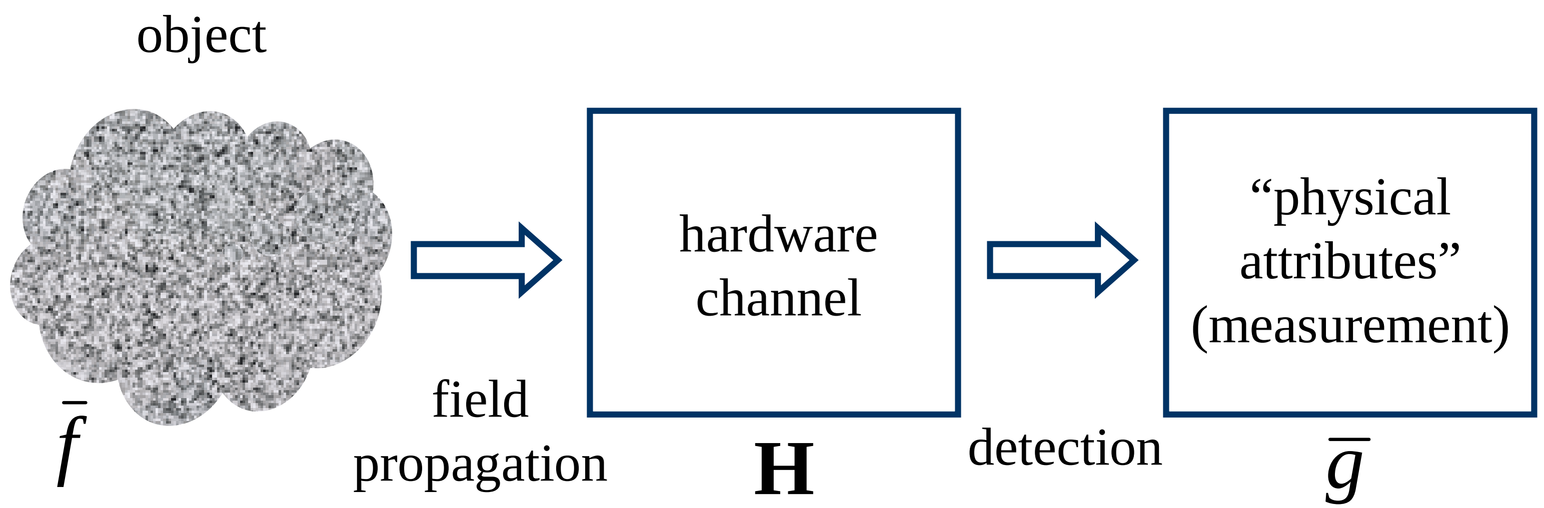
Image Quality Metrics

- Image quality metrics
  - Mutual information (cross-entropy) metric
    - Intuitive definition
    - Rigorous definition using entropy
    - Example: two-point resolution problem
    - Example: confocal microscopy
  - Square error metric
  - Receiver Operator Characteristic (ROC)
- Heterodyne detection

Linear inversion model



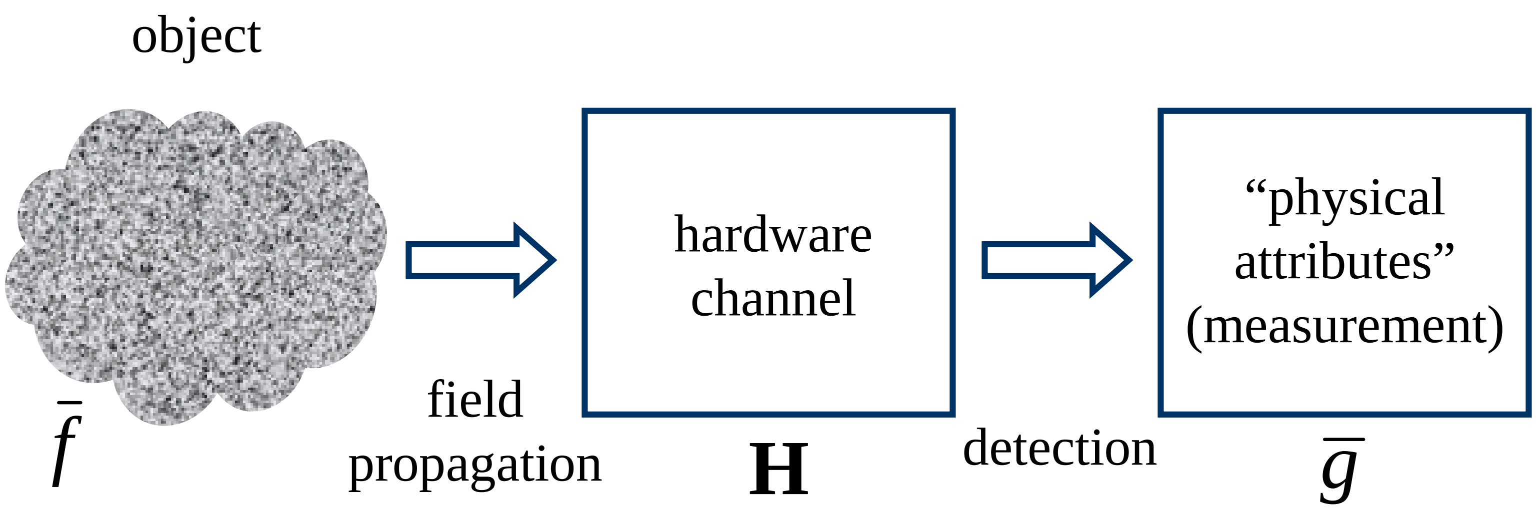
inversion problem:

determine *f*, given the measurement $\quad \bar{g} = \mathbf{H}\,\bar{f}$

noise-to-signal ratio $\quad (\text{NSR}) = \dfrac{(\text{noise variance})}{(\text{average signal power})} = \dfrac{\sigma^2}{1} = \sigma^2$

normalizing signal power to 1

Mutual information (cross-entropy)

