Queues

Lecturer: Stanley B. Gershwin

Stochastic processes

- t is time.
- $X()$ is a *stochastic process* if $X(t)$ is a random variable for every t.
- t is a scalar — it can be discrete or continuous.
- $X(t)$ can be discrete or continuous, scalar or vector.

- A *Markov process* is a stochastic process in which the probability of finding X at some value at time $t + \delta t$ depends only on the value of X at time t.

- Or, let $x(s), s \leq t$, be the history of the values of X before time t and let A be a possible value of X. Then
  prob$\{X(t+\delta t) = A | X(s) = x(s), s \leq t\}$ =
  prob$\{X(t+\delta t) = A | X(t) = x(t)\}$

Markov processes

- In words: if we know what X was at time t, we don't gain any more useful information about $X(t+\delta t)$ by *also* knowing what X was at any time earlier than t.

- *This is the definition of a class of mathematical models. It is NOT a statement about reality!!* That is, not everything is a Markov process.

Markov processes

Example

- I have \$100 at time $t = 0$.
- At every time $t \geq 1$, I have \$$N(t)$.
  - ⋆ A (possibly biased) coin is flipped.
  - ⋆ If it lands with H showing, $N(t+1) = N(t) + 1$.
  - ⋆ If it lands with T showing, $N(t+1) = N(t) - 1$.

$N(t)$ is a Markov process. *Why?*

Markov processes

States and transitions

- States can be numbered 0, 1, 2, 3, ... *(or with multiple indices if that is more convenient)*.
- Time can be numbered 0, 1, 2, 3, ... *(or 0, Δ, 2Δ, 3Δ, ... if more convenient)*.
- The probability of a transition from j to i in one time unit is often written P_{ij}, where

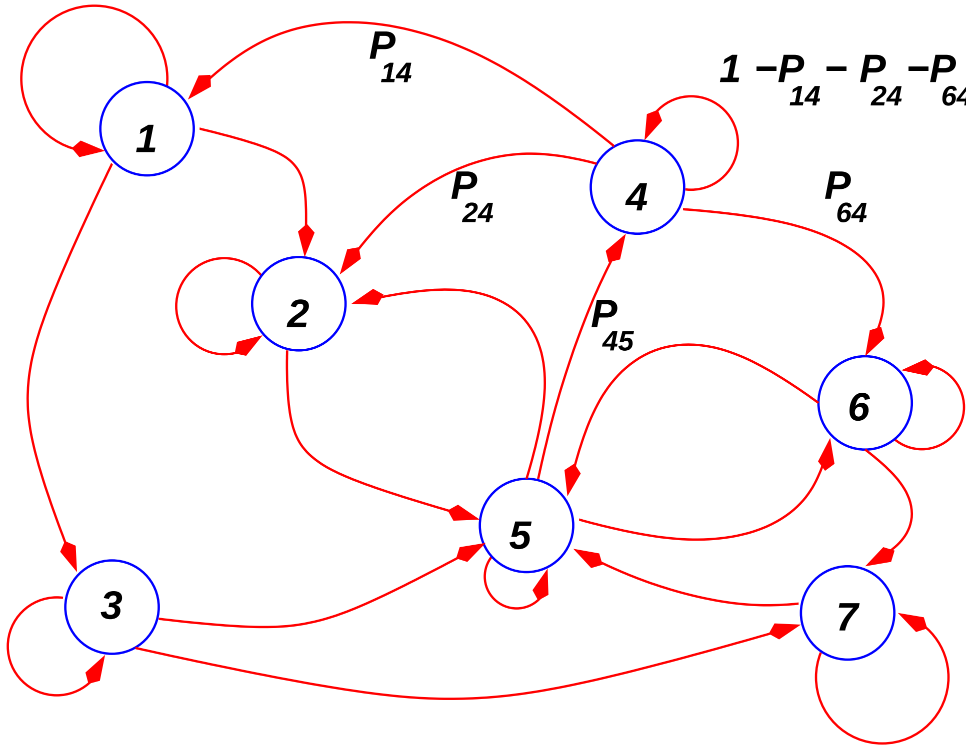
$$P_{ij} = \text{prob}\{X(t+1) = i | X(t) = j\}$$

Markov processes

Discrete state, discrete time

States and transitions

Transition graph



P_{ij} is a probability. Note that $P_{ii} = 1 - \sum_{m, m \neq i} P_{mi}$.

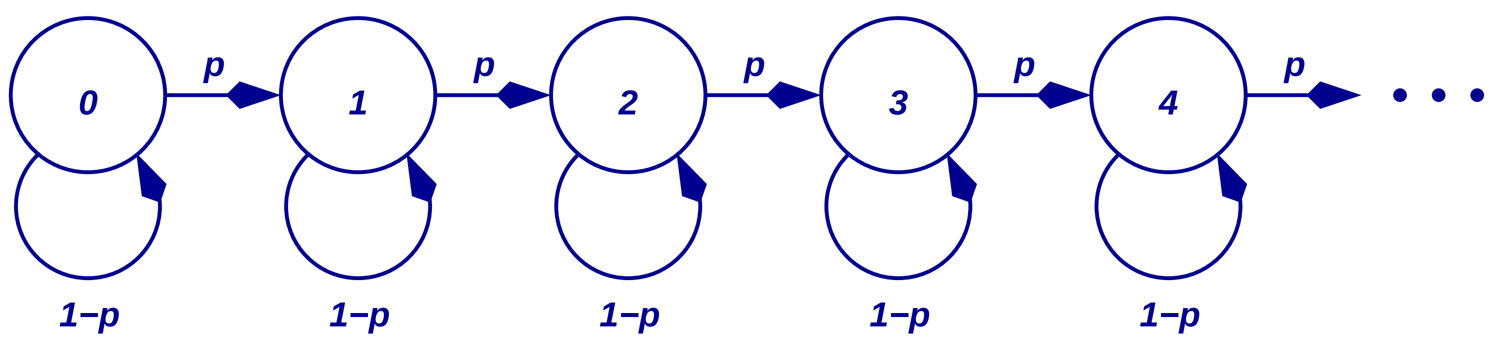
Markov processes

Discrete state, discrete time

States and transitions

Example : $H(t)$ is the number of Hs after t coin flips.

Assume probability of H is p.



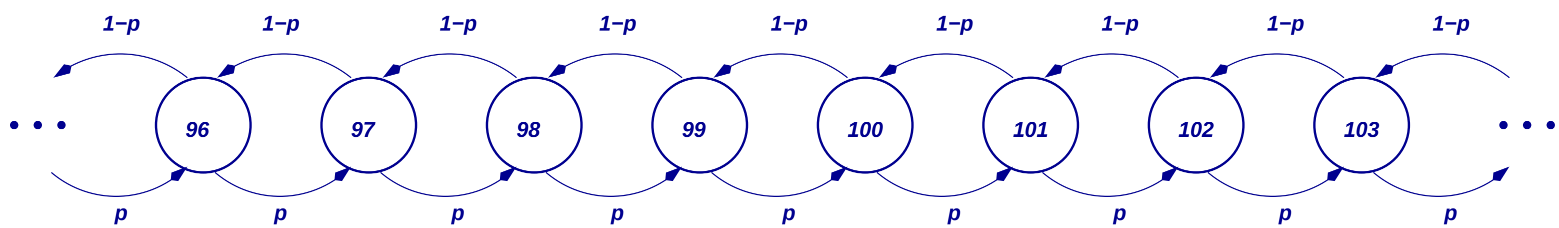
Markov processes

Discrete state, discrete time

States and transitions

Example : Coin flip bets on Slide 5.

Assume probability of H is p.



Markov processes

Discrete state, discrete time

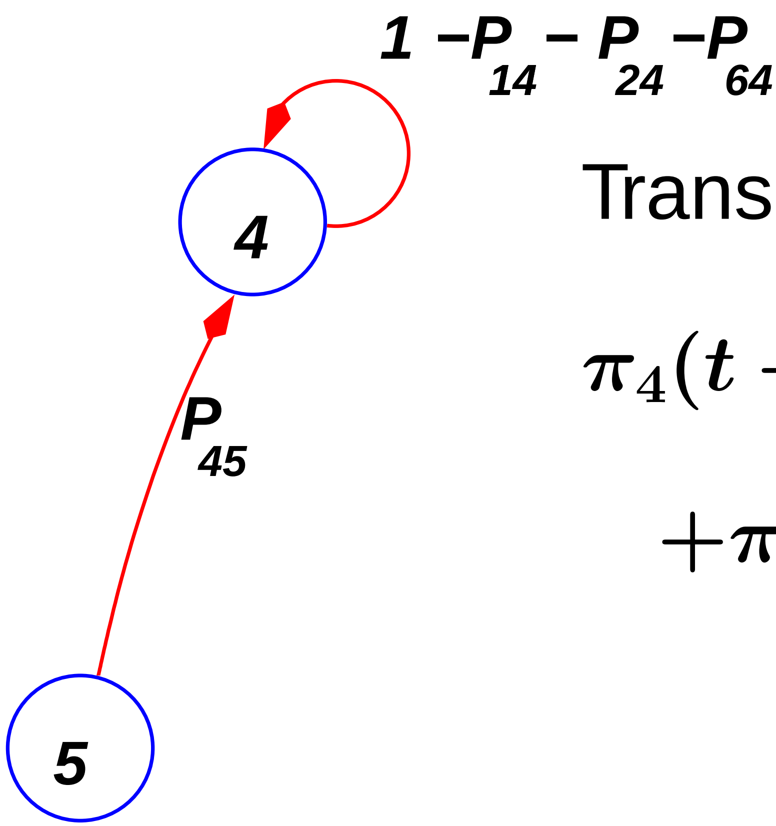
States and transitions

- Define $\pi_i(t) = \text{prob}\{X(t) = i\}$.
- Transition equations: $\pi_i(t+1) = \sum_j P_{ij}\pi_j(t)$.
  (Law of Total Probability)
- Normalization equation: $\sum_i \pi_i(t) = 1$.

