Introduction to Inverse Problems

- What is an image? Attributes and Representations
- Forward *vs* Inverse
- Optical Imaging as Inverse Problem
    - Incoherent and Coherent limits
    - Dimensional mismatch: continuous *vs* discrete
    - Singular *vs* ill-posed
- Ill-posedness: a 2×2 example

Basic premises

- What you “see” or imprint on photographic film is a very narrow interpretation of the word image
- Image is a representation of a physical object having certain attributes
- Examples of attributes
  - Optical image: absorption, emission, scatter, color wrt light
  - Acoustic image: absorption, scatter wrt sound
  - Thermal image: temperature (black-body radiation)
  - Magnetic resonance image: oscillation in response to radio-frequency EM field
- Representation: a transformation upon a matrix of attribute values
  - Digital image (e.g. on a computer file)
  - Analog image (e.g. on your retina)

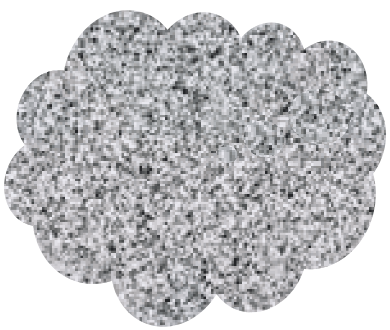
How are images formed

- Hardware
  - elements that operate directly on the physical entity
  - e.g. lenses, gratings, prisms, etc. operate on the optical field
  - e.g. coils, metal shields, etc. operate on the magnetic field
- Software
  - algorithms that transform representations
  - e.g. a radio telescope measures the Fourier transform of the source (representation #1); inverse Fourier transforming leads to a representation in the "native" object coordinates (representation #2); further processing such as iterative and nonlinear algorithms lead to a "cleaner" representation (#3).
  - e.g. a stereo pair measures two aspects of a scene (representation #1); a triangulation algorithm converts that to a binocular image with depth information (representation #2).

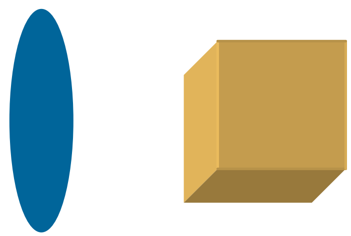
Who does what

- In optics,
  - standard hardware elements (lenses, mirrors, prisms) perform a limited class of operations (albeit very useful ones); these operations are
    - linear in field amplitude for coherent systems
    - linear in intensity for incoherent systems
    - a complicated mix for partially coherent systems
  - holograms and diffractive optical elements in general perform a more general class of operations, but with the same linearity constraints as above
  - nonlinear, iterative, etc. operations are best done with software components (people have used hardware for these purposes but it tends to be power inefficient, expensive, bulky, unreliable – hence these systems seldom make it to real life applications)