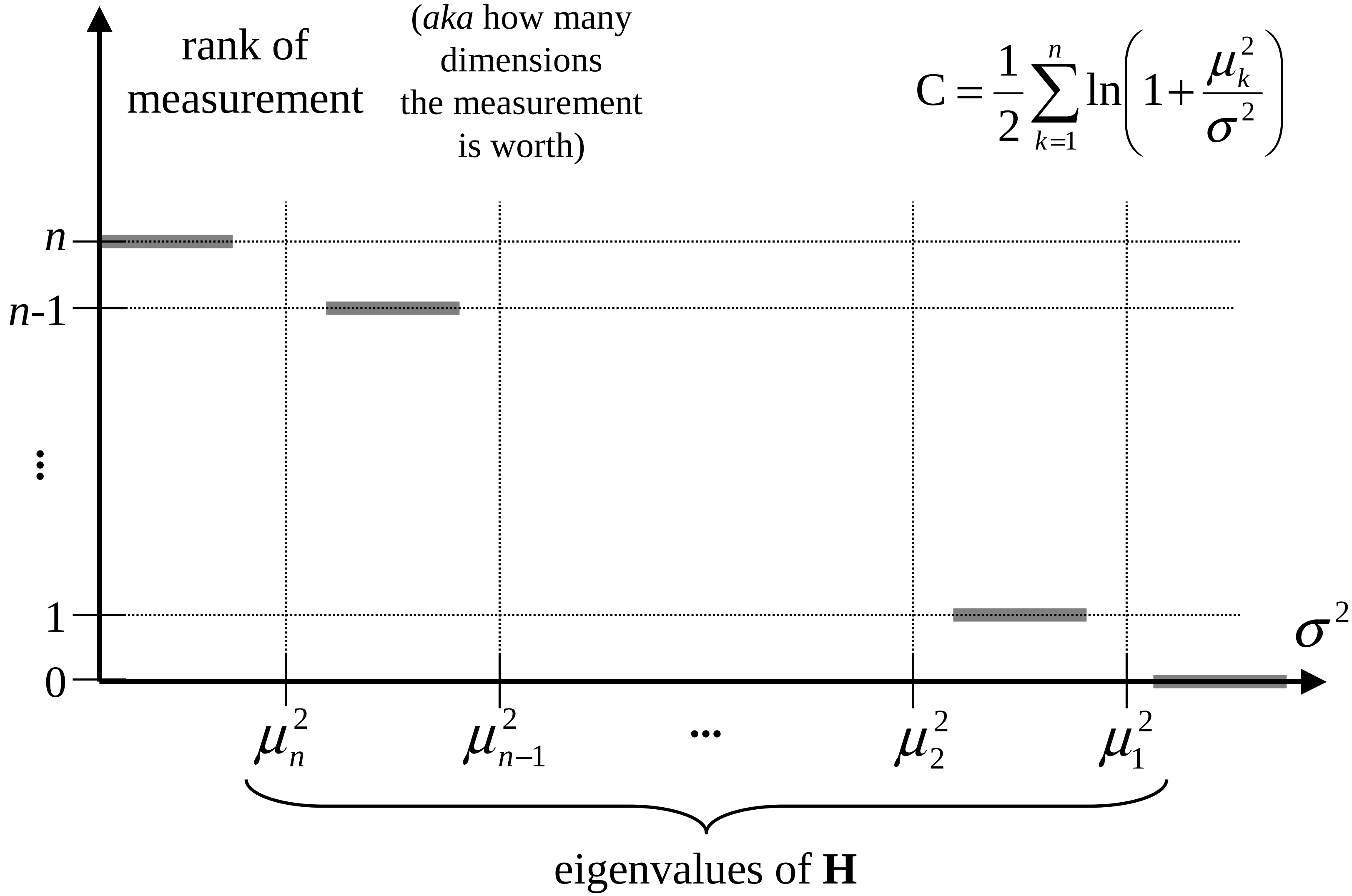


$$C = \frac{1}{2}\sum_{k=1}^{n} \ln\left(1 + \frac{\mu_k^2}{\sigma^2}\right)$$

eigenvalues of **H**

The significance of eigenvalues

$$C = \frac{1}{2}\sum_{k=1}^{n} \ln\left(1 + \frac{\mu_k^2}{\sigma^2}\right)$$



Precision of measurement

$$C = \frac{1}{2}\sum_{k=1}^{n} \ln\left(1+\frac{\mu_k^2}{\sigma^2}\right) =$$

$\mu_t^2 < \sigma^2 < \mu_{t-1}^2$

noise floor

$$\ldots + \ln\left(1+\frac{\mu_{t-2}^2}{\sigma^2}\right) + \ln\left(1+\frac{\mu_{t-1}^2}{\sigma^2}\right) + \ln\left(1+\frac{\mu_t^2}{\sigma^2}\right) + \ldots$$

≈precision
of (t-2)th measurement

this term
≤1

E.g. 0.5470839348

these digits worthless
if $\sigma \approx 10^{-5}$

this term
≈0

Formal definition of cross-entropy (1)

Entropy in thermodynamics (discrete systems):

• $\log_2$[how many are the possible states of the system?]

E.g. two-state system: fair coin, outcome=heads (H) or tails (T)

Entropy=$\log_2 2=1$

Unfair coin: seems more reasonable to "weigh" the two states according to their frequencies of occurence (*i.e.*, probabilities)

$$\text{Entropy} = -\sum_{\text{states}} p(\text{state}) \log_2 p(\text{state})$$

Formal definition of cross-entropy (2)

- Fair coin: p(H)=1/2; p(T)=1/2

$$\text{Entropy} = -\frac{1}{2}\log_2\frac{1}{2} - \frac{1}{2}\log_2\frac{1}{2} = 1\,\text{bit}$$

- Unfair coin: p(H)=1/4; p(T)=3/4

$$\text{Entropy} = -\frac{1}{4}\log_2\frac{1}{4} - \frac{3}{4}\log_2\frac{3}{4} = 0.81\,\text{bits}$$

Maximum entropy ⇔ Maximum uncertainty

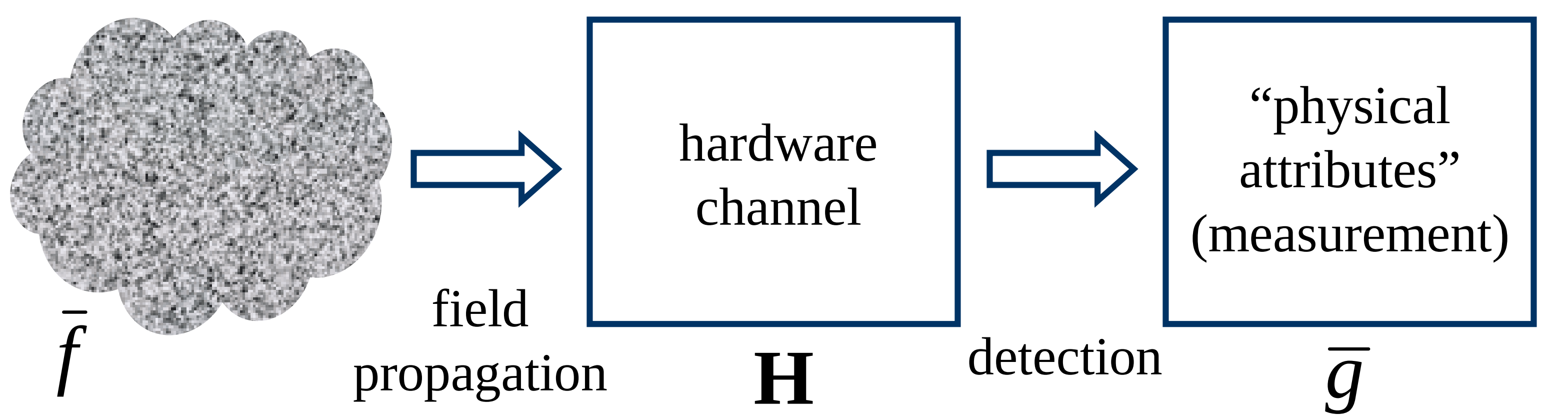
Formal definition of cross-entropy (3)

Joint Entropy

$\log_2$[how many are the possible states of a combined variable obtained from the Cartesian product of two variables?]

$$\text{Joint Entropy}(X,Y) = -\sum_{\substack{\text{states}\\ x\in X}} \sum_{\substack{\text{states}\\ y\in Y}} p(x,y)\log_2 p(x,y)$$

E.g. $\text{Joint Entropy}(\overline{F},\overline{G}) = ?$



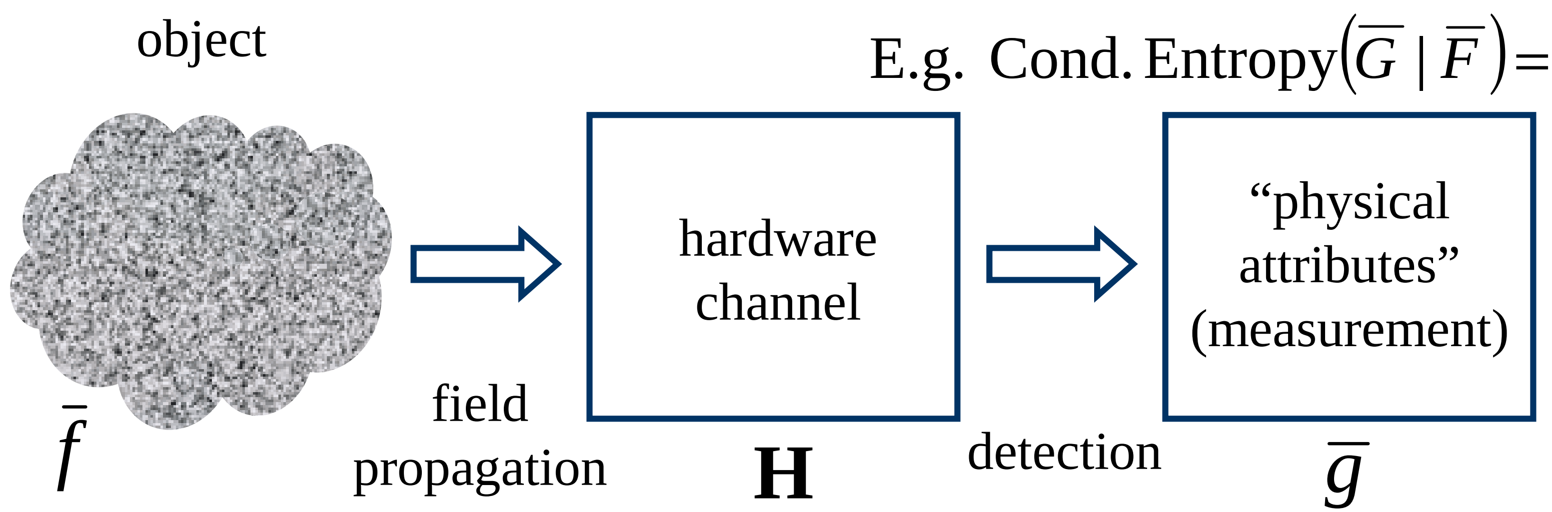
Formal definition of cross-entropy (4)