Markov processes

Discrete state, discrete time

States and transitions



Transition equation:

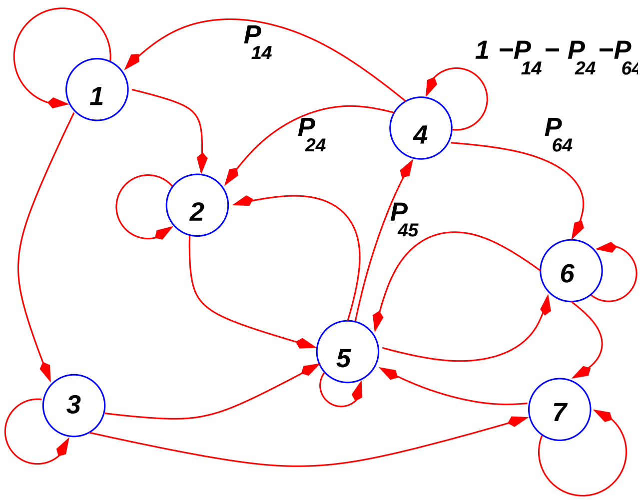
$$\pi_4(t+1) = \pi_5(t)P_{45}$$

$$+\pi_4(t)(1 - P_{14} - P_{24} - P_{64})$$

Markov processes

Discrete state, discrete time

States and transitions



$$\text{prob}\{X(t+1)=2\}$$

$$= \text{prob}\{X(t+1)=2|X(t)=1\}\text{prob}\{X(t)=1\}$$

$$+\text{prob}\{X(t+1)=2|X(t)=2\}\text{prob}\{X(t)=2\}$$

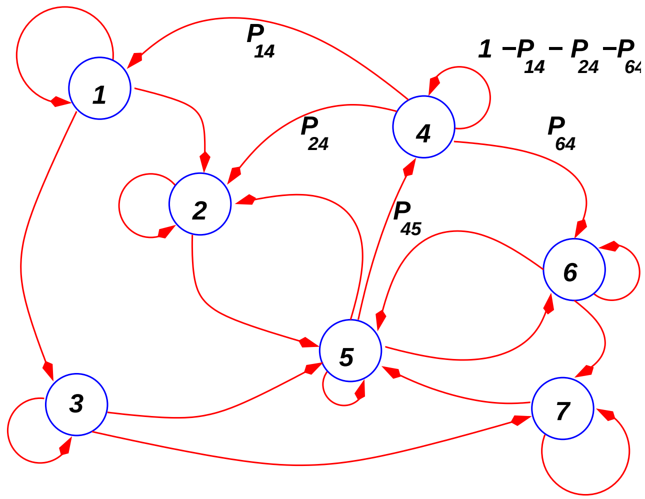
$$+\text{prob}\{X(t+1)=2|X(t)=4\}\text{prob}\{X(t)=4\}$$

$$+\text{prob}\{X(t+1)=2|X(t)=5\}\text{prob}\{X(t)=5\}$$

Markov processes

Discrete state, discrete time

States and transitions



Or, since

$$P_{ij} = \text{prob}\{X(t+1) = i | X(t) = j\}$$

and

$$\pi_i(t) = \text{prob}\{X(t) = i\},$$

$$\pi_2(t+1) = P_{21}\pi_1(t) + P_{22}\pi_2(t) + P_{24}\pi_4(t) + P_{25}\pi_5(t)$$

Note that $P_{22} = 1 - P_{52}$.

Markov processes

Discrete state, discrete time

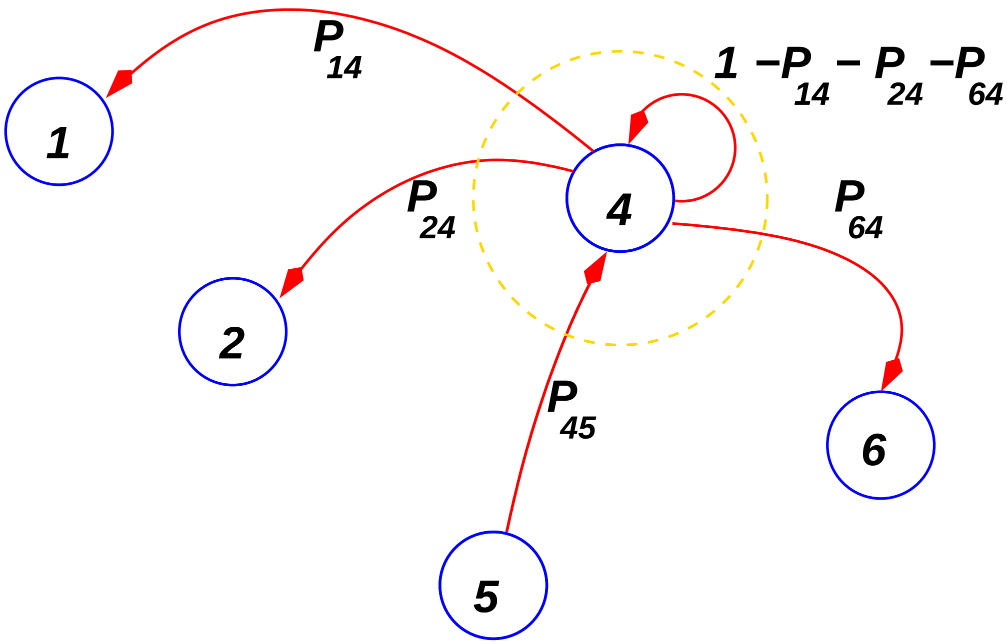
States and transitions

- *Steady state:* $\pi_i = \lim_{t\to\infty} \pi_i(t)$, if it exists.
- Steady-state transition equations: $\pi_i = \sum_j P_{ij}\pi_j$.
- *Alternatively,* steady-state balance equations:
  $\pi_i \sum_{m,m\neq i} P_{mi} = \sum_{j,j\neq i} P_{ij}\pi_j$
- Normalization equation: $\sum_i \pi_i = 1$.

Markov processes

Discrete state, discrete time

States and transitions



Balance equation:

$$\pi_4(P_{14} + P_{24} + P_{64})$$

$$= \pi_5 P_{45}$$

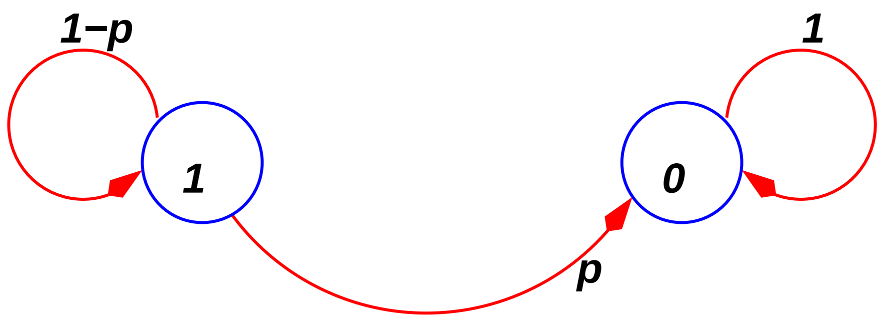
in steady state *only* .

Markov processes

Discrete state, discrete time

Geometric distribution

Consider a two-state system. The system can go from 1 to 0, but not from 0 to 1.

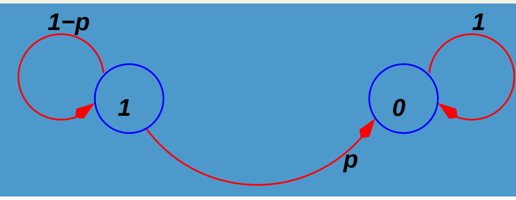


Let p be the conditional probability that the system is in state 0 at time $t+1$, given that it is in state 1 at time t. Then

$$p = \text{ prob } [\alpha(t+1) = 0 | \alpha(t) = 1] \,.$$

Markov processes

Discrete state, discrete time



Let $\pi(\alpha, t)$ be the probability of being in state α at time t.

Then, since

$$\pi(0, t+1) = \text{prob } [\alpha(t+1) = 0 | \alpha(t) = 1] \text{ prob } [\alpha(t) = 1]$$
$$+ \text{ prob } [\alpha(t+1) = 0 | \alpha(t) = 0] \text{ prob } [\alpha(t) = 0],$$

we have

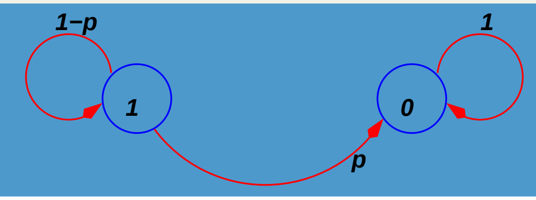
$$\pi(0, t+1) = p\pi(1, t) + \pi(0, t),$$
$$\pi(1, t+1) = (1-p)\pi(1, t),$$

and the normalization equation

$$\pi(1, t) + \pi(0, t) = 1.$$

Markov processes

Discrete state, discrete time



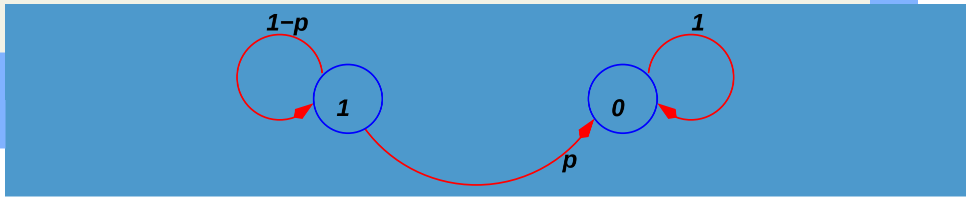
Assume that $\pi(1,0) = 1$. Then the solution is