Imaging channels

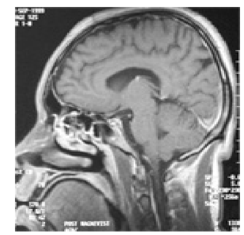


Information generators
- Wave sources
- Wave scatterers



Physics
Algorithms

Processing elements



Humans
Humanoids

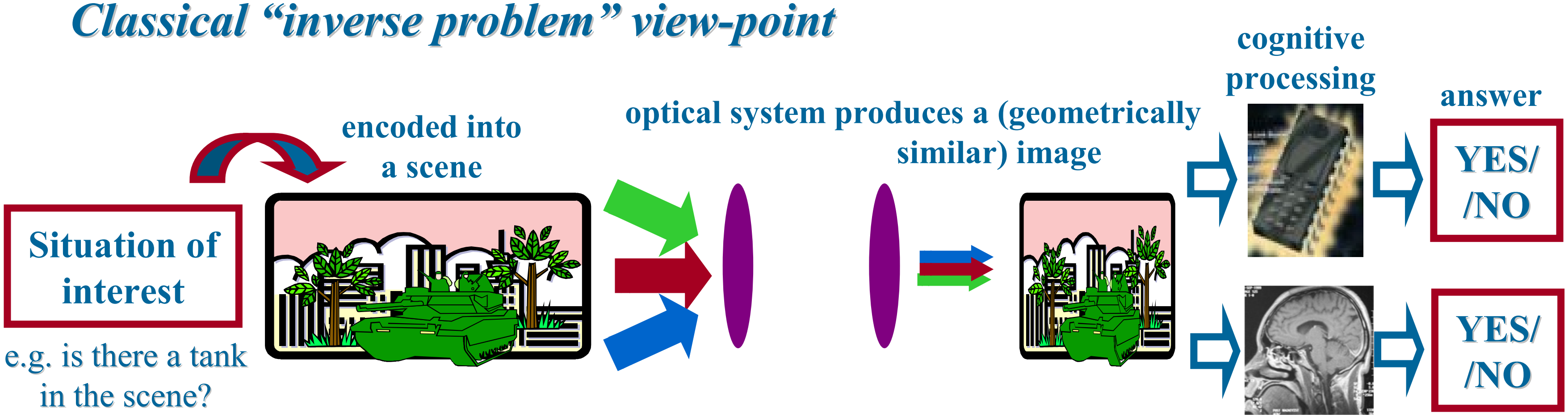
Users

- Imaging
- Communication
- Storage

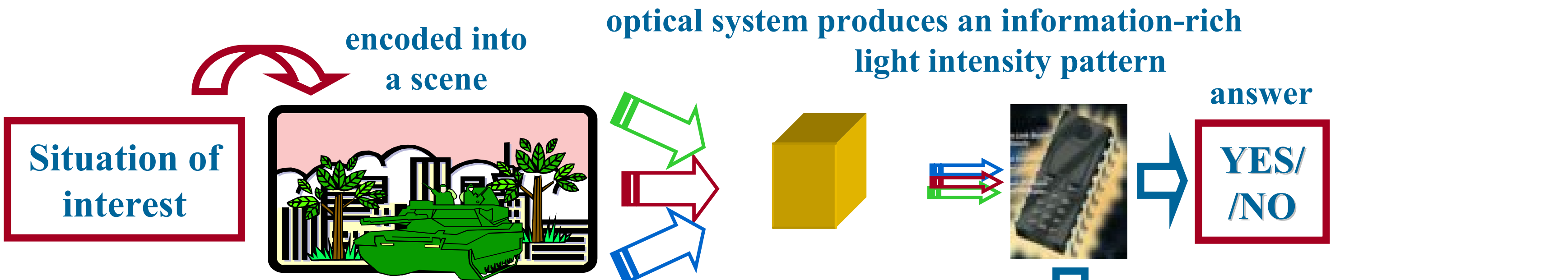
GOAL: Maximize information flow

