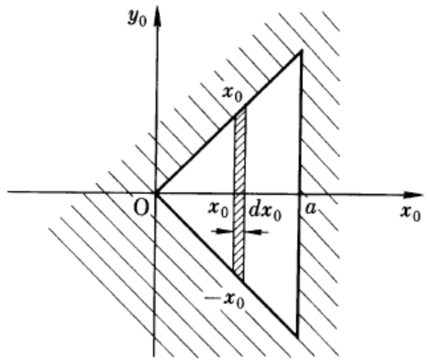
Practice problem: Find the Fraunhofer diffraction pattern of a triangular aperture as shown in the following figure**.** The edges of the triangle are expressed at x=a, y=x, and y=-x, respectively. The screen is placed at $z=z_0$.



Solution: In this case, the aperture along the y direction depends on the position x. so we may integrate first along the y direction, and then along x-direction in the next.

$$E(k_x, k_y) = \frac{\exp\,(ikz)}{z}\int_0^a \exp\,(-ik_x x)\left(\int_{-x}^{x} \exp\left(-ik_y y\right) dy\right) dx$$

$$E(k_x, k_y) = \frac{\exp\,(ikz)}{z}\int_0^a \exp\,(-ik_x x)\left(\frac{\exp\left(\mathrm{i}k_y x\right) - \exp\left(-\mathrm{i}k_y x\right)}{ik_y}\right) dx$$

$$E(k_x, k_y) = \frac{\exp\,(ikz)}{ik_y z}\int_0^a \left[\exp\,(-i(k_x - k_y)x) - \exp\,(-i(k_x + k_y)x)\right] dx$$

$$\int_0^a \left[\exp\,(-i(k_x - k_y)x)\right] dx = \frac{\left[1 - \exp\,(-i(k_x - k_y)a)\right]}{i(k_x - k_y)} = 2a\exp\,(-i\frac{(k_x - k_y)a}{2}) sinc(\frac{(k_x - k_y)a}{2\pi})$$

$$\int_0^a \left[\exp\,(-i(k_x + k_y)x)\right] dx = \frac{\left[1 - \exp\,(-i(k_x + k_y)a)\right]}{i(k_x + k_y)} = 2a\exp\,(-i\frac{(k_x + k_y)a}{2}) sinc(\frac{(k_x + k_y)a}{2\pi})$$

$$E(k_x, k_y) = \frac{2a\exp(ikz)}{ik_y z}[\exp\left(-i\frac{(k_x - k_y)a}{2}\right) sinc\left(\frac{(k_x - k_y)a}{2\pi}\right) - \exp\left(-i\frac{(k_x + k_y)a}{2}\right) sinc\left(\frac{(k_x + k_y)a}{2\pi}\right)]$$

Alternative solution:

$$E(k_x,k_y) = \frac{\exp(ikz)}{z}\int_0^a \exp(-ik_x x)\left(\int_{-x}^{x}\exp(-ik_y y)\,dy\right)dx$$

1) you may verify that:

$$\boxed{\left(\int_{-x}^{x}\exp(-ik_y y)\,dy\right) = \int_{-\infty}^{\infty} rect(\frac{y}{2x})\exp(-ik_y y)\,dy}$$

$$E(k_x,k_y) = \frac{\exp(ikz)}{z}\int_0^a 2|x|sinc\left(\frac{xk_y}{\pi}\right)\exp(-ik_x x)dx$$

2) again, we can see that:

$$\boxed{\int_0^a \exp(-ik_x x)dx = \int_{-\infty}^{\infty}\mathrm{rect}(\frac{\mathrm{x}-\frac{a}{2}}{a})\exp(-ik_x x)dx}$$

$$E(k_x,k_y) = \frac{2\exp(ikz)}{z}\int_{-\infty}^{\infty} sgn(x)xsinc\left(\frac{xk_y}{\pi}\right)\mathrm{rect}(\frac{\mathrm{x}-\frac{a}{2}}{a})\exp(-ik_x x)dx$$

Using shift theorem for $\int_{-\infty}^{\infty}\mathrm{rect}(\frac{\mathrm{x}-\frac{a}{2}}{a})\exp(-ik_x x)dx = \mathrm{a}\exp(-ik_x\frac{a}{2})sinc(\frac{ak_x}{2\pi})$

$$E(k_x,k_y) = \frac{2\,\mathrm{a}\exp(ikz)}{z}\left[\exp(-ik_x\frac{a}{2})sinc(\frac{ak_x}{2\pi})\right]\otimes\int_{-\infty}^{\infty} sgn(x)xsinc\left(\frac{xk_y}{2\pi}\right)\exp(-ik_x x)dx$$

$$\int_{-\infty}^{\infty} xsinc\left(\frac{xk_y}{\pi}\right)\exp(-ik_x x)dx = -i\frac{\partial}{\partial k_x}\left[\int_{-\infty}^{\infty} sinc\left(\frac{xk_y}{\pi}\right)\exp(-ik_x x)dx\right]$$

$$= \frac{-i}{k_y}\frac{\partial}{\partial k_x}rect\left(\frac{k_x}{k_y}\right) = \frac{-i}{k_y}\left(\delta(k_x-k_y)-\delta(k_x+k_y)\right)$$

$$E(k_x,k_y) = \frac{-i2a\exp(ikz)}{zk_y}\left[\exp(-ik_x\frac{a}{2})sinc(\frac{ak_x}{2\pi})\right]\otimes\left(\delta(k_x-k_y)-\delta(k_x+k_y)\right)$$

$$E(k_x,k_y) = \frac{-i2a\exp(ikz)}{zk_y}\left[\exp\left(-i(k_x-k_y)\frac{a}{2}\right)sinc\left(\frac{a(k_x-k_y)}{2\pi}\right) - \exp\left(-i(k_x+k_y)\frac{a}{2}\right)sinc\left(\frac{a(k_x+k_y)}{2\pi}\right)\right]$$

2.71 / 2.710 Optics
Spring 2014