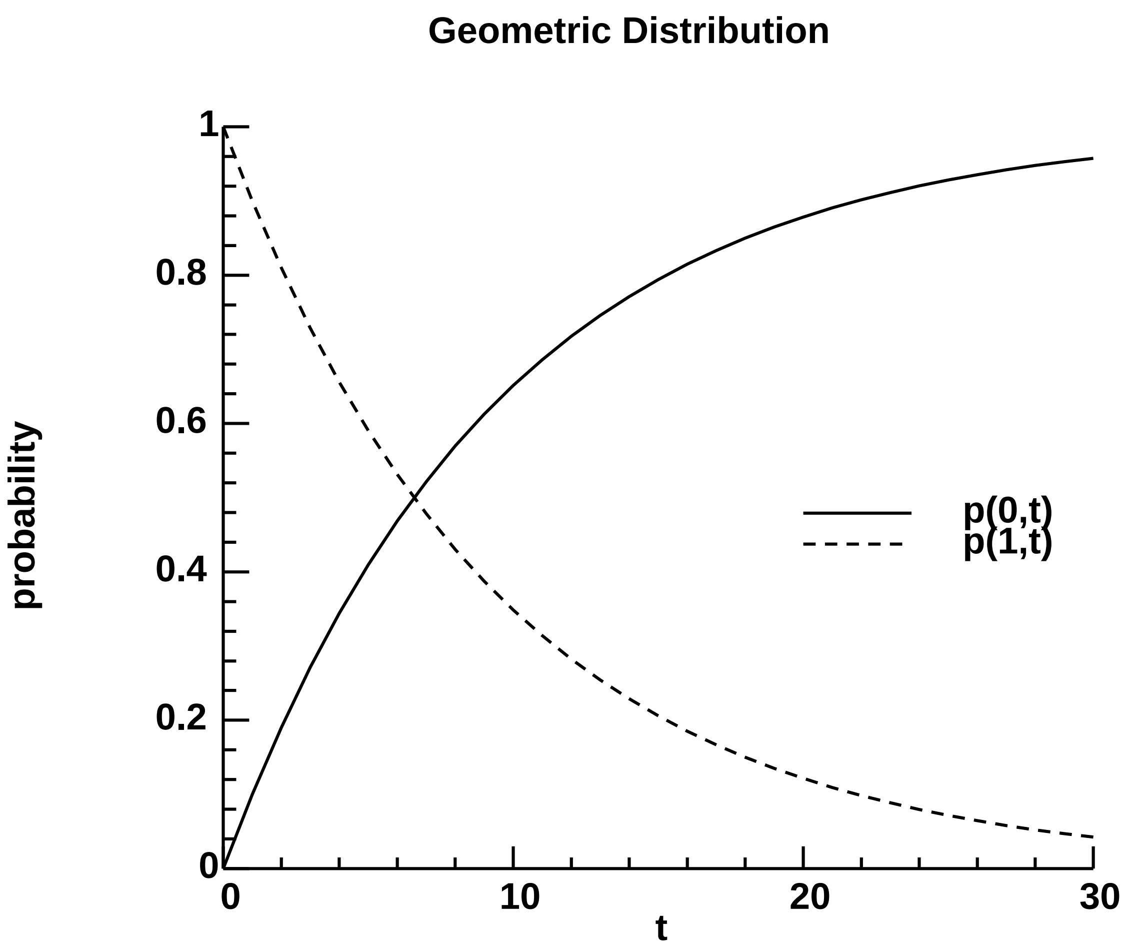
$$\pi(0,t) = 1 - (1-p)^t,$$

$$\pi(1,t) = (1-p)^t.$$

Markov processes

Discrete state, discrete time



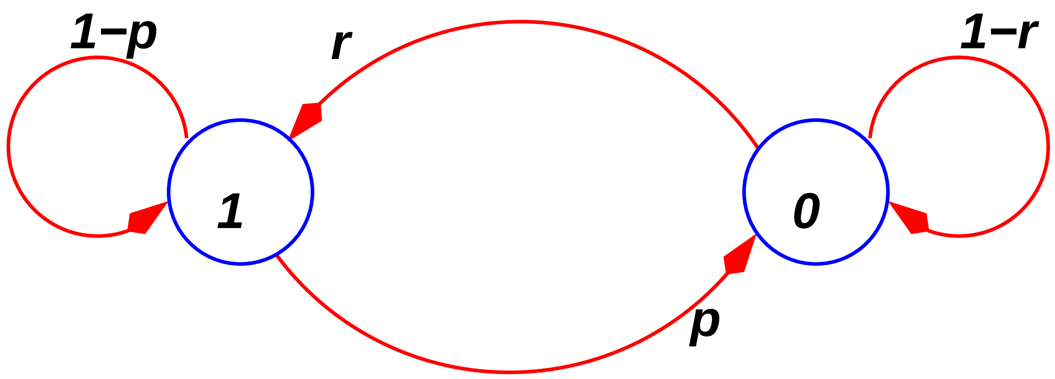


Markov processes

Discrete state, discrete time

Unreliable machine

1=up; 0=down.



Markov processes

Discrete state, discrete time

Unreliable machine

The probability distribution satisfies

$$\pi(0,t+1) = \pi(0,t)(1-r) + \pi(1,t)p,$$

$$\pi(1,t+1) = \pi(0,t)r + \pi(1,t)(1-p).$$

Markov processes

Discrete state, discrete time

Unreliable machine

It is not hard to show that

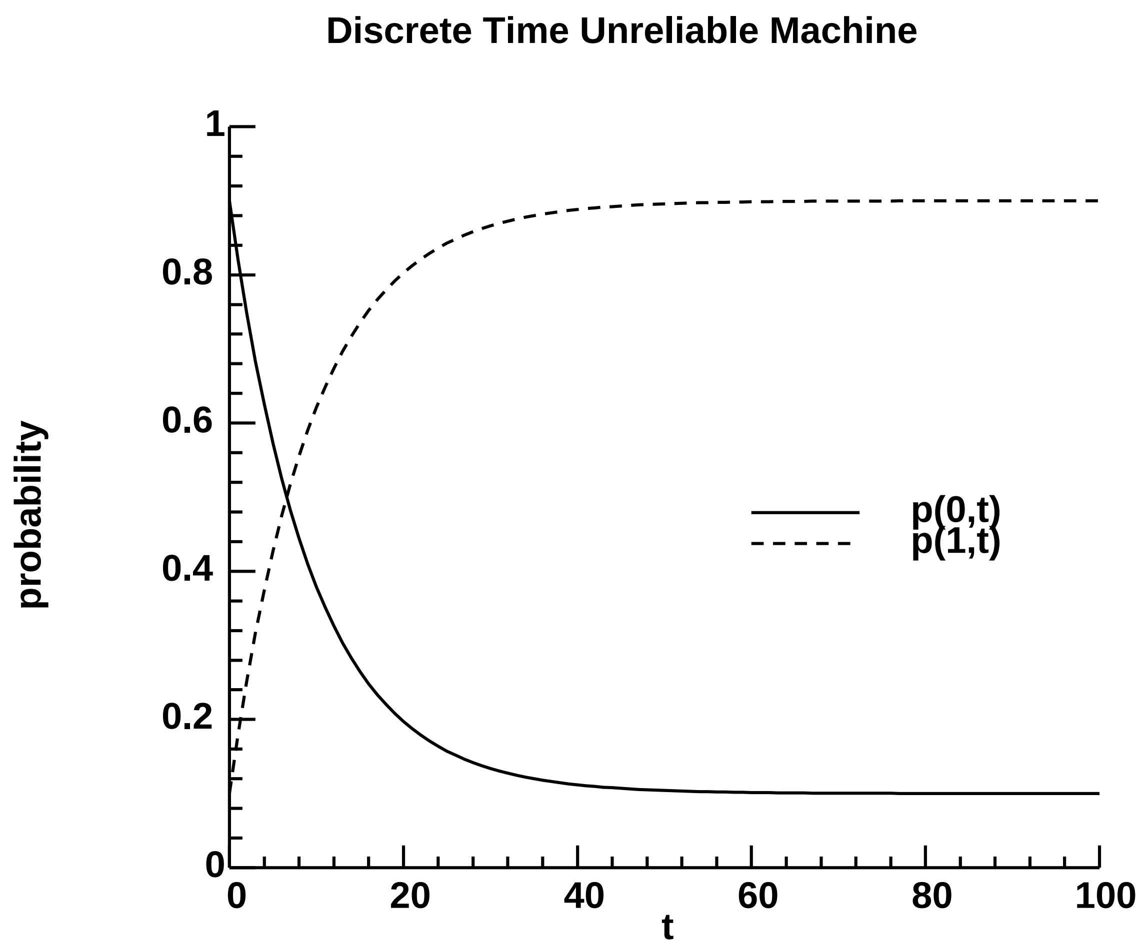
$$\pi(0,t) = \pi(0,0)(1-p-r)^t + \frac{p}{r+p}\left[1-(1-p-r)^t\right],$$

$$\pi(1,t) = \pi(1,0)(1-p-r)^t + \frac{r}{r+p}\left[1-(1-p-r)^t\right].$$

Markov processes

Discrete state, discrete time

Unreliable machine



Markov processes

Discrete state, discrete time

Unreliable machine

As $t \longrightarrow \infty$,

$$\pi(0) \longrightarrow \frac{p}{r+p},$$

$$\pi(1) \longrightarrow \frac{r}{r+p}$$

which is the solution of

$$\pi(0) = \pi(0)(1-r) + \pi(1)p,$$

$$\pi(1) = \pi(0)r + \pi(1)(1-p).$$

If the machine makes one part per time unit when it is operational, the average production rate is

$$\pi(1) = \frac{r}{r+p}$$

Markov processes

Discrete state, continuous time

States and transitions

- States can be numbered 0, 1, 2, 3, ... *(or with multiple indices if that is more convenient)*.
- Time is a real number, defined on $(-\infty, \infty)$ or a smaller interval.
- The probability of a transition from j to i during $[t, t + \delta t]$ is approximately $\lambda_{ij}\delta t$, where δt is small, and

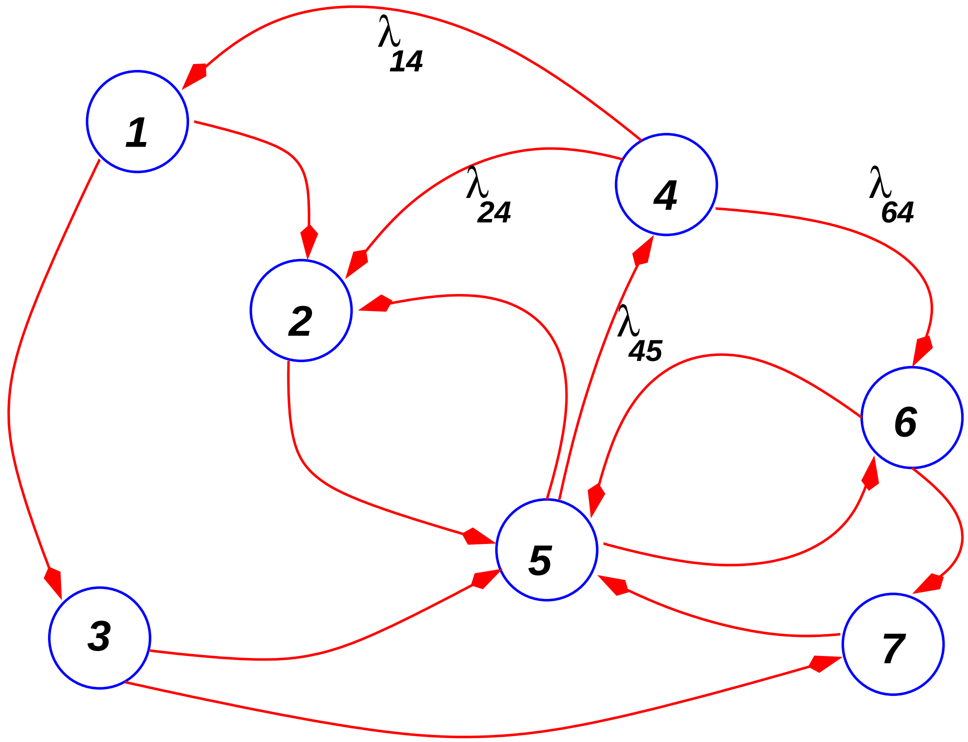
$$\lambda_{ij}\delta t \approx \text{prob}\{X(t+\delta t) = i | X(t) = j\} \text{ for } i \neq j$$

Markov processes

Discrete state, continuous time

States and transitions

Transition graph



$\boldsymbol{\lambda}_{ij}$ is a probability *rate.* $\boldsymbol{\lambda}_{ij}\delta t$ is a probability.