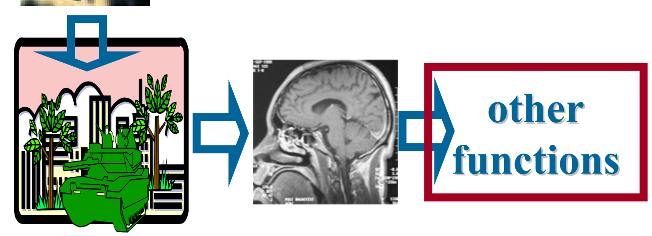
Generalized (cognitive) representations

Classical "inverse problem" view-point



"Non-imaging" or "generalized" sensor view-point

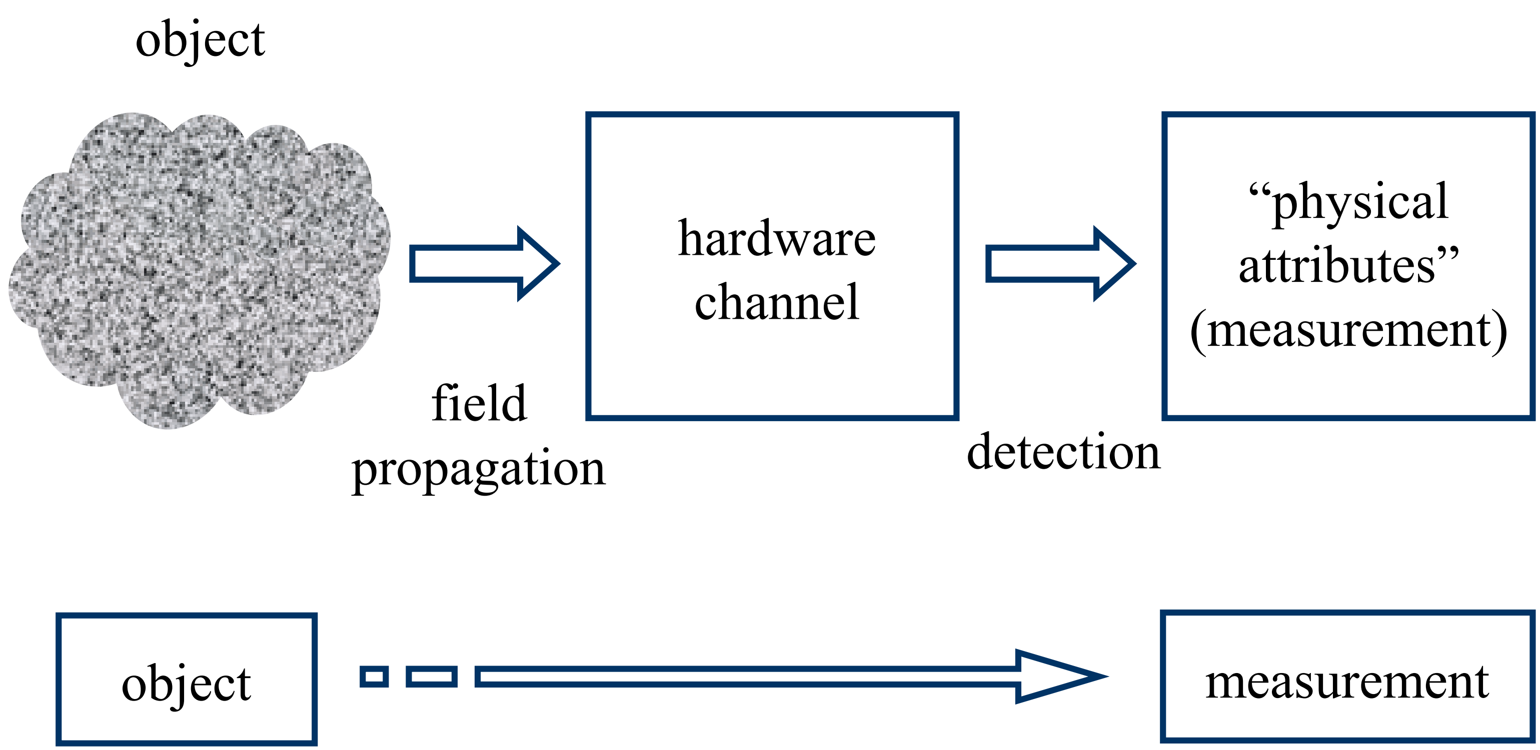




if necessary (requires resource reallocation)

Advantages: - optimum resource allocation
- better reliability
- adaptive, attentive operation

