x,z)$ is only a function of x, then we find:

$$\left(\frac{\partial\Phi}{\partial x}\right)^2+\left(\frac{\partial\Phi}{\partial z}\right)^2=n^2(x) \tag{39}$$

Since there is the index in independent of z, we may assume the slope of phase change in z direction is linear:

$$\left(\frac{\partial\Phi}{\partial z}\right)=C(const) \tag{40}$$

This allows us to find

$$\frac{\partial\Phi}{\partial x}=\sqrt{n^2(x)-C^2} \tag{41}$$

From Fermat's principle, we can visualize that direction of rays follow the gradient of phase front:

$$n\frac{d\vec{r}}{dl}=\nabla\Phi \tag{42}$$

z-direction: $$n(x)\frac{dz}{dl} = \mathrm{C} \quad (43)$$

x-direction: $$n(x)\frac{dx}{dl} = \sqrt{n^2(x) - C^2} \quad (44)$$

Therefore, the light path (x, z) is determined by:

$$\frac{dz}{dx} = \frac{\mathrm{C}}{\sqrt{n^2(x)-C^2}} \quad (45)$$

Hence

$$z - z_0 = \int_{x_0}^{x} \frac{\mathrm{C}}{\sqrt{n^2(x)-C^2}} dx \quad (46)$$

Without loss of generality, we may assume a quadratic index profile along the x direction, such as found in gradient index optical fibers or rods:

$$n^2(x) = n_0^2(1 - \alpha x^2) \quad (47)$$

$$z - z_0 = \int_{x_0}^{x} \frac{\mathrm{C}}{\sqrt{n_0^2(1-\alpha x^2)-C^2}} dx \quad (48)$$

To find the integral explicitly we may take the following transformation of the variable x:

$$x = \sqrt{\frac{n_0^2-C^2}{n_0^2\alpha}}\, sin\theta \quad (49)$$

Therefore,

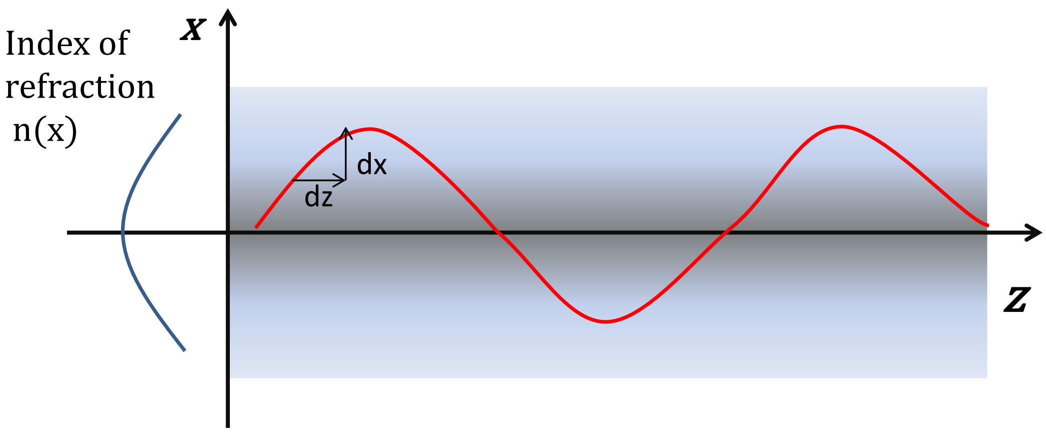
$$z - z_0 = \int_{\theta_0}^{\theta} \frac{\mathrm{C}}{n_0\sqrt{\alpha}} d\theta \quad (50)$$

$$z = z_0 + \frac{\mathrm{C}}{n_0\sqrt{\alpha}}(\theta - \theta_0) \quad (51)$$

Or more commonly,

$$x\sqrt{\frac{n_0^2\alpha}{n_0^2-C^2}} = sin\theta = sin\left(\theta_0 + \sqrt{\frac{n_0^2\alpha}{C^2}}(z - z_0)\right) \quad (52)$$

As you can see in this example, ray propagation in the gradient index waveguide follows a sinusoid pattern! The periodicity is determined by a constant $\frac{2\pi C}{n_0\sqrt{\alpha}}$.



Observation: the constant C is related to the original "launching" angle β of the optical ray. To check that we start by:

$$\left.\frac{dz}{dx}\right|_{x=x_0} = \frac{C}{\sqrt{n^2(x_0)-C^2}} \quad (53)$$

If we assume C=$n(x_0)cos\beta$, then

$$\left.\frac{dz}{dx}\right|_{x=x_0} = \frac{\cos\beta}{sin\beta} = cot\beta \quad (54)$$

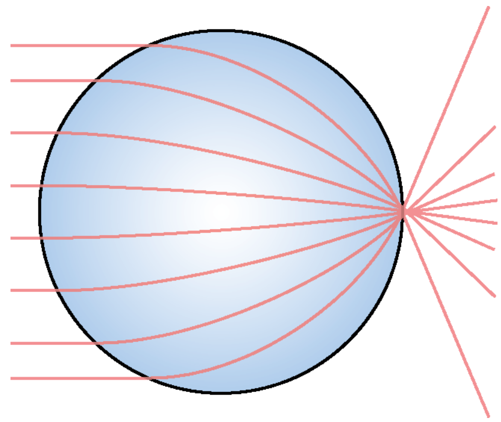
B. Other popular examples: Luneberg Lens

The Luneberg lens is inhomogeneous sphere that brings a collimated beam of light to a focal point at the rear surface of the sphere. For a sphere of radius R with the origin at the center, the gradient index function can be written as:

$$n(r) = \begin{cases} n_0\sqrt{2-\frac{r^2}{R^2}}, & r \le R \\ n_0 & r > R \end{cases} \quad (55)$$

Such lens was mathemateically conceived during the 2nd world war by R. K. Luneberg, (see: R. K. Luneberg, Mathematical Theory of Optics (Brown University, Providence, Rhode Island, 1944), pp. 189-213.) The applications of such Luneberg lens was quickly demonstrated in microwave frequencies, and later for optical communications as well as in acoustics. Recently, such device gained new interests in in phased array communications, in illumination systems, as well as concentrators in solar energy harvesting and in imaging objectives.

Image of Luneberg lens removed due to copyright restrictions.



Left: Picture of an Optical Luneberg Lens (a glass ball 60 mm in diameter) used as spherical retro-reflector on **Meteor-3M** spacecraft. (Nasa.gov)
Right: Ray Schematics of Luneberg Lens with a radially varying index of refraction. All parallel rays (red solid curves) coming from the left-hand side of the Luneberg lens will focus to a point on the edge of the sphere.