Markov processes

Discrete state, continuous time

States and transitions

Transition equation

Define $\pi_i(t) = \text{prob}\{X(t) = i\}$. Then for δt small,

$$\pi_5(t + \delta t) \approx$$

$$(1 - \lambda_{25}\delta t - \lambda_{45}\delta t - \lambda_{65}\delta t)\pi_5(t)$$

$$+\lambda_{52}\delta t\pi_2(t) + \lambda_{53}\delta t\pi_3(t) + \lambda_{56}\delta t\pi_6(t) + \lambda_{57}\delta t\pi_7(t)$$

Markov processes

Discrete state, continuous time

States and transitions

Or,

$$\pi_5(t+\delta t) \approx \pi_5(t)$$

$$-(\lambda_{25}+\lambda_{45}+\lambda_{65})\pi_5(t)\delta t$$

$$+(\lambda_{52}\pi_2(t)+\lambda_{53}\pi_3(t)+\lambda_{56}\pi_6(t)+\lambda_{57}\pi_7(t))\delta t$$

Markov processes

Discrete state, continuous time

States and transitions

Or,

$$\lim_{\delta t \to 0} \frac{\pi_5(t+\delta t) - \pi_5(t)}{\delta t} = \frac{d\pi_5}{dt}(t) =$$

$$-(\lambda_{25} + \lambda_{45} + \lambda_{65})\pi_5(t)$$

$$+\lambda_{52}\pi_2(t) + \lambda_{53}\pi_3(t) + \lambda_{56}\pi_6(t) + \lambda_{57}\pi_7(t)$$

Markov processes

Discrete state, continuous time

States and transitions

- Define $\pi_i(t) = \text{prob}\{X(t) = i\}$
- It is convenient to define $\lambda_{ii} = -\sum_{j \neq i} \lambda_{ji}$
- Transition equations: $\dfrac{d\pi_i(t)}{dt} = \sum_j \lambda_{ij} \pi_j(t)$.
- Normalization equation: $\sum_i \pi_i(t) = 1$.

Markov processes

Discrete state, continuous time

States and transitions

- *Steady state:* $\pi_i = \lim_{t\to\infty} \pi_i(t)$, if it exists.
- Steady-state transition equations: $0 = \sum_j \lambda_{ij}\pi_j$.
- *Alternatively,* steady-state balance equations:
  $\pi_i \sum_{m,m\neq i} \lambda_{mi} = \sum_{j,j\neq i} \lambda_{ij}\pi_j$
- Normalization equation: $\sum_i \pi_i = 1$.

Markov processes

Discrete state, continuous time

States and transitions

Sources of confusion in continuous time models:

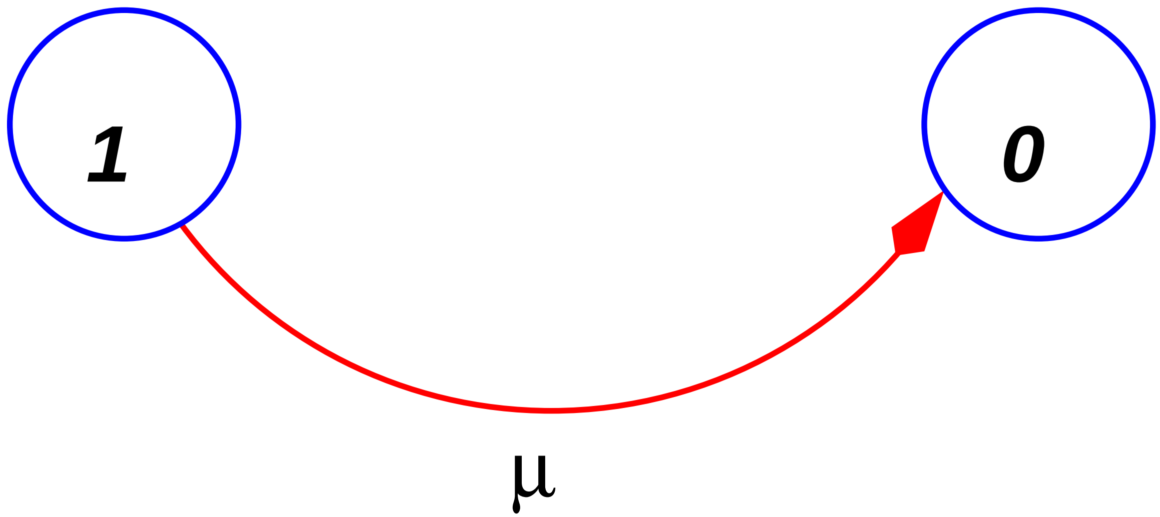
- *Never* Draw self-loops in continuous time markov process graphs.
- *Never* write $\mathbf{1-\lambda_{14}-\lambda_{24}-\lambda_{64}}$. Write
  - ⋆ $\mathbf{1-(\lambda_{14}+\lambda_{24}+\lambda_{64})\delta t}$, or
  - ⋆ $\mathbf{-(\lambda_{14}+\lambda_{24}+\lambda_{64})}$
- $\boldsymbol{\lambda_{ii} = -\sum_{j\neq i}\lambda_{ji}}$ is *NOT* a rate and *NOT* a probability. It is *ONLY* a convenient notation.

Markov processes

Discrete state, continuous time

Exponential

Exponential random variable T: the time to move from state 1 to state 0.



Markov processes

Discrete state, continuous time

Exponential

$$\pi(0, t+\delta t) =$$

prob $[\alpha(t+\delta t)=0|\alpha(t)=1]$ prob $[\alpha(t)=1]+$
prob $[\alpha(t+\delta t)=0|\alpha(t)=0]$ prob$[\alpha(t)=0]$.

or

$$\pi(0, t+\delta t) = p\delta t \pi(1,t) + \pi(0,t) + o(\delta t)$$

or

$$\frac{d\pi(0,t)}{dt} = p\pi(1,t).$$

Markov processes

Discrete state, continuous time

Exponential

Since $\pi(0,t) + \pi(1,t) = 1$,

$$\frac{d\pi(1,t)}{dt} = -p\pi(1,t).$$

If $\pi(1,0) = 1$, then

$$\pi(1,t) = e^{-pt}$$

and

$$\pi(0,t) = 1 - e^{-pt}$$

Markov processes

Discrete state, continuous time

Exponential

The probability that the transition takes place at some $T \in [t, t+\delta t]$ is

$$\text{prob}\,[\alpha(t+\delta t) = 0 \text{ and } \alpha(t) = 1] = e^{-pt} p \delta t.$$

The exponential density function is pe^{-pt}.

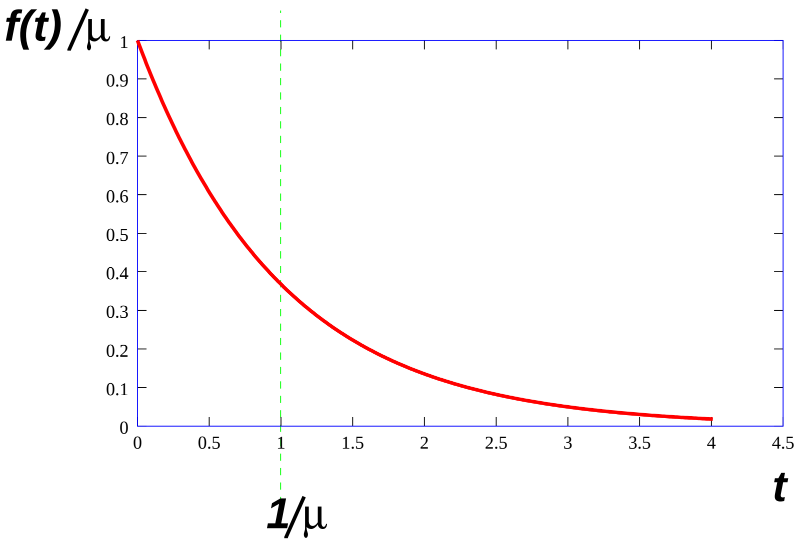
The time of the transition from 1 to 0 is said to be *exponentially distributed* with rate p. The expected transition time is $1/p$. *(Prove it!)*

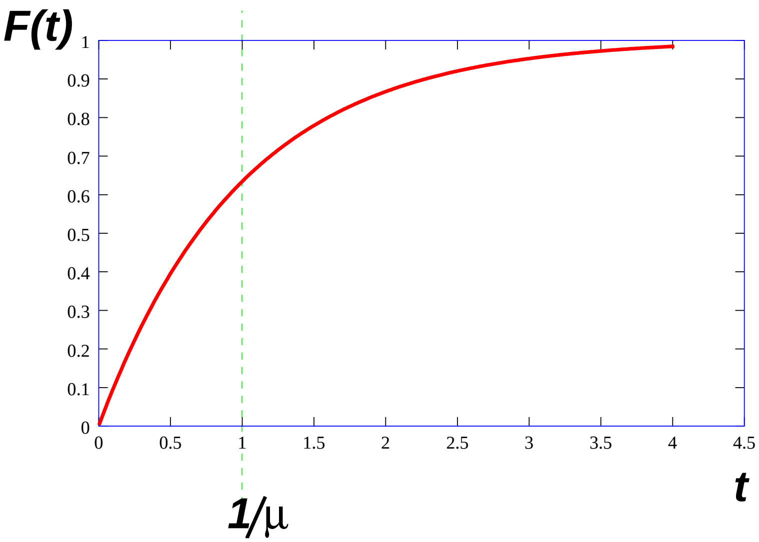
Markov processes

Discrete state, continuous time

Exponential

- $f(t) = \mu e^{-\mu t}$ for $t \geq 0$; $f(t) = 0$ otherwise; $F(t) = 1 - e^{-\mu t}$ for $t \geq 0$; $F(t) = 0$ otherwise.
- $ET = 1/\mu, V_T = 1/\mu^2$. Therefore, cv=1.