Conditional Entropy

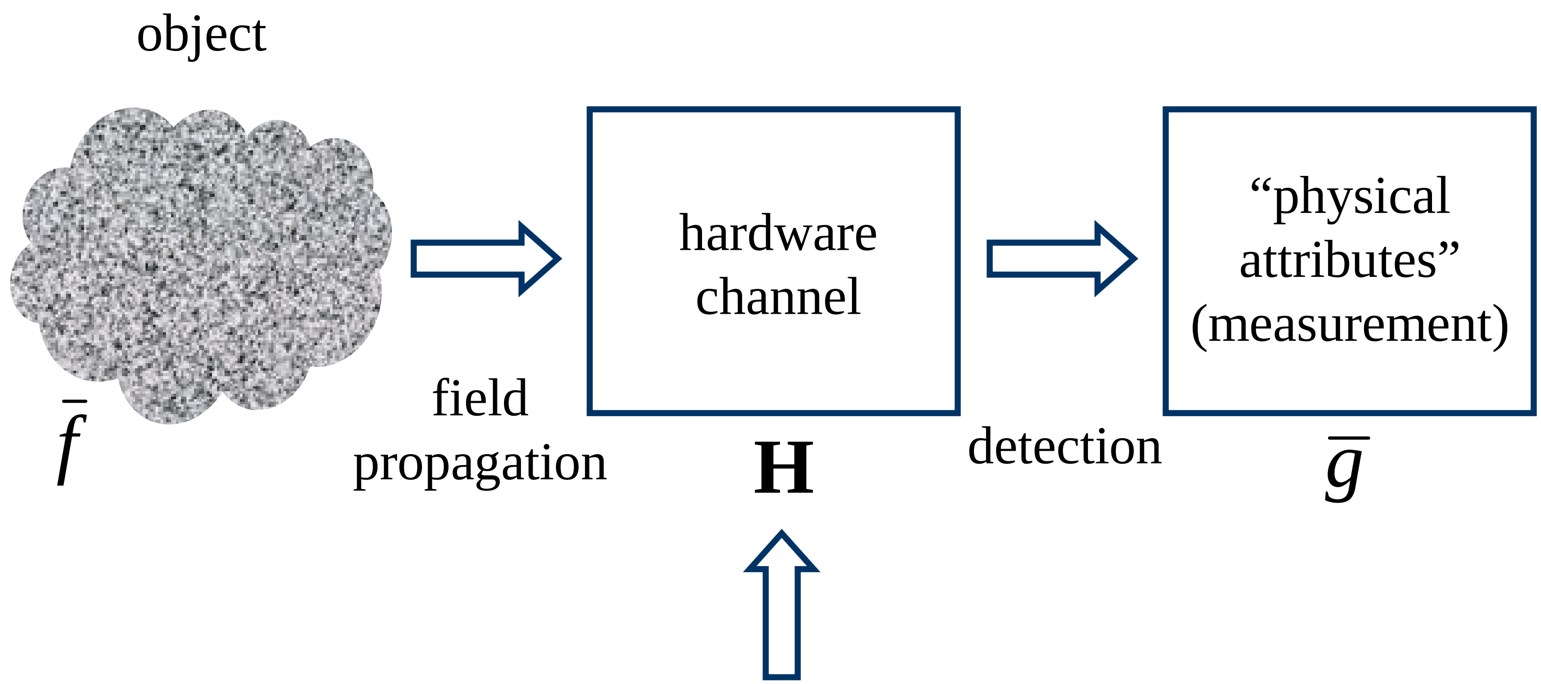
$\log_2$[how many are the possible states of a combined variable given the actual state of one of the two variables?]

$$\text{Cond. Entropy}(Y \mid X) = -\sum_{\substack{\text{states}\\ x \in X}} \sum_{\substack{\text{states}\\ y \in Y}} p(x,y) \log_2 p(y \mid x)$$

E.g. $\text{Cond. Entropy}(\overline{G} \mid \overline{F}) = ?$



Formal definition of cross-entropy (5)



Noise adds uncertainty to the measurement *wrt* the object
⇔ eliminates information from the measurement *wrt* object

Formal definition of cross-entropy (6)

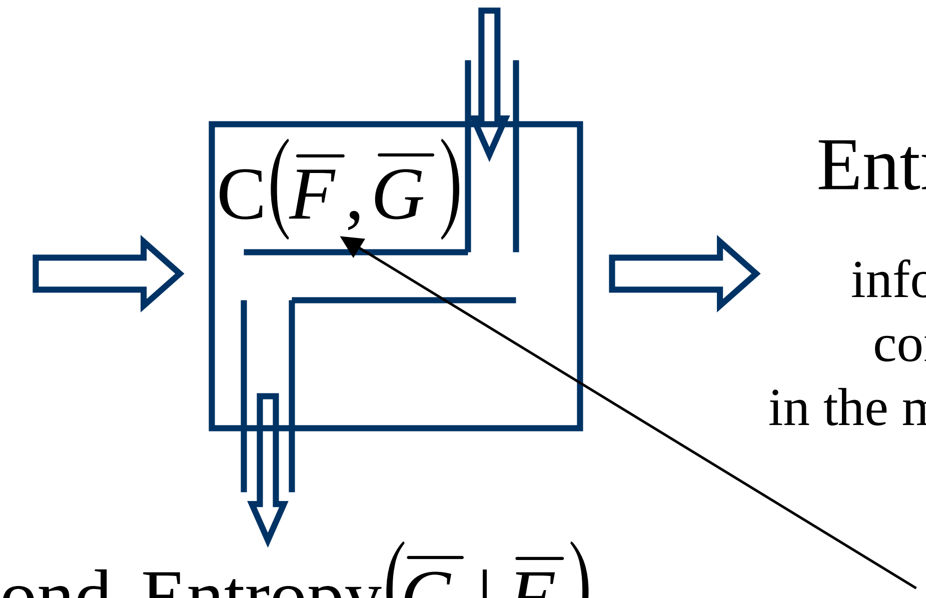
representation by
Seth Lloyd, 2.100

uncertainty added due to noise

$$\text{Cond. Entropy}\left(\overline{F} \mid \overline{G}\right)$$

$$\text{Entropy}\left(\overline{F}\right)$$

information
contained
in the object



$$\text{Entropy}\left(\overline{G}\right)$$

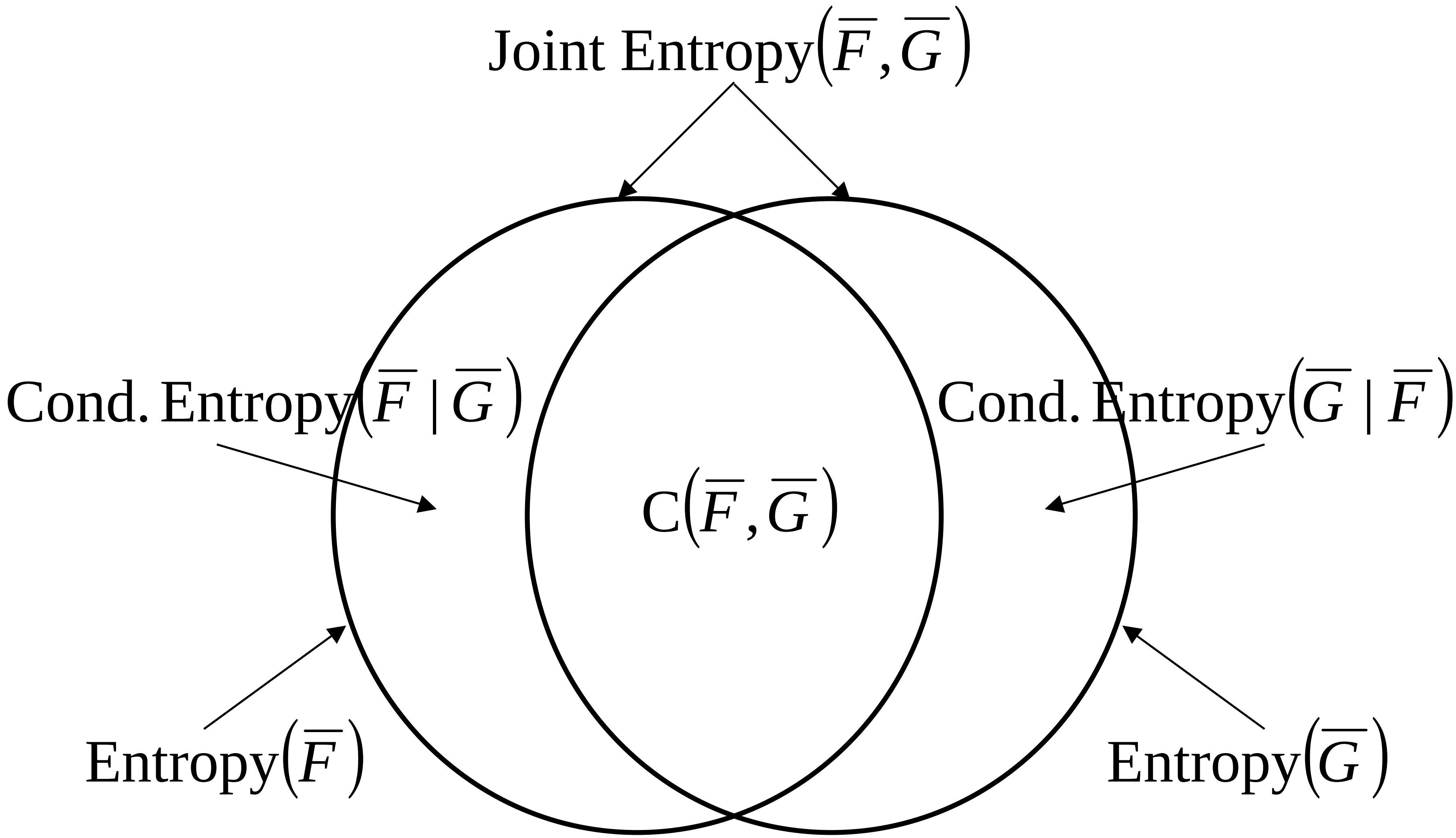
information
contained
in the measurement