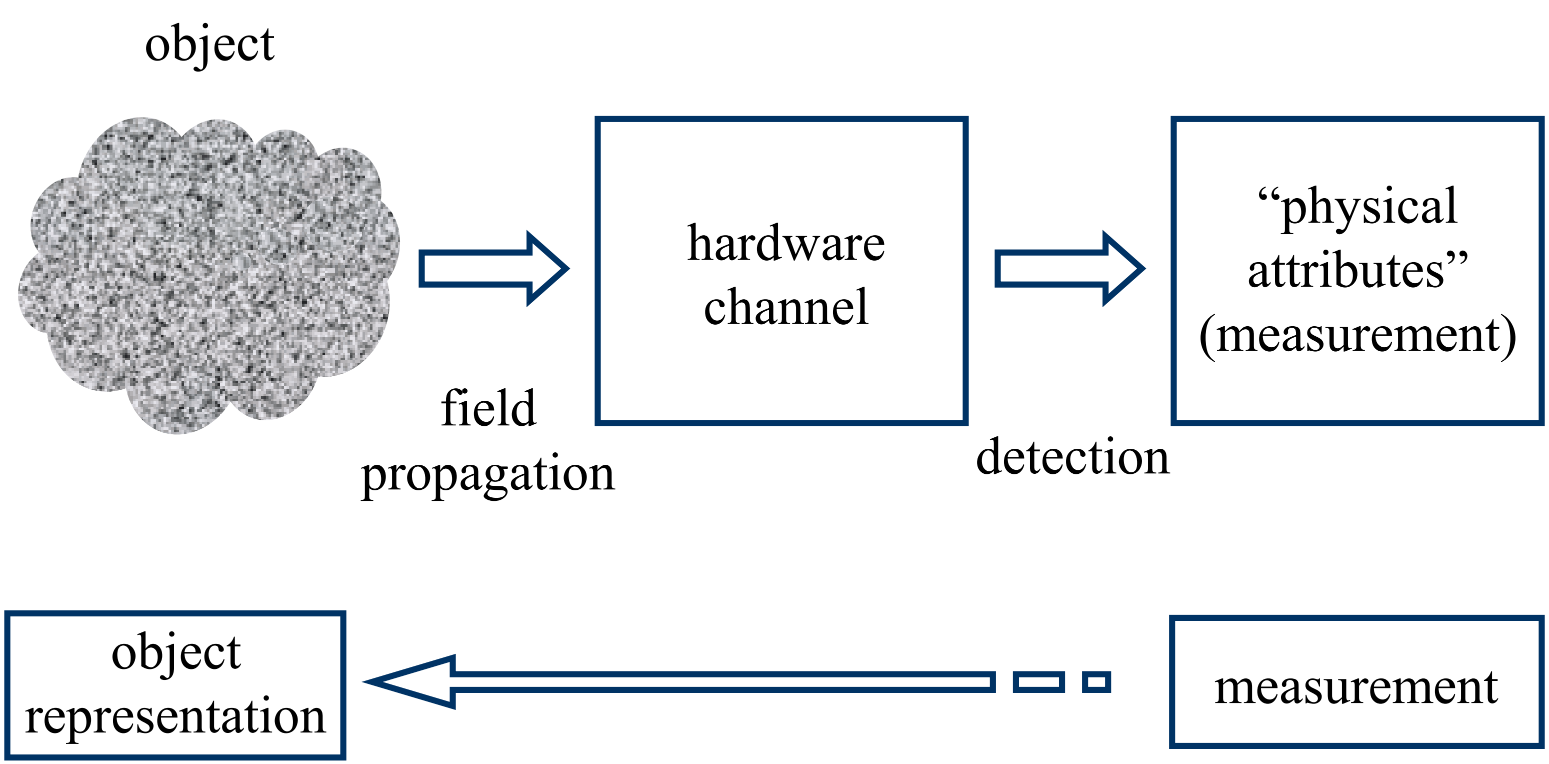
Forward problem



The Forward Problem answers the following question:

- Predict the measurement given the object attributes

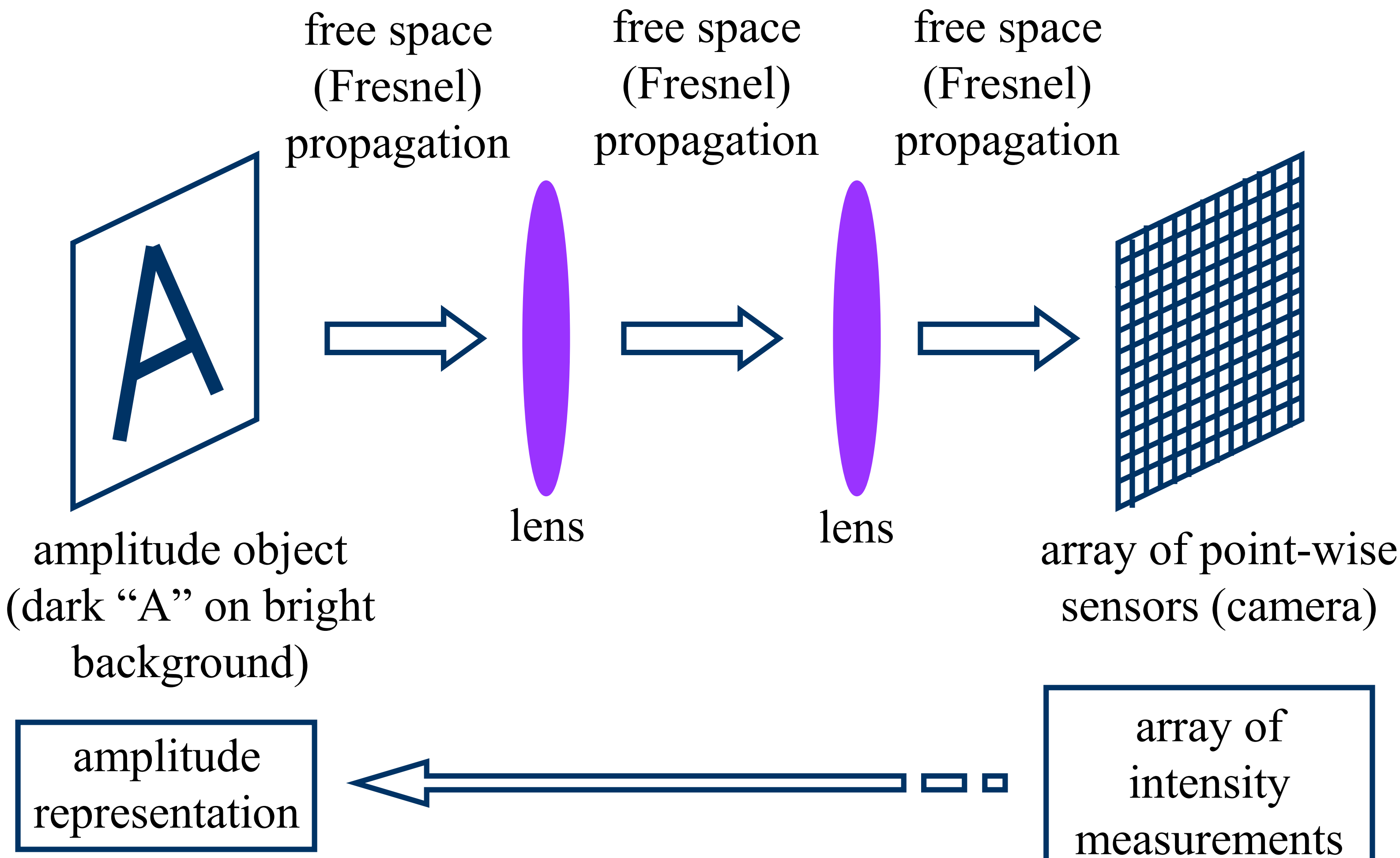
Inverse problem



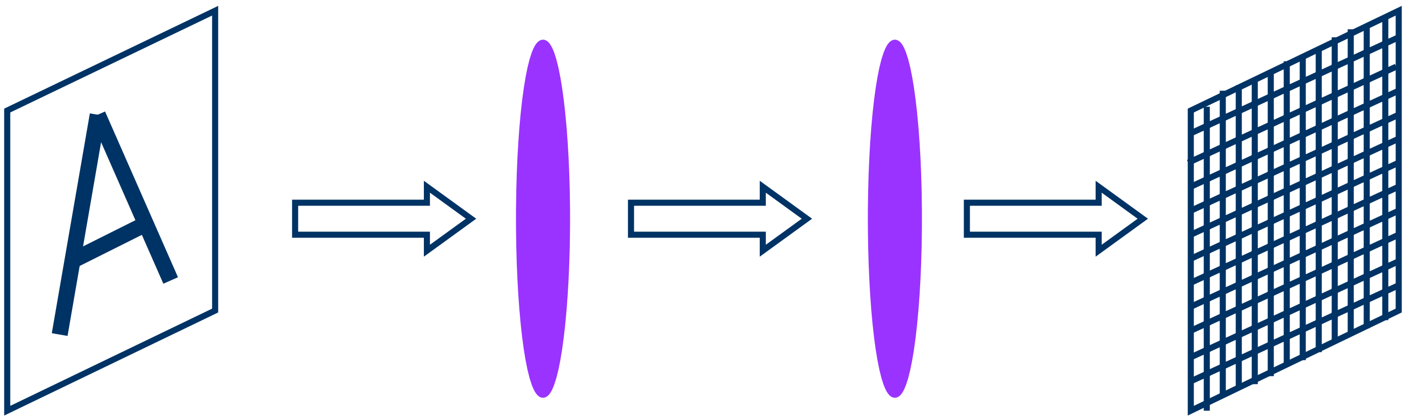
The Inverse Problem answers the following question:

• Form an object representation given the measurement

Optical Inversion



Optical Inversion: coherent



Nonlinear problem

$$f(x,y)$$

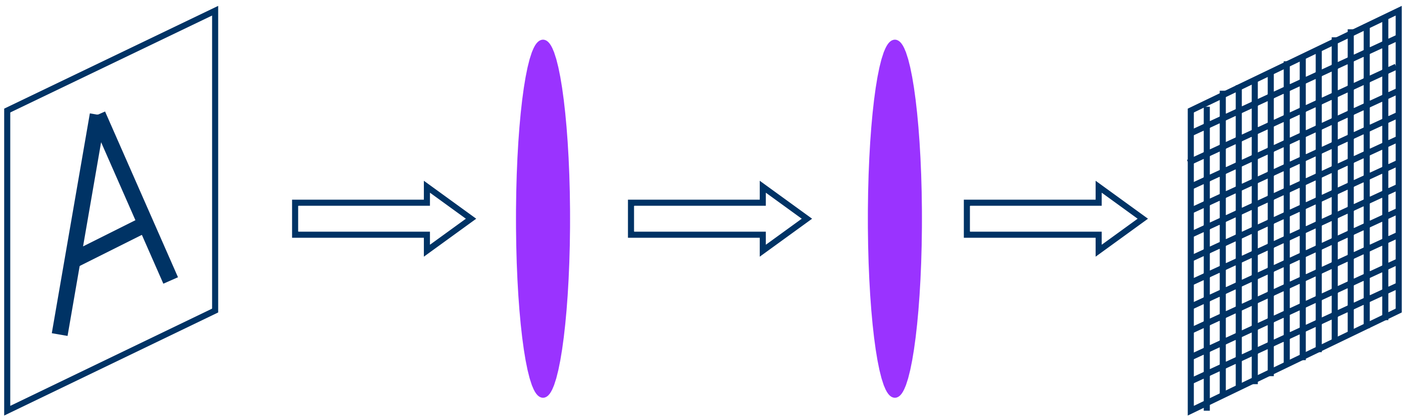
object amplitude

$$I(x',y') = \left| \int f(x,y) h_{\mathrm{coh}}(x'-x, y'-y)\,\mathrm{d}x\mathrm{d}y \right|^2$$

intensity measurement at the output plane

Note: I could make the problem linear if I could measure amplitudes directly (e.g. at radio frequencies)

Optical Inversion: incoherent



Linear problem

$$I_{\text{obj}}(x,y)$$

object intensity

$$I_{\text{meas}}(x',y') = \int I_{\text{obj}}(x,y)\, h_{\text{incoh}}(x'-x, y'-y)\,\mathrm{d}x\,\mathrm{d}y$$

intensity measurement at the output plane

Dimensional mismatch

- The object is a "continuous" function (amplitude or intensity) assuming quantum mechanical effects are at sub-nanometer scales, *i.e.* much smaller than the scales of interest (100nm or more)
  - i.e. the object dimension is uncountably infinite
- The measurement is "discrete," therefore countable and finite
- To be able to create a "1-1" object representation from the measurement, I would need to create a 1-1 map from a finite set of integers to the set of real numbers. This is of course impossible
  - the inverse problem is inherently ill-posed
- We can resolve this difficulty by relaxing the 1-1 requirement
  - therefore, we declare ourselves satisfied if we <u>sample</u> the object with sufficient density (Nyquist theorem)
  - implicitly, we have assumed that the object lives in a finite-dimensional space, although it "looks" like a continuous function