Markov processes

Discrete state, continuous time

Exponential

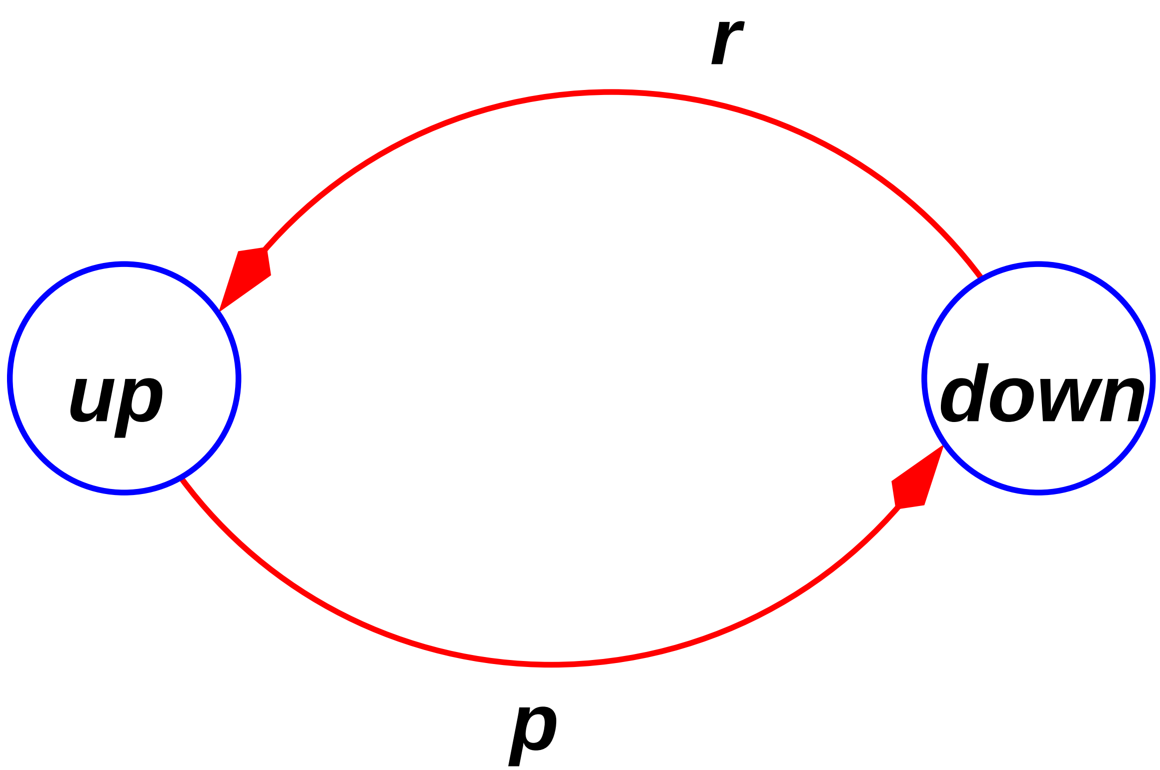
- Memorylessness:
  $P(T > t + x | T > x) = P(T > t)$
- $P(t \leq T \leq t + \delta t | T \geq t) \approx \mu \delta t$ for small δt.
- If $T_1, \ldots, T_n$ are independent exponentially distributed random variables with parameters $\mu_1 \ldots, \mu_n$ and $T = \min(T_1, \ldots, T_n)$, then T is an exponentially distributed random variable with parameter $\mu = \mu_1 + \ldots + \mu_n$.

Markov processes

Discrete state, continuous time

Unreliable machine

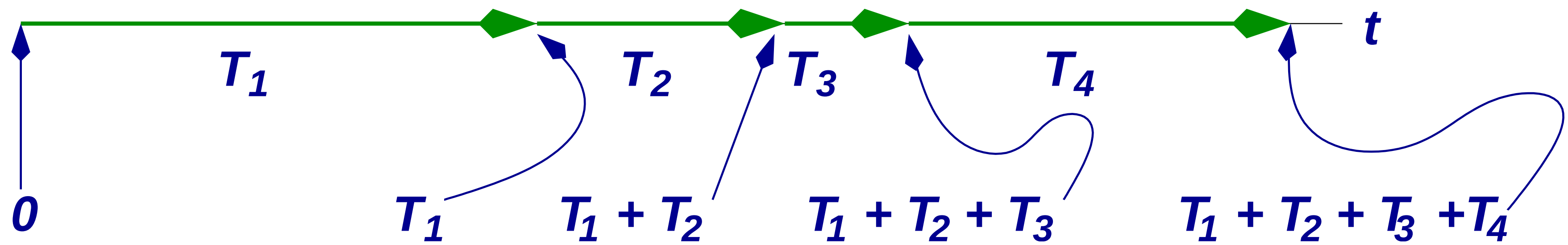
Continuous time unreliable machine. MTTF=$1/p$; MTTR=$1/r$.



Markov processes

Discrete state, continuous time

Poisson Process



Let $T_i, i = 1, \ldots$ be a set of independent exponentially distributed random variables with parameter λ that each represent the time until an event occurs. Then $\sum_{i=0}^{n} T_i$ is the time required for n such events.

Define $N(t) = \begin{cases} 0 \text{ if } T_1 > t \\ n \text{ such that } \sum_{i=0}^{n} T_i \leq t, \;\; \sum_{i=0}^{n+1} T_i > t \end{cases}$

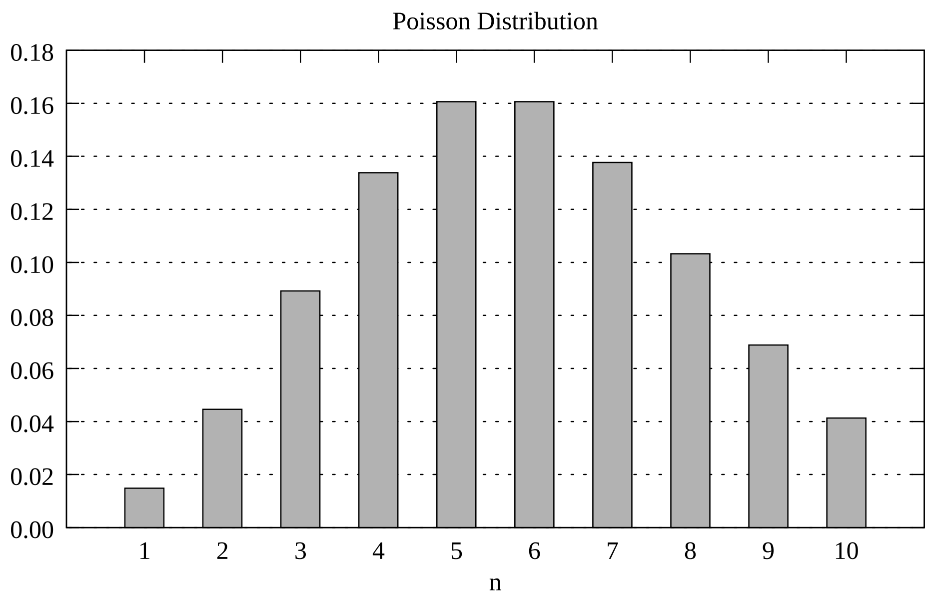
Then $N(t)$ is a *Poisson process* with parameter λ.