$$\text{Cond. Entropy}\left(\overline{G} \mid \overline{F}\right)$$

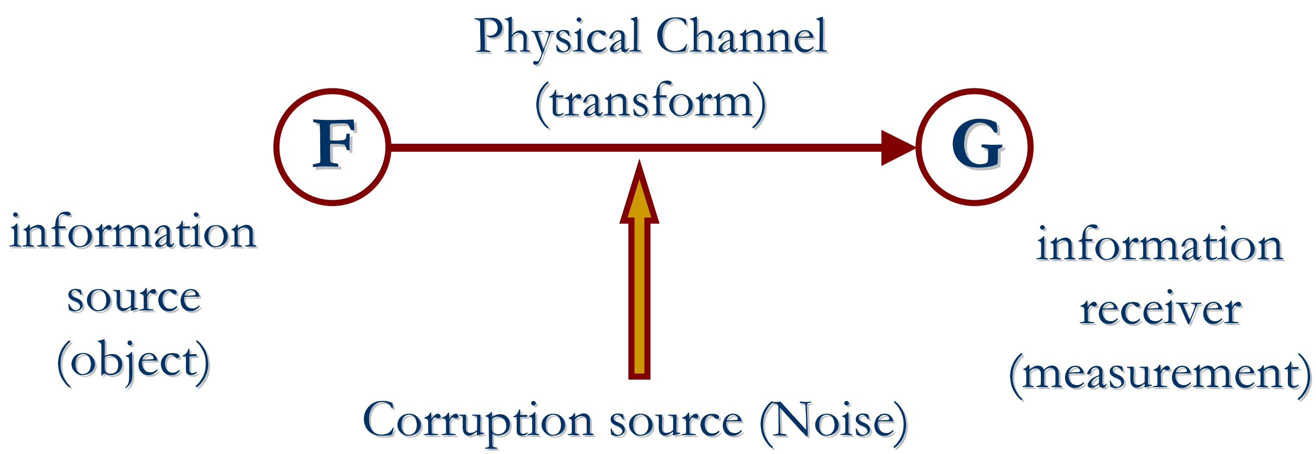
information eliminated due to noise

cross-entropy
(*aka* mutual information)

Formal definition of cross-entropy (7)



Formal definition of cross-entropy (8)



$$
\begin{aligned}
\mathrm{C}(\overline{F},\overline{G}) &= \mathrm{Entropy}(\overline{F}) - \mathrm{Cond.\,Entropy}(\overline{F}\,|\,\overline{G}) \\
&= \mathrm{Entropy}(\overline{G}) - \mathrm{Cond.\,Entropy}(\overline{G}\,|\,\overline{F}) \\
&= \mathrm{Entropy}(\overline{F}) + \mathrm{Entropy}(\overline{G}) - \mathrm{Joint\ Entropy}(\overline{F},\overline{G})
\end{aligned}
$$

Entropy & Differential Entropy

- Discrete objects (can take values among a discrete set of states)
  - definition of entropy

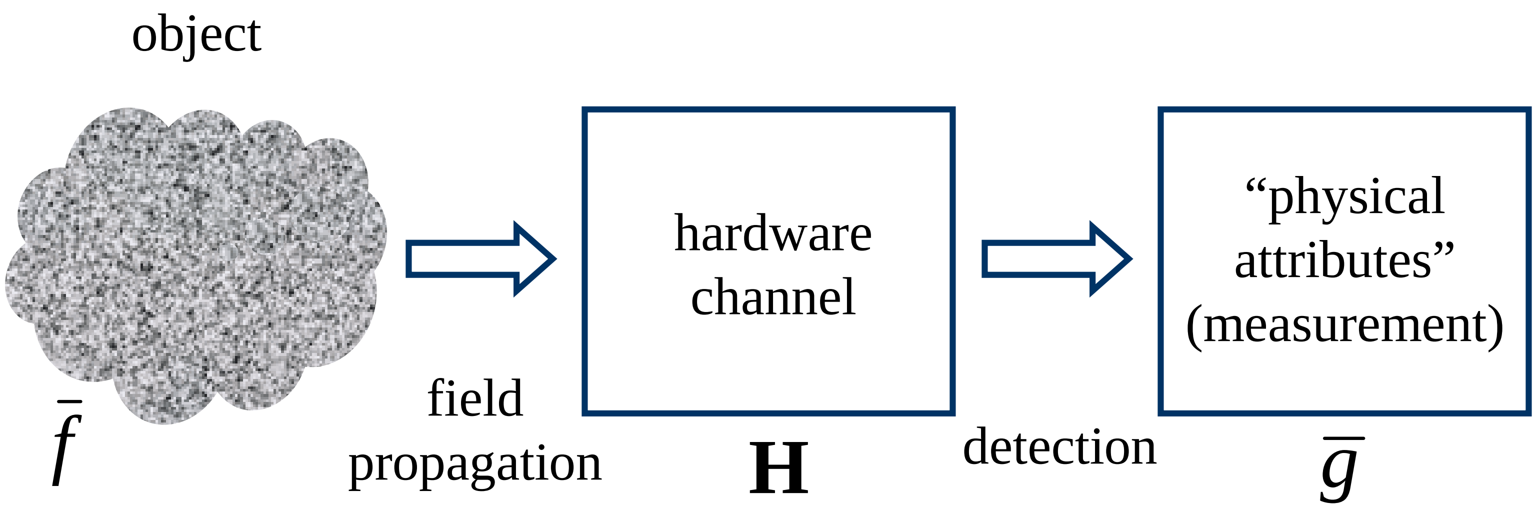
$$\text{Entropy} = -\sum_k p(x_k)\log_2 p(x_k)$$

  - unit: 1 bit (=entropy value of a YES/NO question with 50% uncertainty)
- Continuous objects (can take values from among a continuum)
  - definition of differential entropy

$$\text{Diff. Entropy} = -\int_{\Omega(X)} p(x)\ln p(x)\,\mathrm{d}x$$

  - unit: 1 nat (=diff. entropy value of a significant digit in the representation of a random number, divided by ln10)

Image Mutual Information (IMI)



Assumptions: (a) F has Gaussian statistics

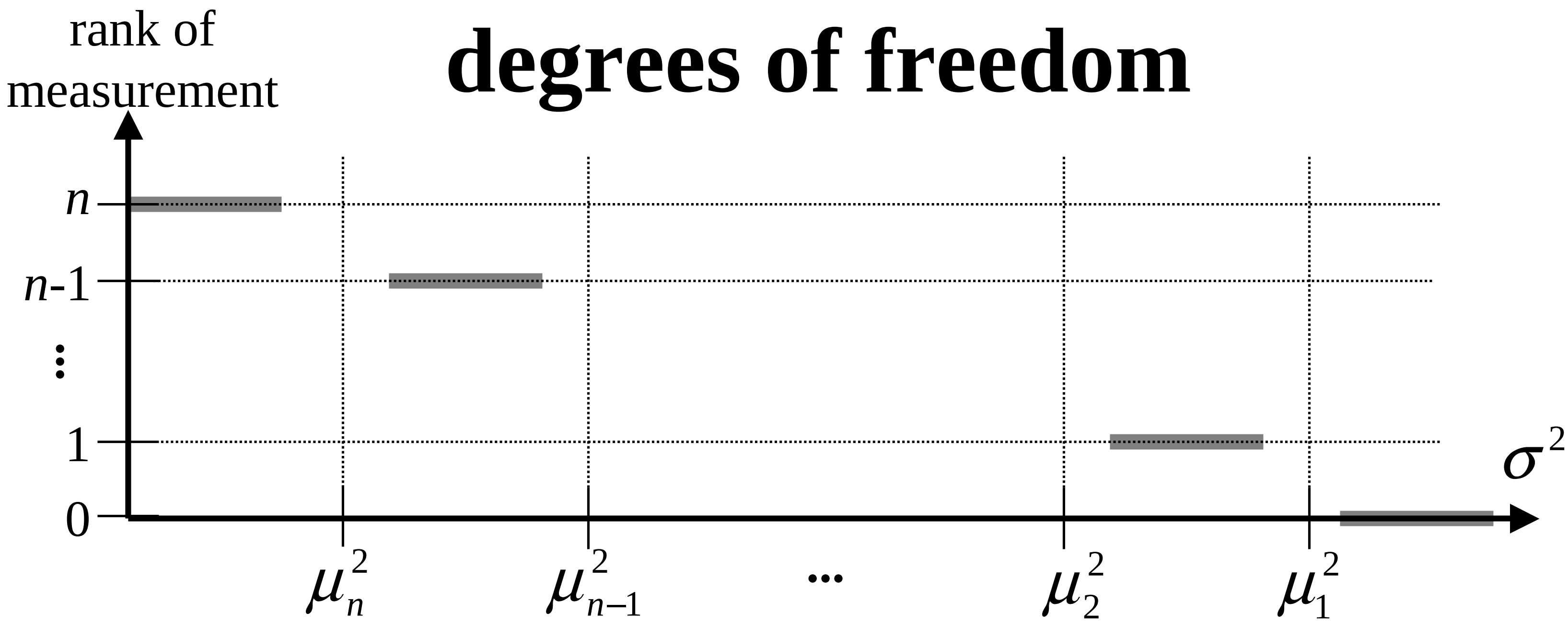
(b) white additive Gaussian noise (waGn)

