X, Y independent $\rightarrow \text{cov}(X, Y) = 0$

Assume random variables X and Y are *discrete*. That is, assume that there is a finite or denumerable sample space which is a set of ω_i and a set of quantities x_i and y_i defined.

Definition X and Y are *independent* if

$$\text{prob}((X = x) \text{ and } (Y = y)) = \text{prob}(X = x)\text{prob}(Y = y)$$

in which x is some x_i and y is some y_j .

Then if X and Y are independent,

$$E(XY) = E(X)E(Y)$$

Proof:

$$E(XY) = \sum_{i,j} x_i y_j \text{prob}(XY = x_i y_j)$$

$$= \sum_{i,j} x_i y_j \text{prob}((X = x_i) \text{ and } (Y = y_j))$$

$$= \sum_{i,j} x_i y_j \text{prob}(X = x_i)\text{prob}(Y = y_j)$$

$$= \sum_{i} x_i \text{prob}(X = x_i) \sum_{j} y_j \text{prob}(Y = y_j) = E(X)E(Y)$$

Then if X and Y are independent,

$$\text{cov}(X, Y) = E[(X - E(X))(Y - E(Y))]$$

$$= E[XY - XE(Y) - YE(X) + E(X)E(Y)]$$

$$= E[XY] - E(X)E(Y) - E(Y)E(X) + E(X)E(Y) = 0$$

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