LP Example

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Consider the factory in Figure 1 that consists of three parallel machines. It makes a single product which can be produced using any one of the machines. The possible material flows are indicated.

Assume that the cost (\$/part) of using machine M_i is c_i, and that the maximum rate that M_i can operate is μ_i. Assume that $c_3 > c_2 > c_1 > 0$ and let $\mu_3 = \infty$. The total demand is D.

Problem: How should the demand be allocated among the machines to minimize cost?

Intuitive answer: We want to use the least expensive machine as much as possible, and the most expensive machine as little as possible. Therefore

If $D \leq \mu_1$,	$x_1 = D, x_2 = x_3 = 0$	cost $= c_1 D$
If $\mu_1 < D \leq \mu_1 + \mu_2$,	$x_1 = \mu_1, x_2 = D - \mu_1, x_3 = 0$	cost $= c_1\mu_1 + c_2(D - \mu_1)$
If $\mu_1 + \mu_2 < D$,	$x_1 = \mu_1, x_2 = \mu_2, x_3 = D - \mu_1 - \mu_2$	cost $= c_1\mu_1 + c_2\mu_2 + c_3(D - \mu_1 - \mu_2)$

LP formulation:

$$\min c_1x_1 + c_2x_2 + c_3x_3$$

such that

$$x_1 + x_2 + x_3 = D$$
$$x_1 \leq \mu_1$$
$$x_2 \leq \mu_2$$
$$x_i \geq 0, i = 1, 2, 3$$

The constraint space is illustrated in Figure 2 for two different values of D. The arrows indicate the solution points.