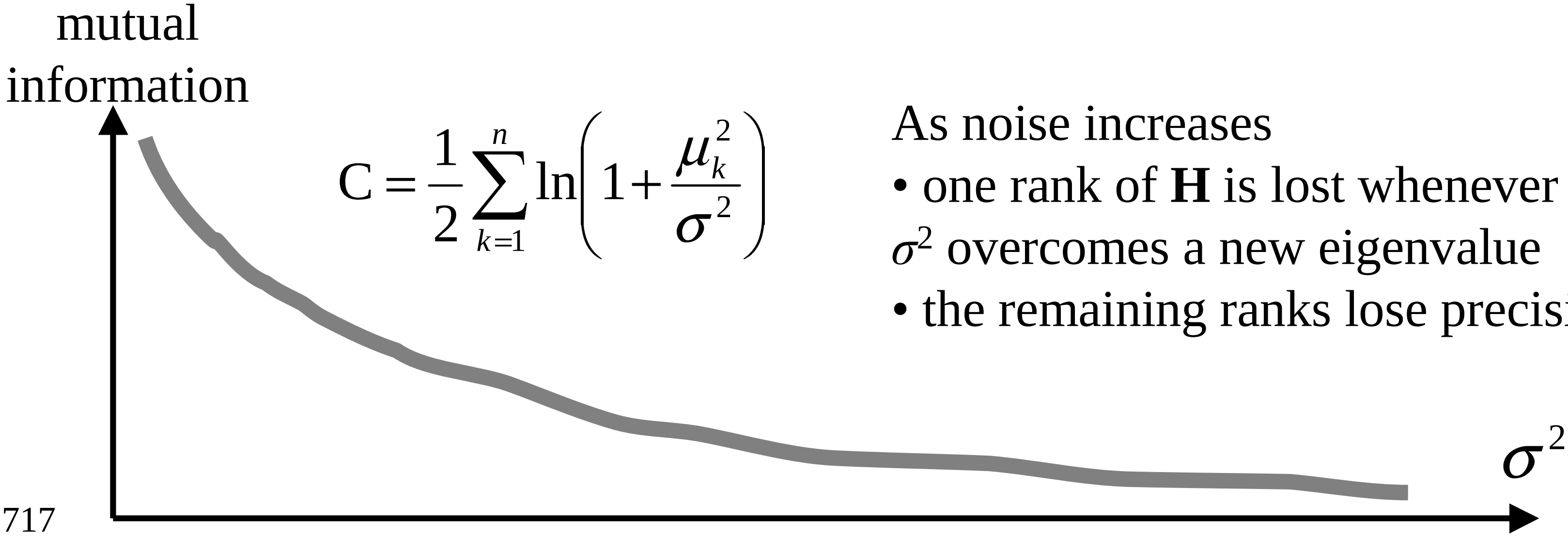
i.e. $g = \mathbf{H} f + w$

where W is a Gaussian random vector with diagonal correlation matrix

Then $$C\left(\overline{F}, \overline{G}\right) = \frac{1}{2} \sum_{k=1}^{n} \ln\left(1 + \frac{\mu_k^2}{\sigma^2}\right)$$ μ_k : eigenvalues of $\mathbf{H}$

Mutual information & degrees of freedom





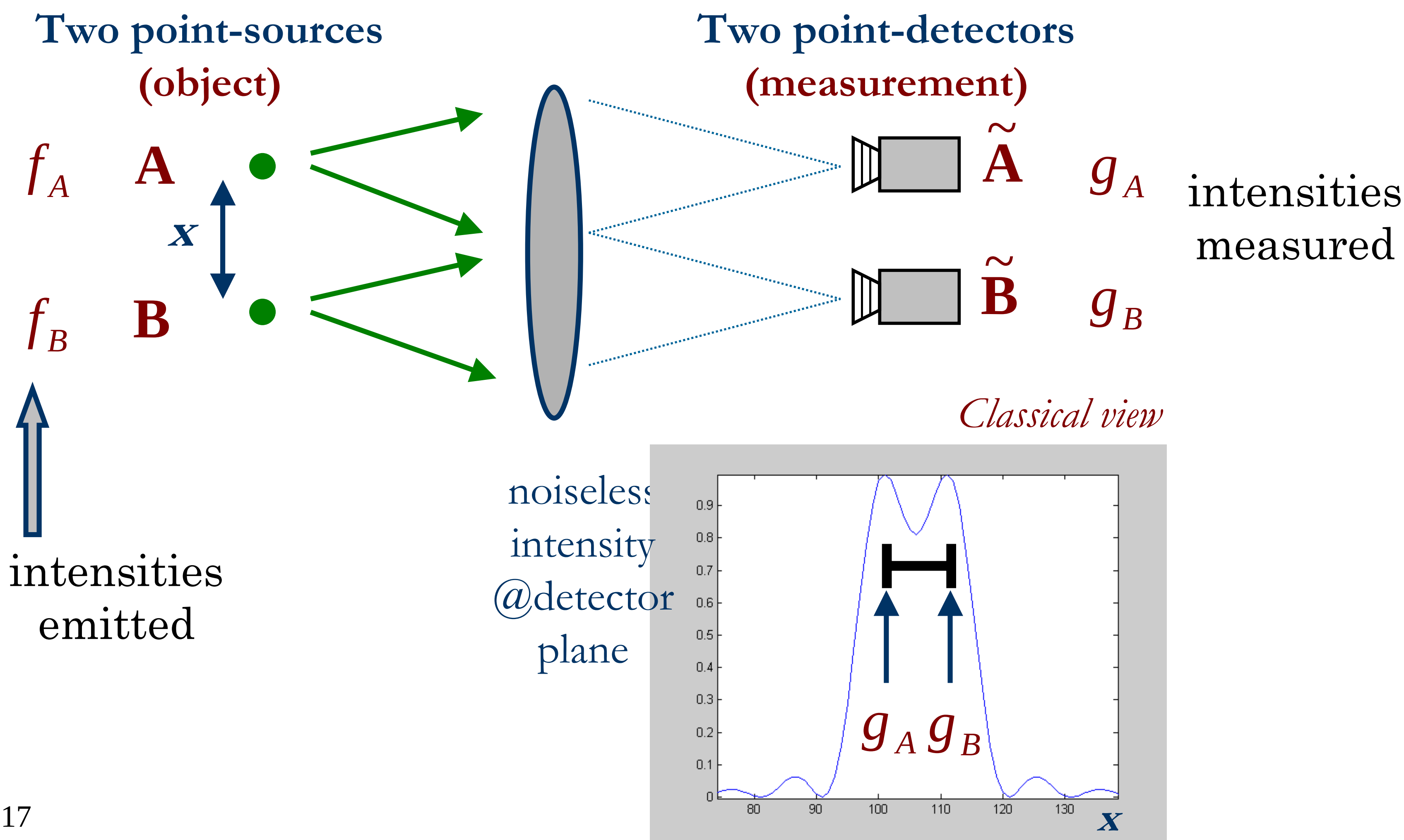
$$C = \frac{1}{2}\sum_{k=1}^{n} \ln\left(1+\frac{\mu_k^2}{\sigma^2}\right)$$