Markov processes

Discrete state, continuous time

Poisson Distribution

$$P(N(t) = n) = e^{-\lambda t}\frac{(\lambda t)^n}{n!}$$



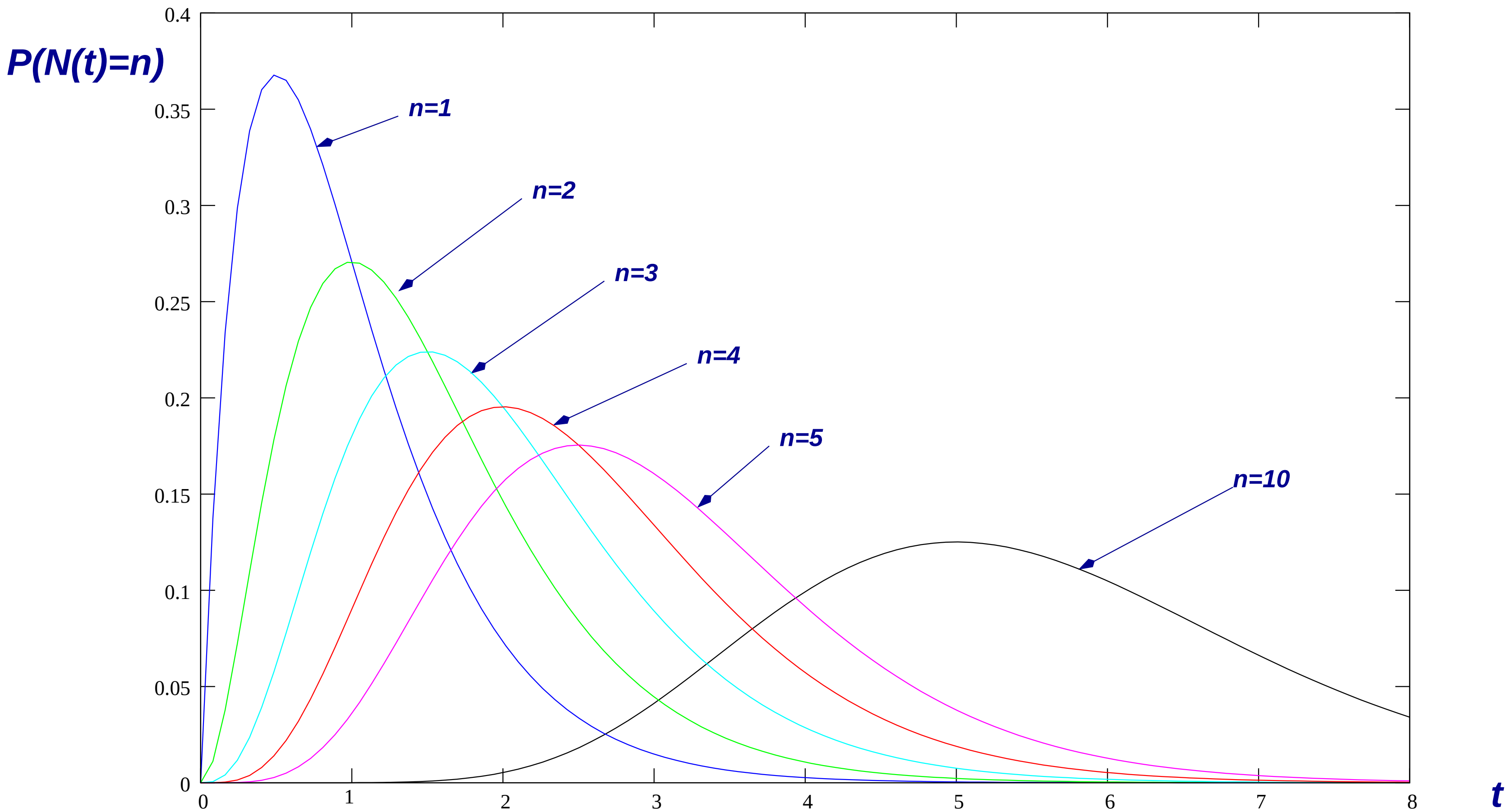
$$\lambda t = 6$$

Markov processes

Discrete state, continuous time

Poisson Distribution

$$P(N(t) = n) = e^{-\lambda t}\frac{(\lambda t)^n}{n!}, \qquad \lambda = 2$$



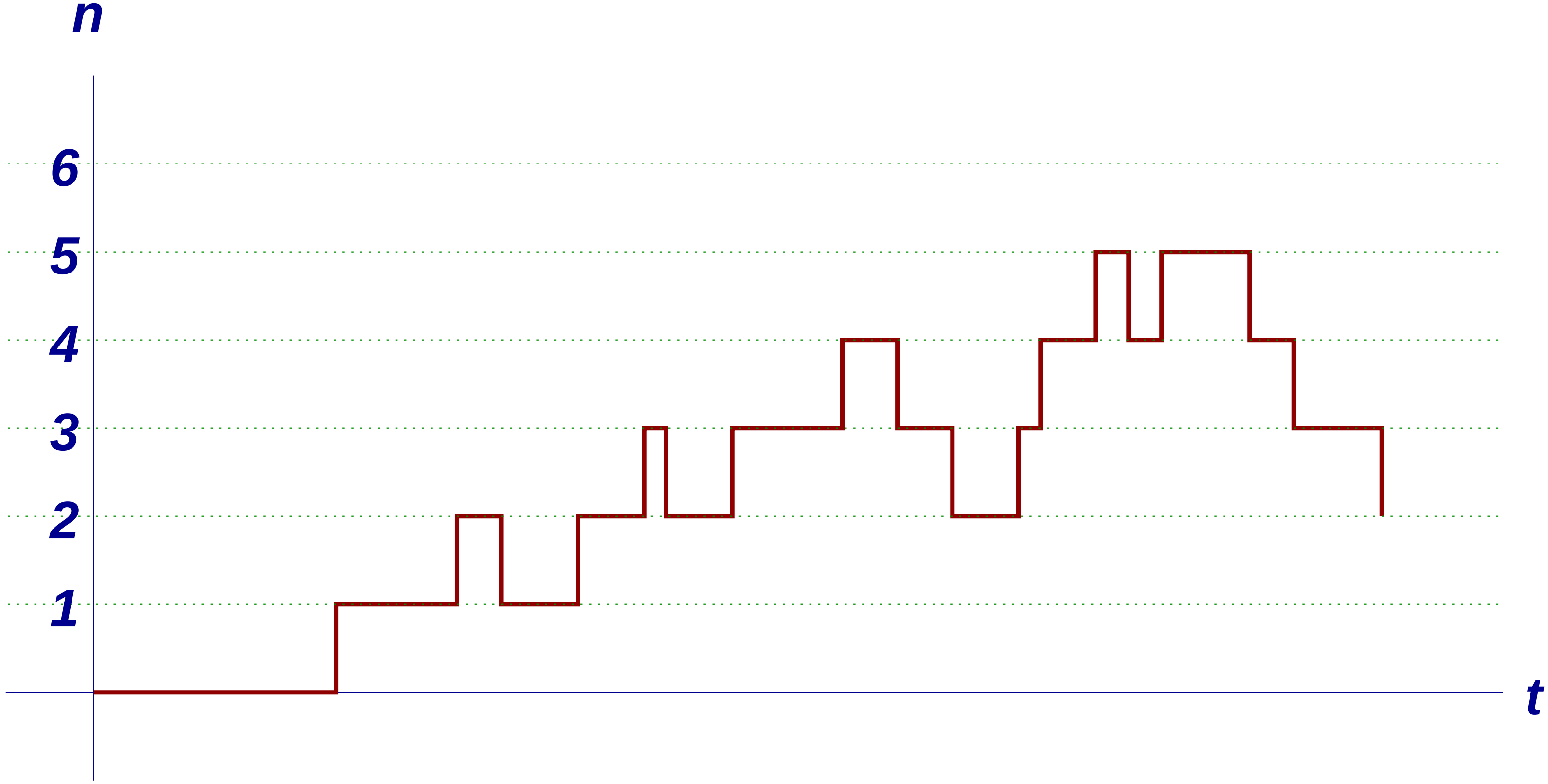
$M/M/1$ Queue

- Simplest model is the $\boldsymbol{M/M/1}$ queue:
  - ⋆ Exponentially distributed inter-arrival times — mean is $\mathbf{1}/\boldsymbol{\lambda}$; $\boldsymbol{\lambda}$ is *arrival rate* (customers/time). *(Poisson arrival process.)*
  - ⋆ Exponentially distributed service times — mean is $\mathbf{1}/\boldsymbol{\mu}$; $\boldsymbol{\mu}$ is *service rate* (customers/time).
  - ⋆ 1 server.
  - ⋆ Infinite waiting area.
- Define the *utilization* $\boldsymbol{\rho = \lambda/\mu}$.

$M/M/1$ Queue

Sample path

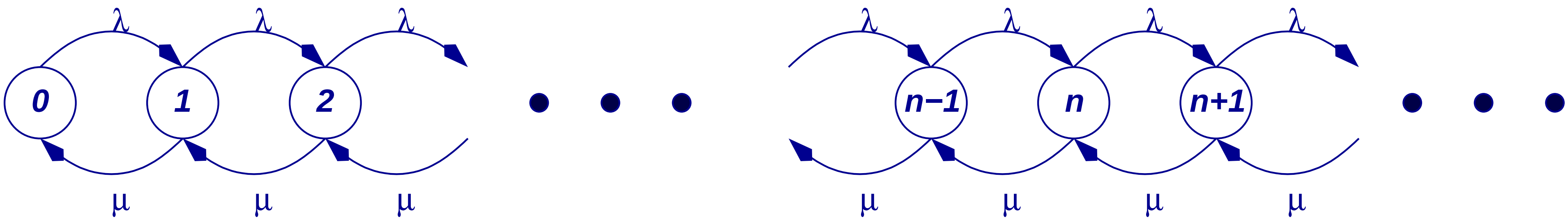
Number of customers in the system as a function of time.



Queueing theory

$M/M/1$ Queue

State Space



Queueing theory

$M/M/1$ Queue

Performance of $M/M/1$ queue

Let $P(n,t)$ be the probability that there are n parts in the system at time t. Then,

$$P(n,t+\delta t) = P(n-1,t)\lambda\delta t + P(n+1,t)\mu\delta t$$
$$+P(n,t)(1-(\lambda\delta t+\mu\delta t)) + o(\delta t)$$

for $n > 0$

and

$$P(0,t+\delta t) = P(1,t)\mu\delta t + P(0,t)(1-\lambda\delta t) + o(\delta t).$$

Queueing theory

$M/M/1$ Queue

Performance of $M/M/1$ queue

Or,

$$\frac{dP(n,t)}{dt} = P(n-1,t)\lambda + P(n+1,t)\mu - P(n,t)(\lambda+\mu),$$
$$n > 0$$
$$\frac{dP(0,t)}{dt} = P(1,t)\mu - P(0,t)\lambda.$$

If a steady state distribution exists, it satisfies

$$0 = P(n-1)\lambda + P(n+1)\mu - P(n)(\lambda+\mu), n > 0$$
$$0 = P(1)\mu - P(0)\lambda.$$

Why "if"?

Performance of $M/M/1$ queue

Let $\rho = \lambda/\mu$. These equations are satisfied by

$$P(n) = (1-\rho)\rho^n, n \geq 0$$

if $\rho < 1$. The average number of parts in the system is

$$\bar{n} = \sum_n nP(n) = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu - \lambda}.$$

Queueing theory

$M/M/1$ Queue

Little's Law

- True for most systems of practical interest.
- Steady state only.
- L = the average number of customers in a system.
- W = the average delay experienced by a customer in the system.