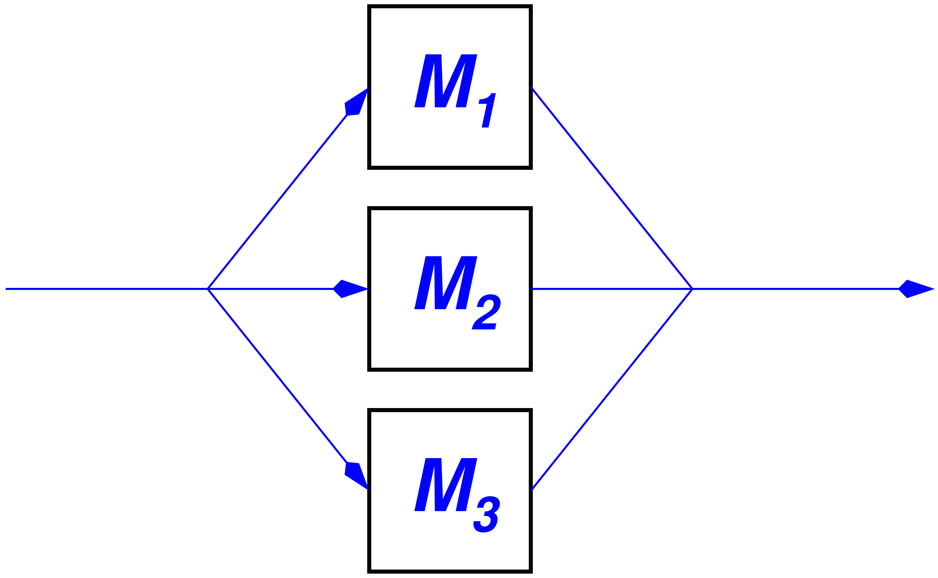

Figure 1: Factory

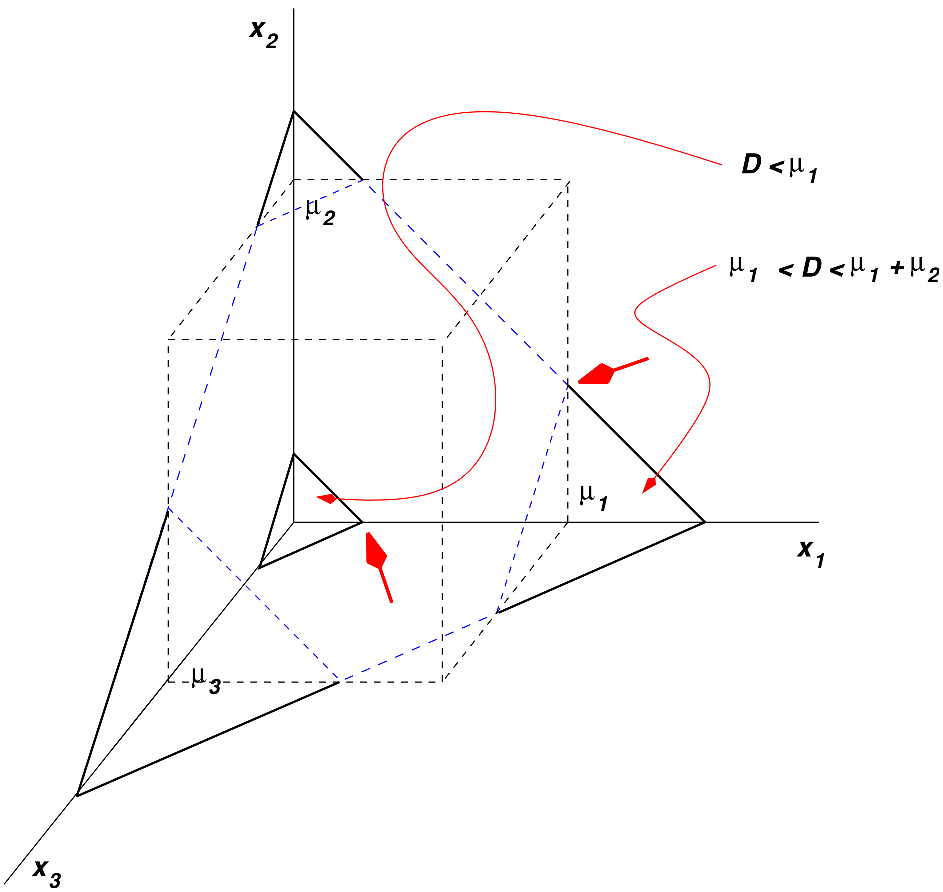

Figure 2: Constraint space

LP in Standard Form: Define x_4 and x_5 as the slack variables associated with the upper bounds on x_1 and x_2. Then

$$\min c_1x_1 + c_2x_2 + c_3x_3$$

such that

$$x_1 + x_2 + x_3 = D$$
$$x_1 + x_4 = \mu_1$$
$$x_2 + x_5 = \mu_2$$
$$x_i \geq 0, i = 1, ..., 5$$

In that case

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$
$$b = \begin{pmatrix} D \\ \mu_1 \\ \mu_2 \end{pmatrix}$$
$$c^T = \begin{pmatrix} c_1 & c_2 & c_3 & 0 & 0 \end{pmatrix}$$

Verification of solution guess:

1. $D \leq \mu_1$:

We have guessed that when D is small, $x_1 = D, x_2 = x_3 = 0$. Then, also, $x_4 = \mu_1 - D, x_5 = \mu_2$. This is a feasible solution. It is also basic, in which the basic variables are x_1, x_4, x_5 and

$$A_B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad A_N = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$c_B^T = \begin{pmatrix} c_1 & 0 & 0 \end{pmatrix} \quad c_N^T = \begin{pmatrix} c_2 & c_3 \end{pmatrix}$$

It is easy to show that

$$A_B^{-1}A_N = \begin{pmatrix} 1 & 1 \\ -1 & -1 \\ 1 & 0 \end{pmatrix}$$

This is demonstrated at the end of this note.

Therefore,

$$c_R^T = c_N^T - c_B^T A_B^{-1} A_N = \begin{pmatrix} c_2 & c_3 \end{pmatrix} - \begin{pmatrix} c_1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} c_2 & c_3 \end{pmatrix} - \begin{pmatrix} c_1 & c_1 \end{pmatrix} = \begin{pmatrix} c_2 - c_1 & c_3 - c_1 \end{pmatrix}$$

and, by assumption,

$$c_2 - c_1 > 0$$
$$c_3 - c_1 > 0$$

Therefore, since both components of c_R are positive, the solution we guessed is correct.

2. $\mu_1 < D \leq \mu_1 + \mu_2$

We have guessed that $x_1 = \mu_1, x_2 = D - \mu_1, x_3 = 0$. Then $x_4 = 0, x_5 = \mu_2 - D + \mu_1$. The basic variables are x_1, x_2, x_5 and

$$A_B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad A_N = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$c_B^T = \begin{pmatrix} c_1 & c_2 & 0 \end{pmatrix} \quad c_N^T = \begin{pmatrix} c_3 & 0 \end{pmatrix}$$

Then

$$A_B^{-1} A_N = \begin{pmatrix} 0 & 1 \\ -1 & -1 \\ 1 & -1 \end{pmatrix}$$

and

$$c_R^T = c_N^T - c_B^T A_B^{-1} A_N = \begin{pmatrix} -c_2 + c_3 & c_1 - c_2 + c_3 \end{pmatrix}$$

which is also componentwise positive.