As noise increases

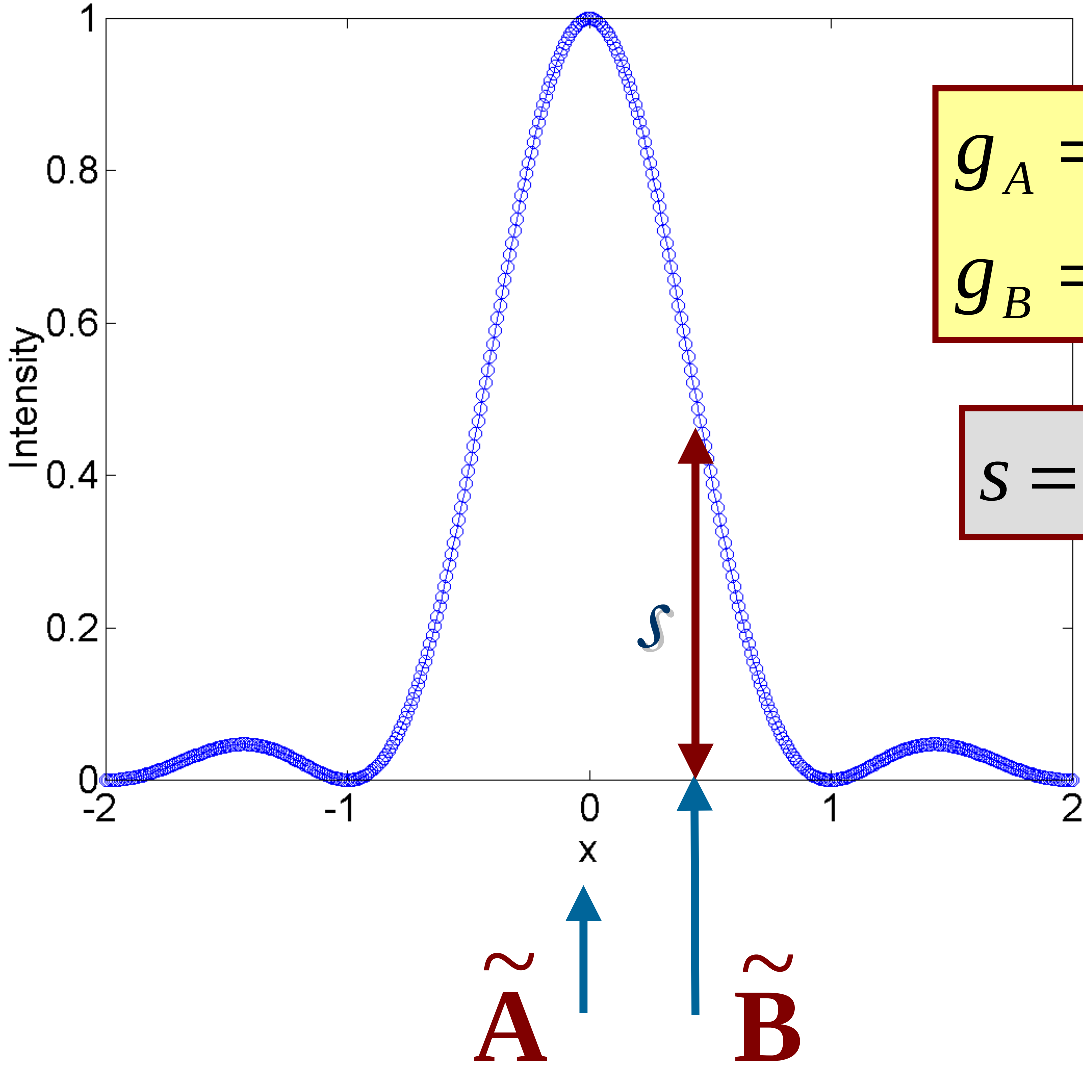
- one rank of **H** is lost whenever σ^2 overcomes a new eigenvalue
- the remaining ranks lose precision

Example: two-point resolution

<u>**Finite-NA imaging system, unit magnification**</u>



Cross-leaking power



$$g_A = f_A + s f_B$$

$$g_B = s f_A + f_B$$

$$s = \mathrm{sinc}^2(x)$$

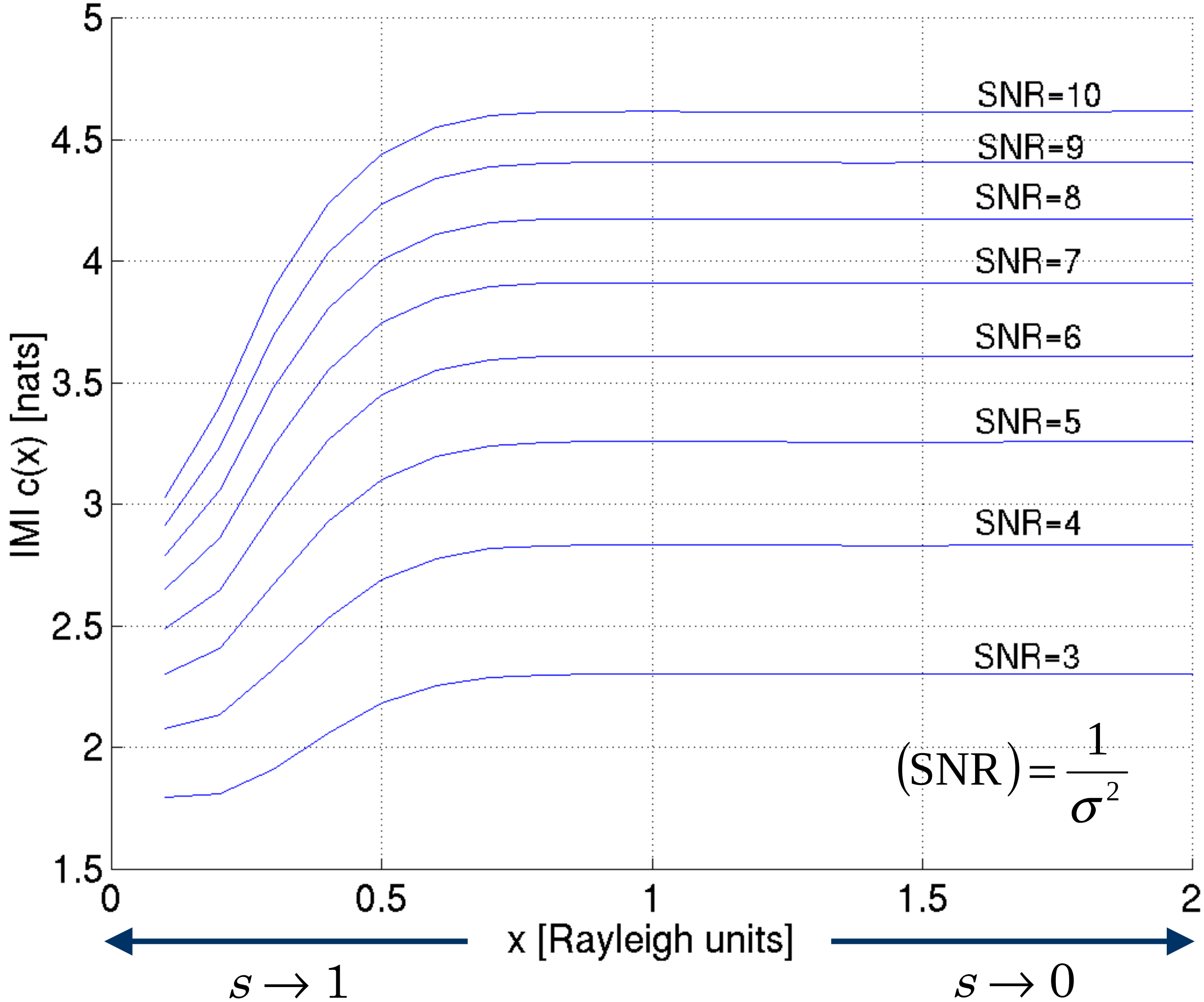
IMI for two-point resolution problem

$$\mathbf{H} = \begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix} \qquad \det(\mathbf{H}) = 1 - s^2 \qquad \begin{matrix} \mu_1 = 1 + s \\ \mu_2 = 1 - s \end{matrix}$$

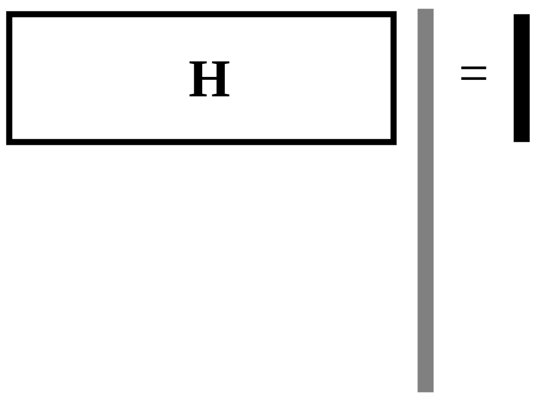
$$\mathbf{H}^{-1} = \frac{1}{1 - s^2} \begin{pmatrix} 1 & -s \\ -s & 1 \end{pmatrix}$$

$$\mathrm{C}(\overline{F}, \overline{G}) = \frac{1}{2} \ln\left(1 + \frac{(1 - \mathrm{s})^2}{\sigma^2} \right) + \frac{1}{2} \ln\left(1 + \frac{(1 + \mathrm{s})^2}{\sigma^2} \right)$$

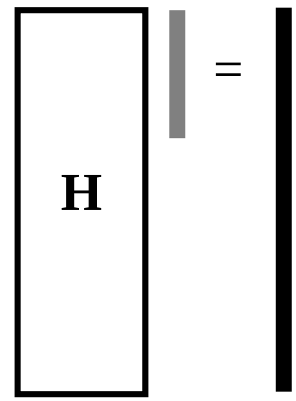
IMI *vs* source separation



IMI for rectangular matrices (1)