$$L = \lambda W$$

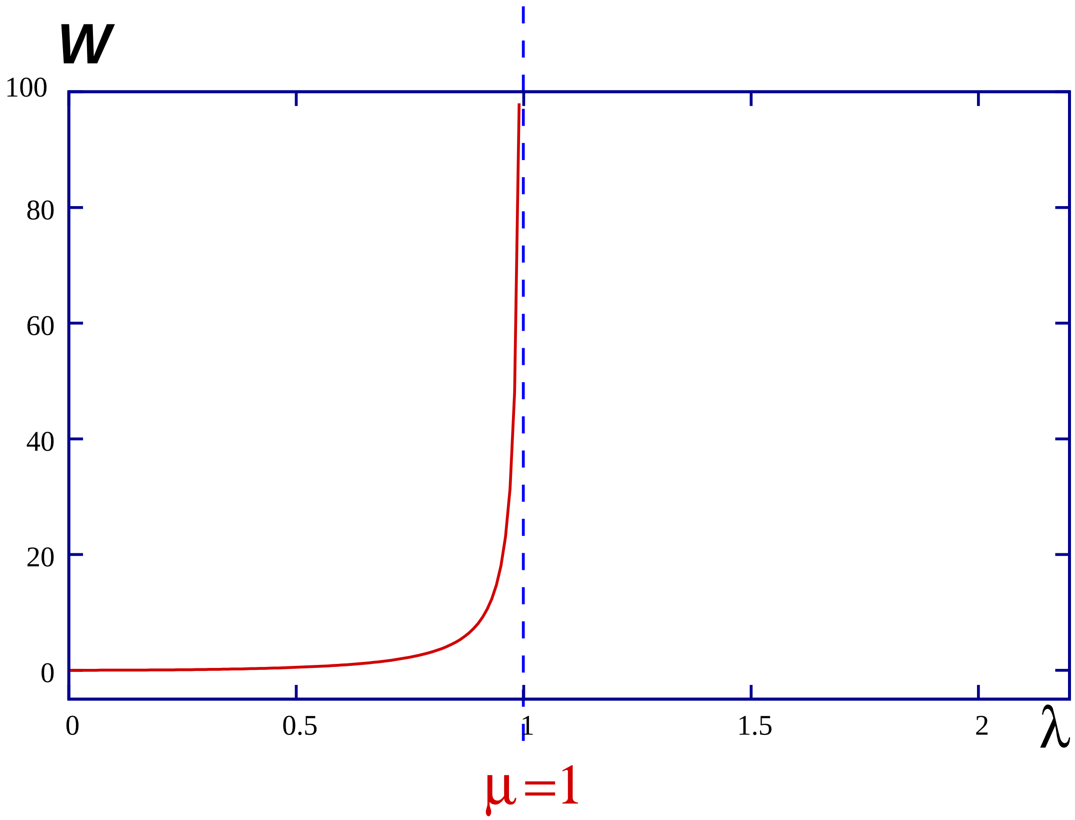
In the $M/M/1$ queue, $L = \bar{n}$ and

$$W = \frac{1}{\mu - \lambda}.$$

Queueing theory

$M/M/1$ Queue

Capacity

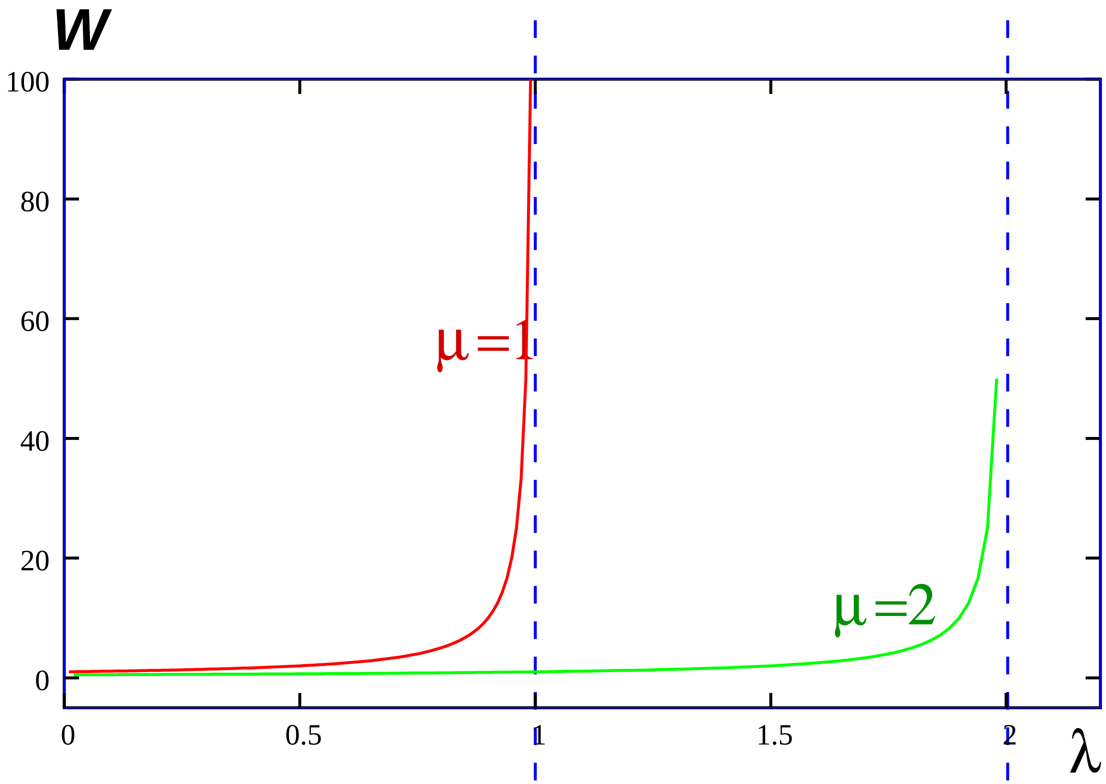


- μ is the *capacity* of the system.
- If $\lambda < \mu$, system is stable and waiting time remains bounded.
- If $\lambda > \mu$, waiting time grows over time.

Queueing theory

$M/M/1$ Queue

Capacity



- To increase capacity, increase μ.
- To decrease delay for a given λ, increase μ.

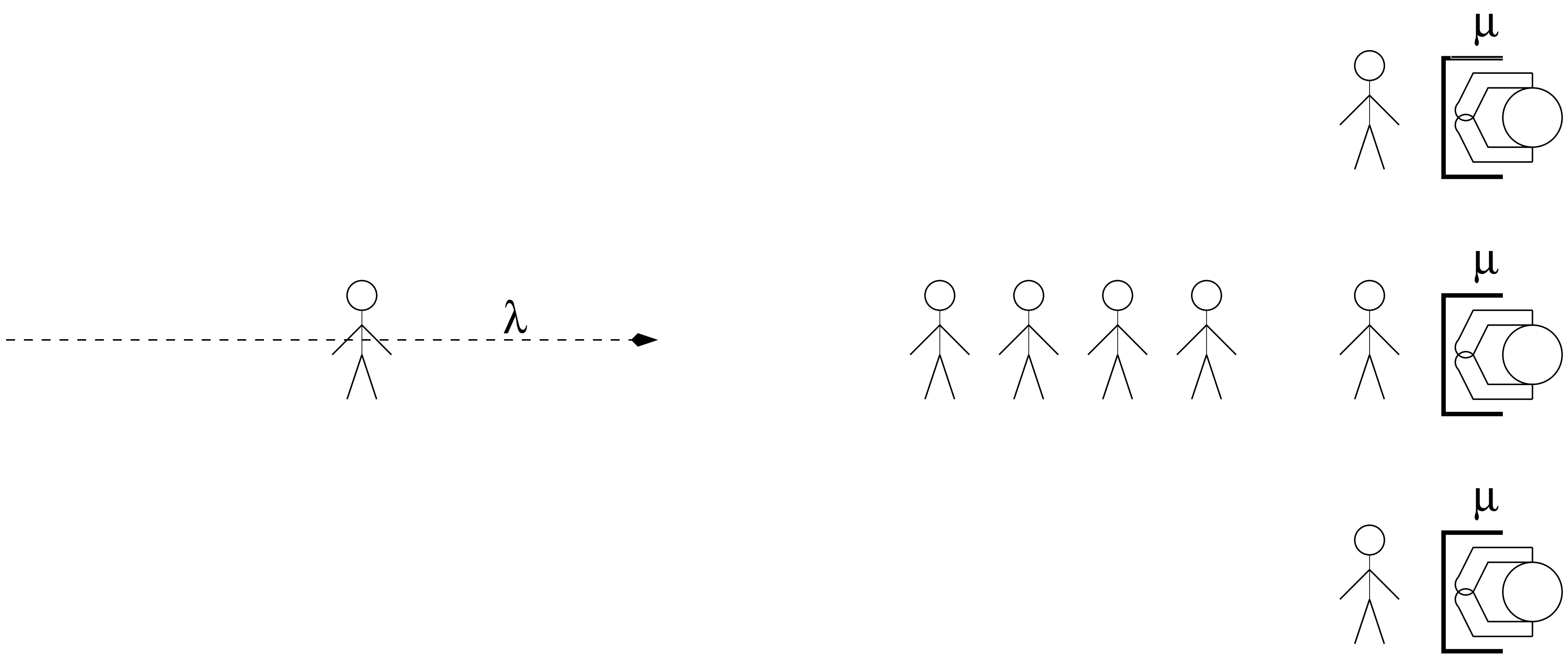
Things get more complicated when:

- There are multiple servers.
- There is finite space for queueing.
- The arrival process is not Poisson.
- The service process is not exponential.

Closed formulas and approximations exist for some cases.

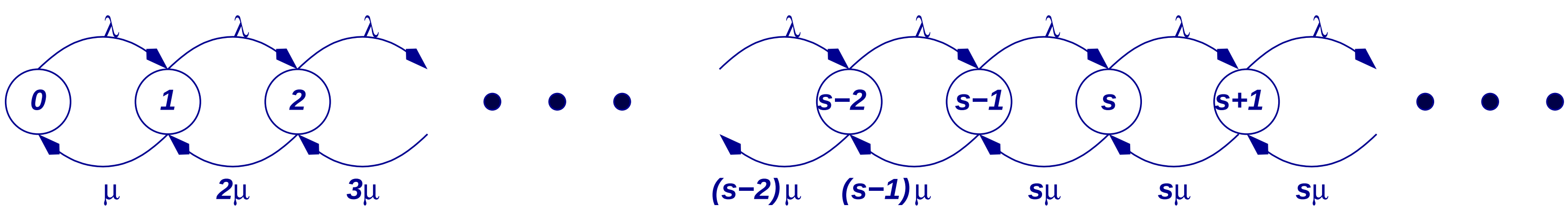
Queueing theory

$M/M/s$ Queue



s-Server Queue, $s = 3$

- The service rate when there are $k > s$ customers in the system is $s\mu$ since all s servers are always busy.
- The service rate when there are $k \leq s$ customers in the system is $k\mu$ since only k of the servers are busy.