Singularity and ill-posedness

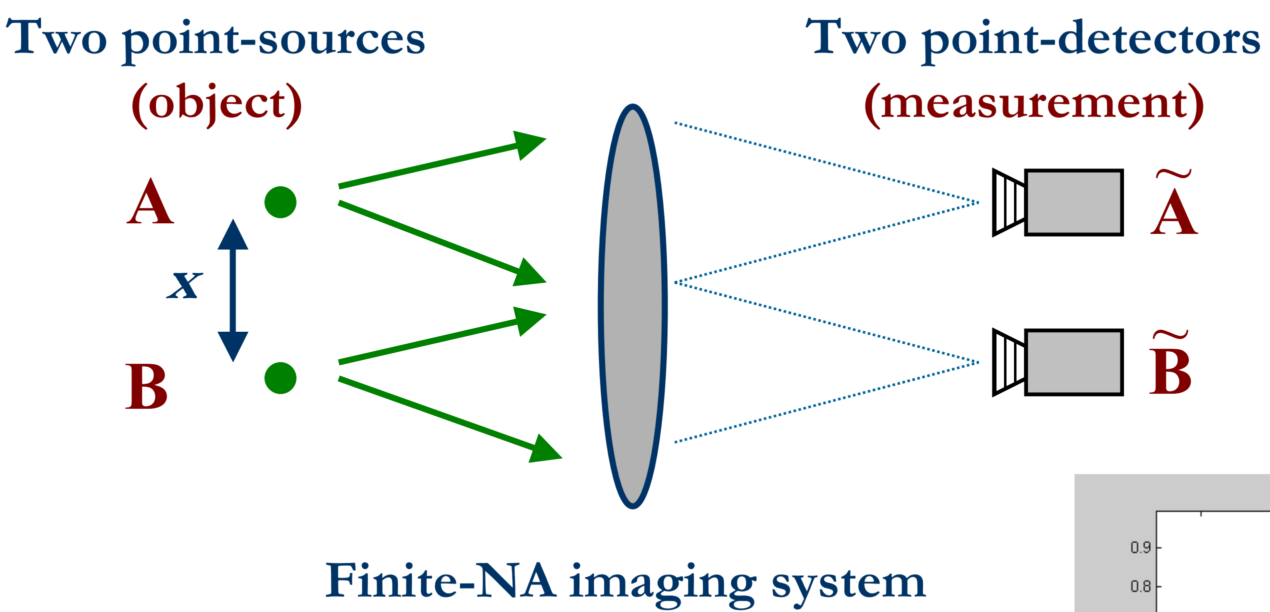
Under the finite-dimensional object assumption, the linear inverse problem is converted from an integral equation to a matrix equation

$$g(x',y') = \int f(x,y)\, h(x'-x, y'-y)\, \mathrm{d}x\, \mathrm{d}y$$

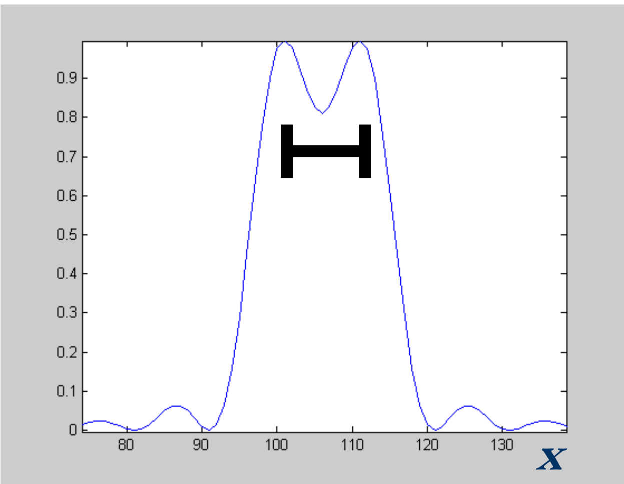
$$\Leftrightarrow \quad \bar{g} = \mathbf{H}\, \bar{f}$$

- If the matrix **H** is rectangular, the problem may be <u>overconstrained</u> or <u>underconstrained</u>
- If the matrix **H** is square and has det(**H**)=0, the problem is <u>singular</u>; it can only be solved partially by giving up on some object dimensions (*i.e.* leaving them indeterminate)
- If the matrix **H** is square and det(**H**) is non-zero but small, the problem may be <u>ill-posed</u> or <u>unstable</u>: it is extremely sensitive to errors in the measurement f