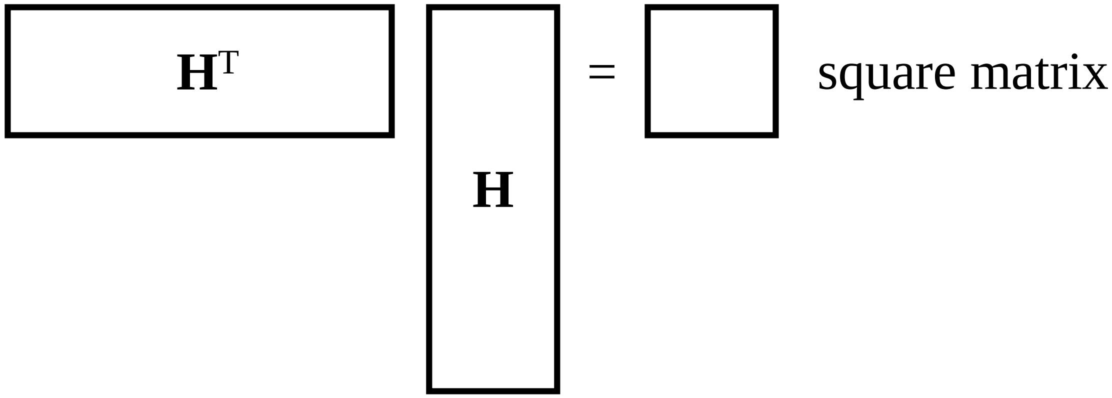
underdetermined
(more unknowns than measurements)



overdetermined
(more measurements than unknowns)

eigenvalues cannot be computed, but instead we compute the singular values of the rectangular matrix

IMI for rectangular matrices (2)



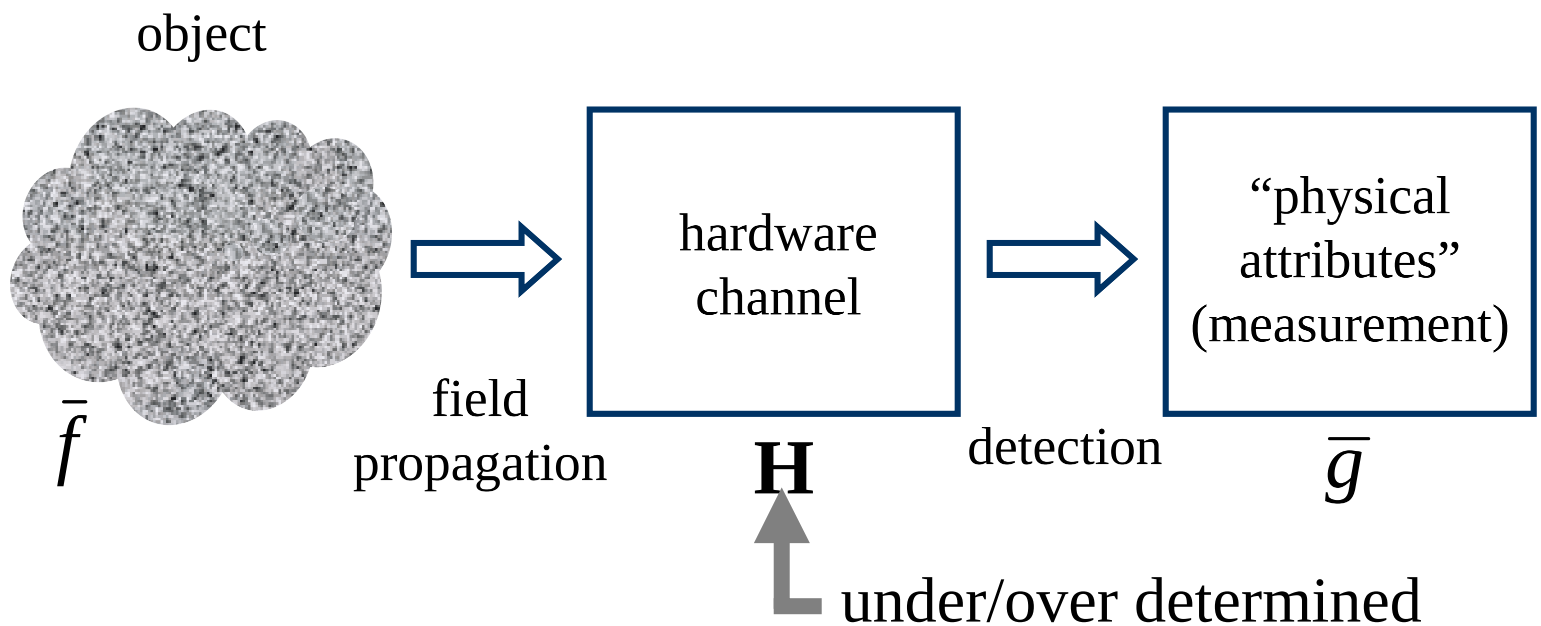
recall pseudo-inverse

$$\hat{f} = \underbrace{\left(\mathbf{H}^{\mathrm{T}}\mathbf{H}\right)}^{-1}\mathbf{H}^{\mathrm{T}}g$$

inversion operation associated with rank of

$$\text{eigenvalues}\left(\mathbf{H}^{\mathrm{T}}\mathbf{H}\right) \equiv \text{singular values}\left(\mathbf{H}\right)$$

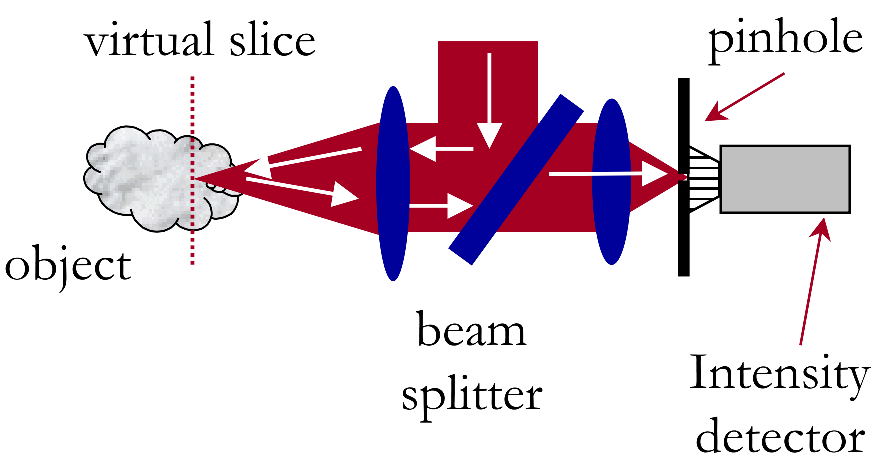
IMI for rectangular matrices (3)



$$C = \frac{1}{2}\sum_{k=1}^{n} \ln\left(1 + \frac{\tau_k}{\sigma^2}\right)$$

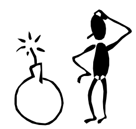
← singular values of **H**

Confocal microscope



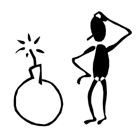
Small pinhole:

Depth resolution



Light efficiency

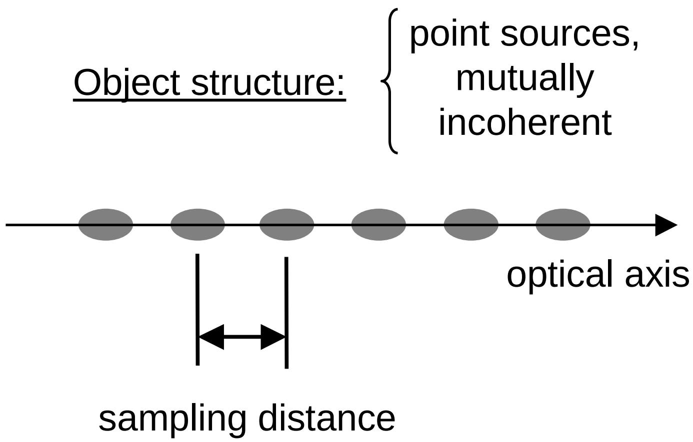
Large pinhole:

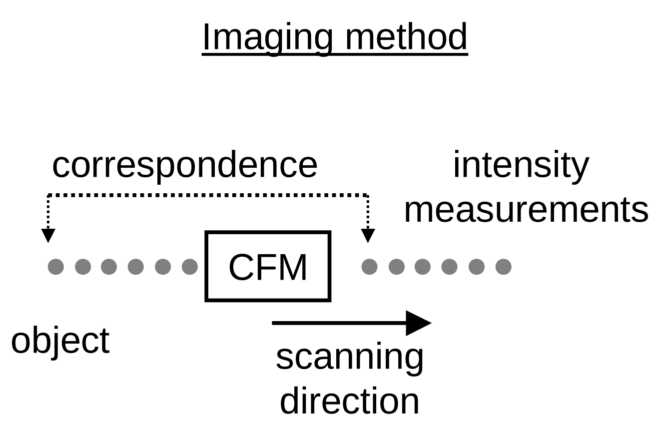


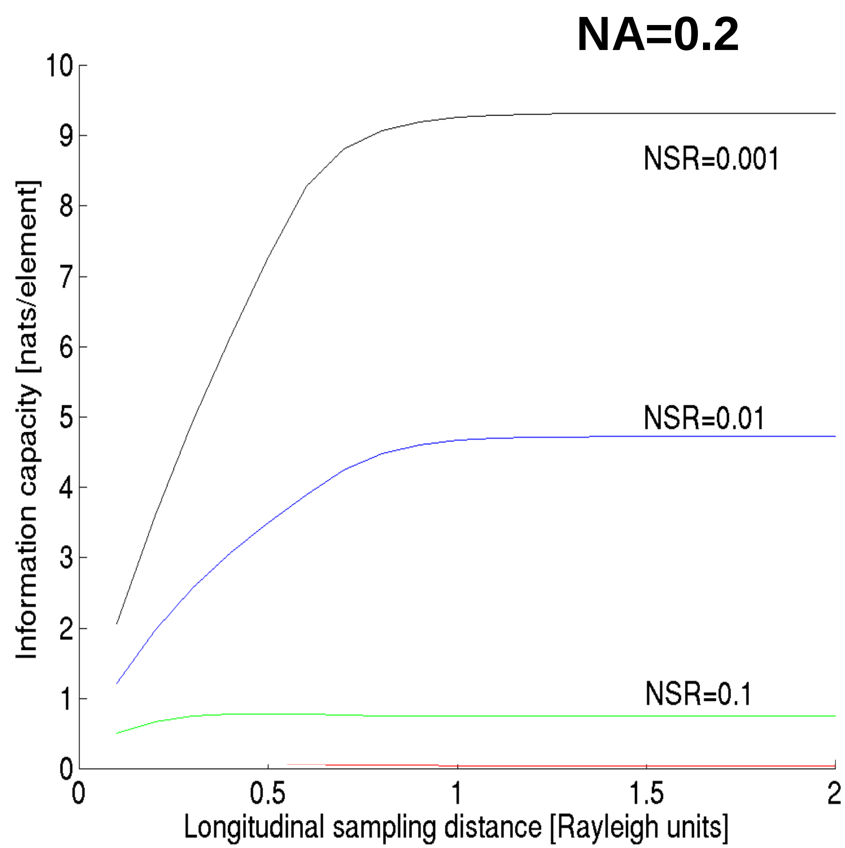
Depth resolution

Light efficiency

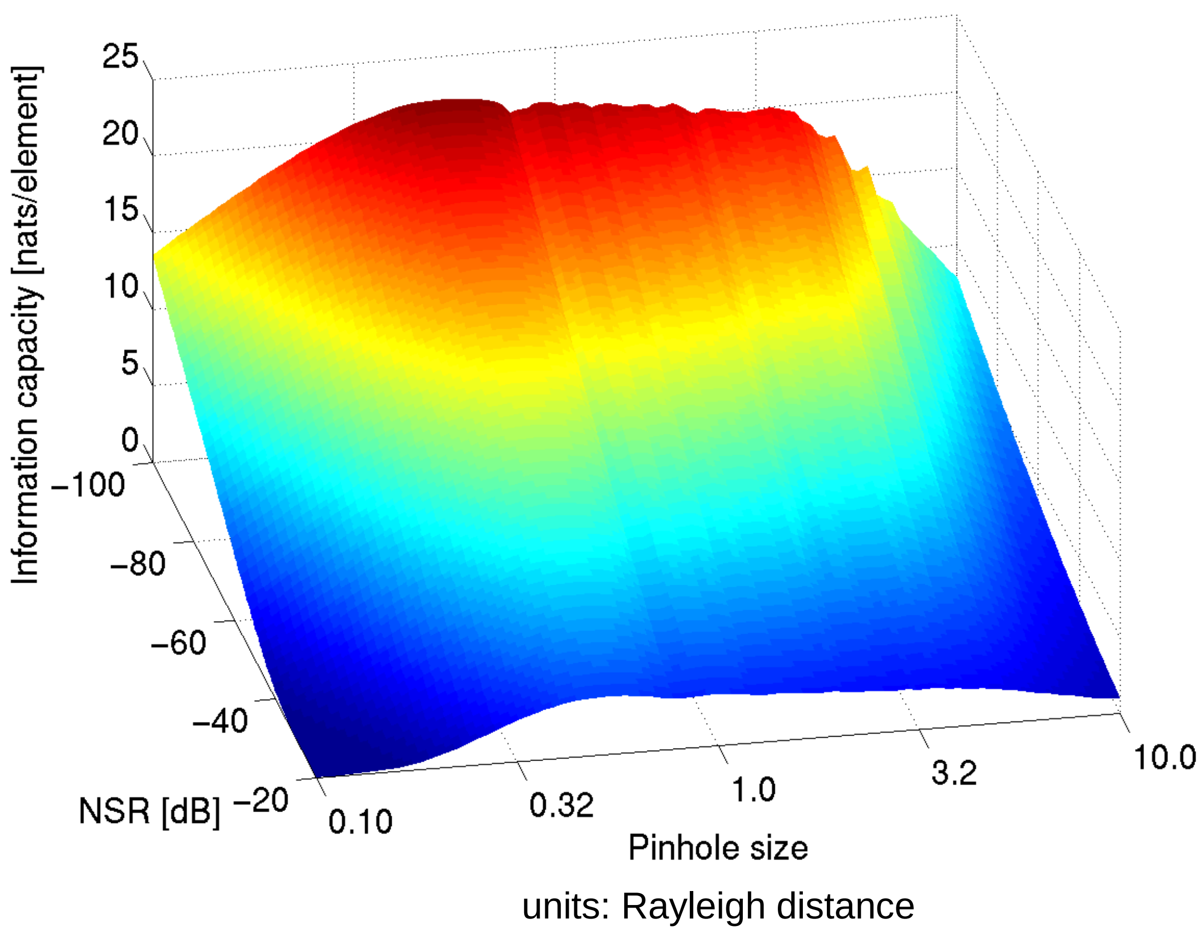
Depth “resolution” *vs.* noise







Depth "resolution" *vs.* noise & pinhole size



IMI summary

- It quantifies the number of possible states of the object that the imaging system can successfully discern; this includes
  - the rank of the system, i.e. the number of object dimensions that the system can map
  - the precision available at each rank, i.e. how many significant digits can be reliably measured at each available dimension
- An alternative interpretation of IMI is the game of "20 questions:" how many questions about the object can be answered reliably based on the image information?
- IMI is intricately linked to image exploitation for applications, e.g. medical diagnosis, target detection & identification, etc.
- Unfortunately, it can be computed in closed form only for additive Gaussian statistics of both object and image; other more realistic models are usually intractable

Other image quality metrics

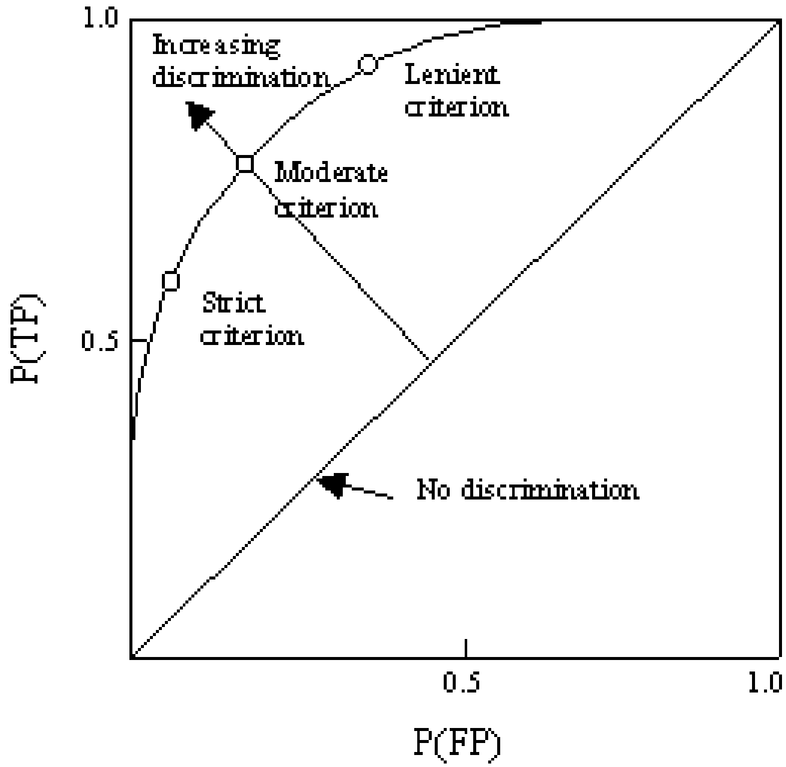
- Mean Square Error (MSQ) between object and image

$$E = \sum_{\substack{\text{object}\\ \text{samples}}} \left(f_k - \hat{f}_k \right)^2 \qquad \hat{f}_k = \begin{pmatrix} \text{result of} \\ \text{inversion} \end{pmatrix}$$

  - e.g. pseudoinverse minimizes MSQ in an overdetermined problem
  - obvious problem: most of the time, we don't know what *f* is!
  - more when we deal with Wiener filters and regularization

- Receiver Operator Characteristic
  - measures the performance of a cognitive system (human or computer program) in a detection or estimation task based on the image data

Receiver Operator Characteristic



Target detection task

Example: medical diagnosis,
• H0 (null hypothesis) = no tumor
• H1 = tumor

TP = true positive (*i.e.* correct identification of tumor)
FP = false positive (*aka* false alarm)