3. $\mu_1 + \mu_2 < D$**:**

We have guessed that $x_1 = \mu_1, x_2 = \mu_2, x_3 = D - \mu_1 - \mu_2$. Then $x_4 = x_5 = 0$ and x_1, x_2, x_3 are the basic variables. Therefore,

$$A_B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad A_N = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$c_B^T = \begin{pmatrix} c_1 & c_2 & c_3 \end{pmatrix} \quad c_N^T = \begin{pmatrix} 0 & 0 \end{pmatrix}$$

Then

$$A_B^{-1} A_N = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix}$$

and

$$c_R^T = c_N^T - c_B^T A_B^{-1} A_N = -\begin{pmatrix} c_1 & c_2 & c_3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix} = (c_3 - c_1 \quad c_3 - c_2)$$

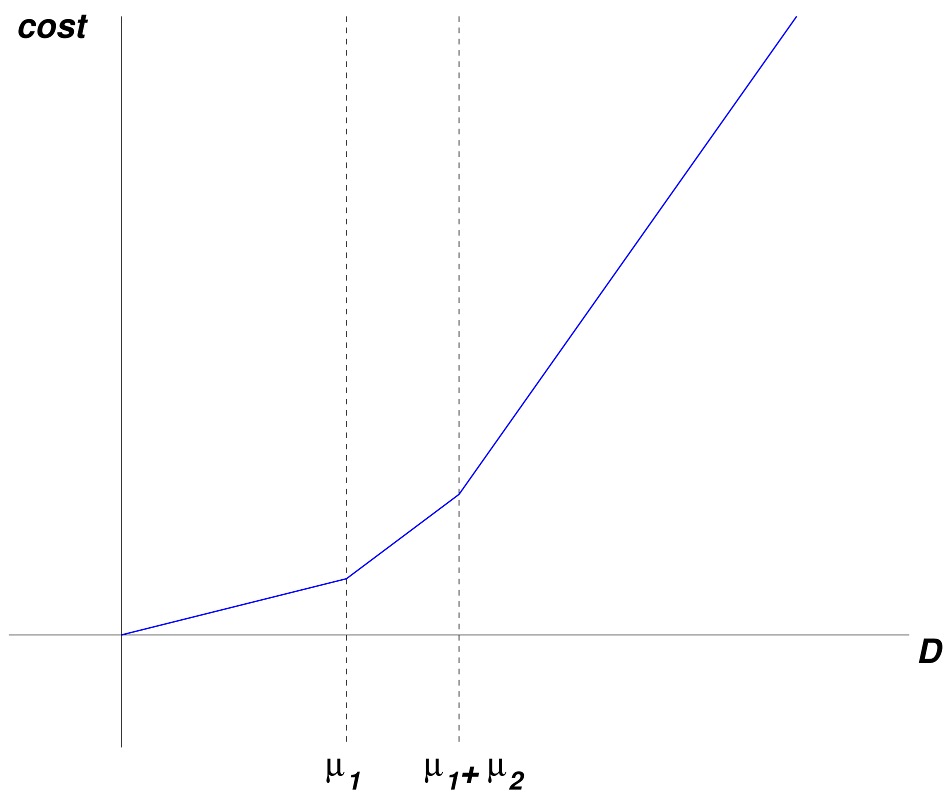

Figure 3: Cost

which is also componentwise positive.

Note that the cost as a function of D is shown in Figure 3.

To show that, in the first case,

$$A_B^{-1}A_N = \begin{pmatrix} 1 & 1 \\ -1 & -1 \\ 1 & 0 \end{pmatrix}$$

$A_B^{-1}A_N$ is a matrix of two columns. The first column satisfies

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}$$

and the second satisfies

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix}$$

The equation for the first column is equivalent to

$$\begin{aligned} 1 &= s_1 \\ 0 &= s_1 + s_2 \\ 1 &= s_3 \end{aligned}$$

or

$$\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

and the equation for the second column is

$$\begin{aligned} 1 &= t_1 \\ 0 &= t_1 + t_2 \\ 0 &= s_3 \end{aligned}$$

or

$$\begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

Putting the columns together,

$$A_B^{-1}A_N = \begin{pmatrix} 1 & 1 \\ -1 & -1 \\ 1 & 0 \end{pmatrix}$$