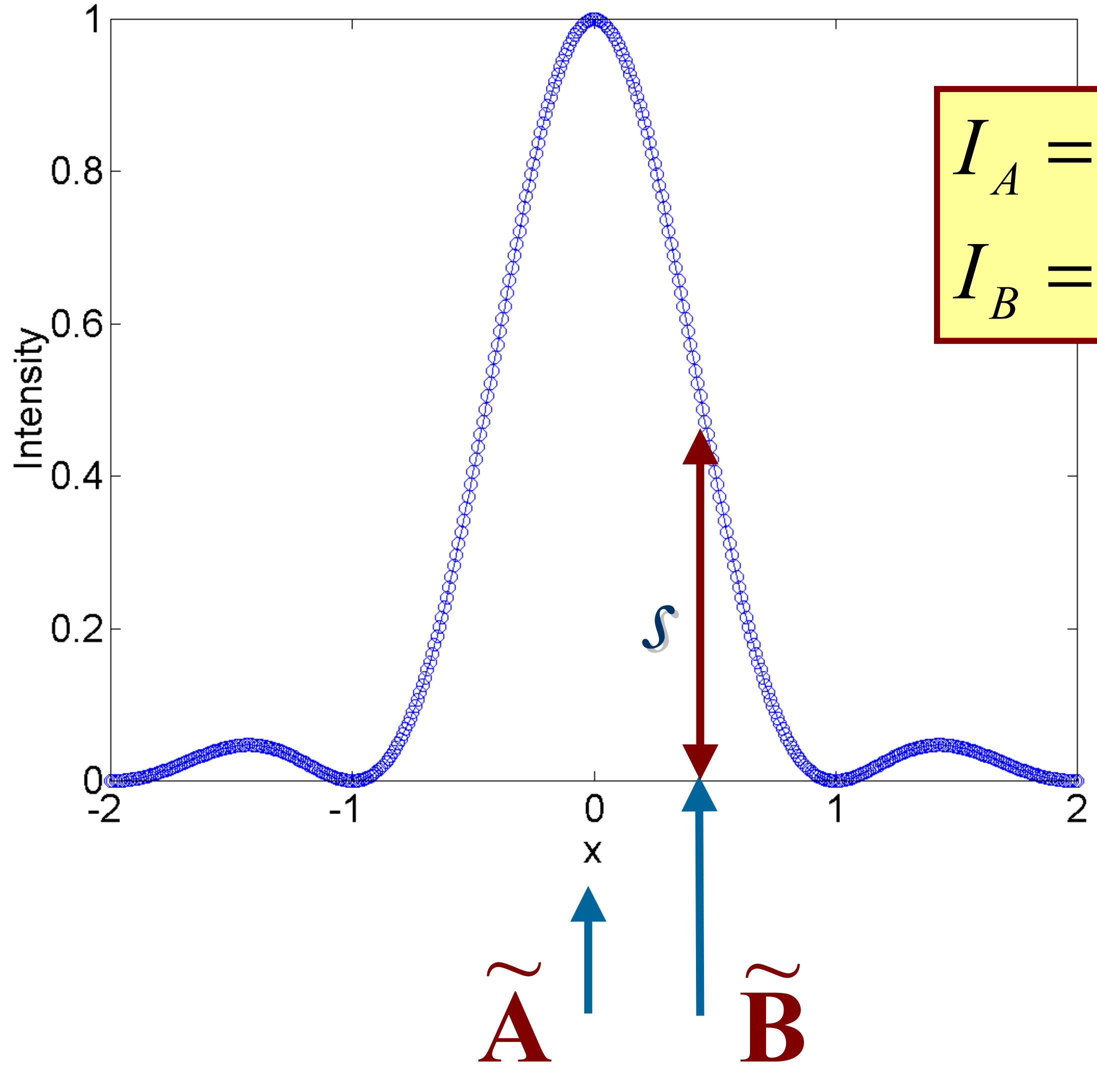
Resolution: a toy problem



Classical view



Cross-leaking power



$$I_A = J_A + sJ_B$$
$$I_B = sJ_A + J_B$$

Ill-posedness in two-point inversion

$$\mathbf{H} = \begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix}$$

$$\det(\mathbf{H}) = 1 - s^2$$

$$\mathbf{H}^{-1} = \frac{1}{1-s^2} \begin{pmatrix} 1 & -s \\ -s & 1 \end{pmatrix}$$