Queueing theory

$M/M/s$ Queue

Steady-State Probability Distribution

$$P(k) = \begin{cases} P(0)\dfrac{s^k \rho^k}{k!}, & k \le s \\ P(0)\dfrac{s^s \rho^k}{s!}, & k > s \end{cases}$$

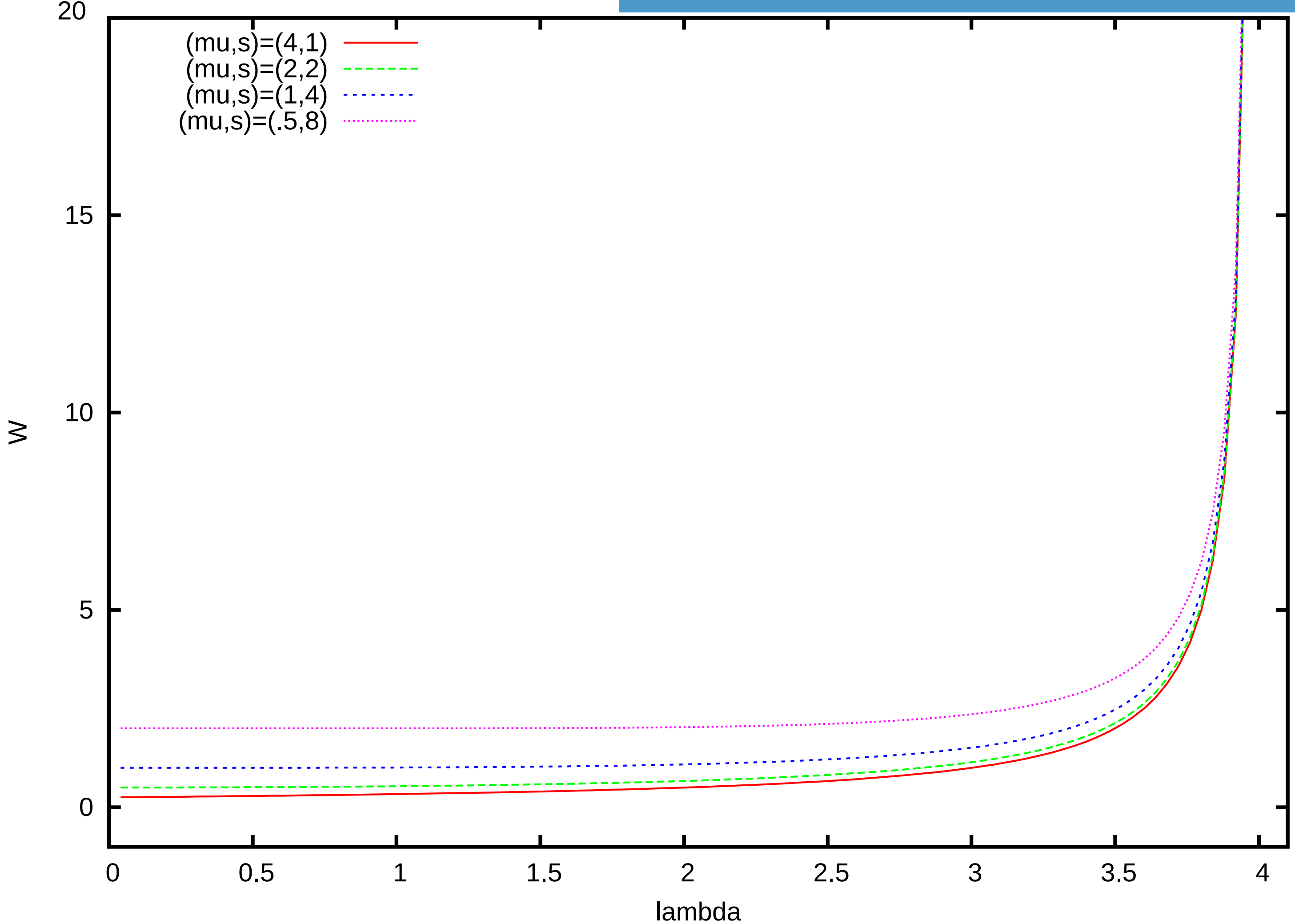
where

$$\rho = \frac{\lambda}{s\mu} < 1; \qquad P(0) \text{ chosen so that } \sum_k P(k) = 1$$

Queueing theory

$M/M/s$ Queue

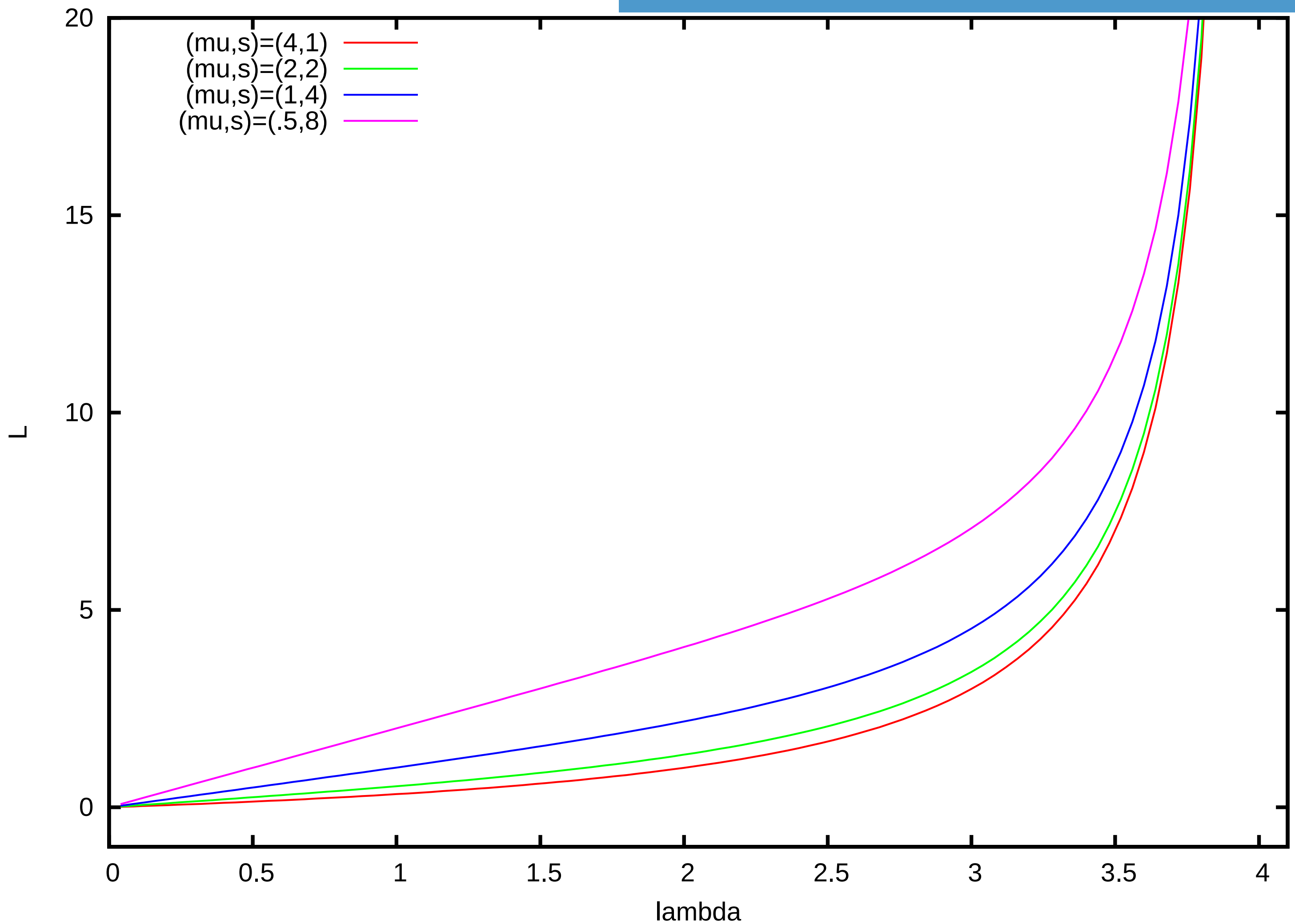
Performance



Queueing theory

$M/M/s$ Queue

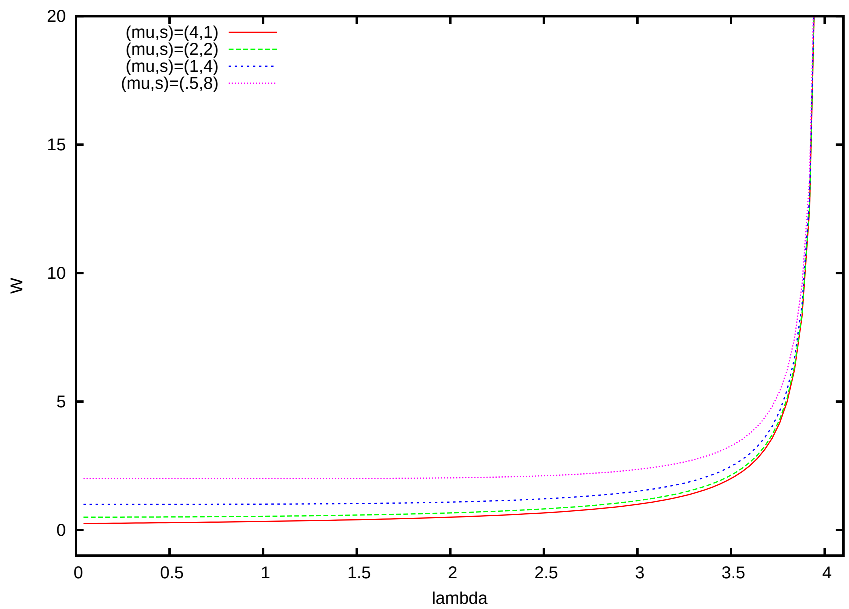
Performance

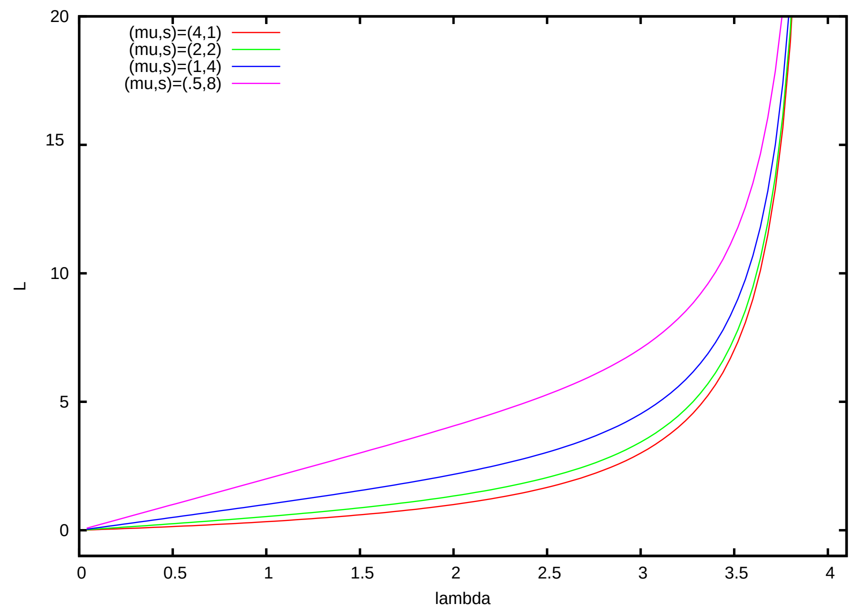


Queueing theory

$M/M/s$ Queue

Performance





- Why do the curves go to infinity at the same value of λ?
- Why is the $(\mu, s) = (.5, 8)$ curve the highest, followed by $(\mu, s) = (1, 4)$, etc.?

Networks of Queues

- Set of queues where customers can go to another queue after completing service at a queue.
- *Open network:* where customers enter and leave the system. λ is known and we must find L and W.
- *Closed network:* where the population of the system is constant. L is known and we must find λ and W.

Networks of Queues

Examples

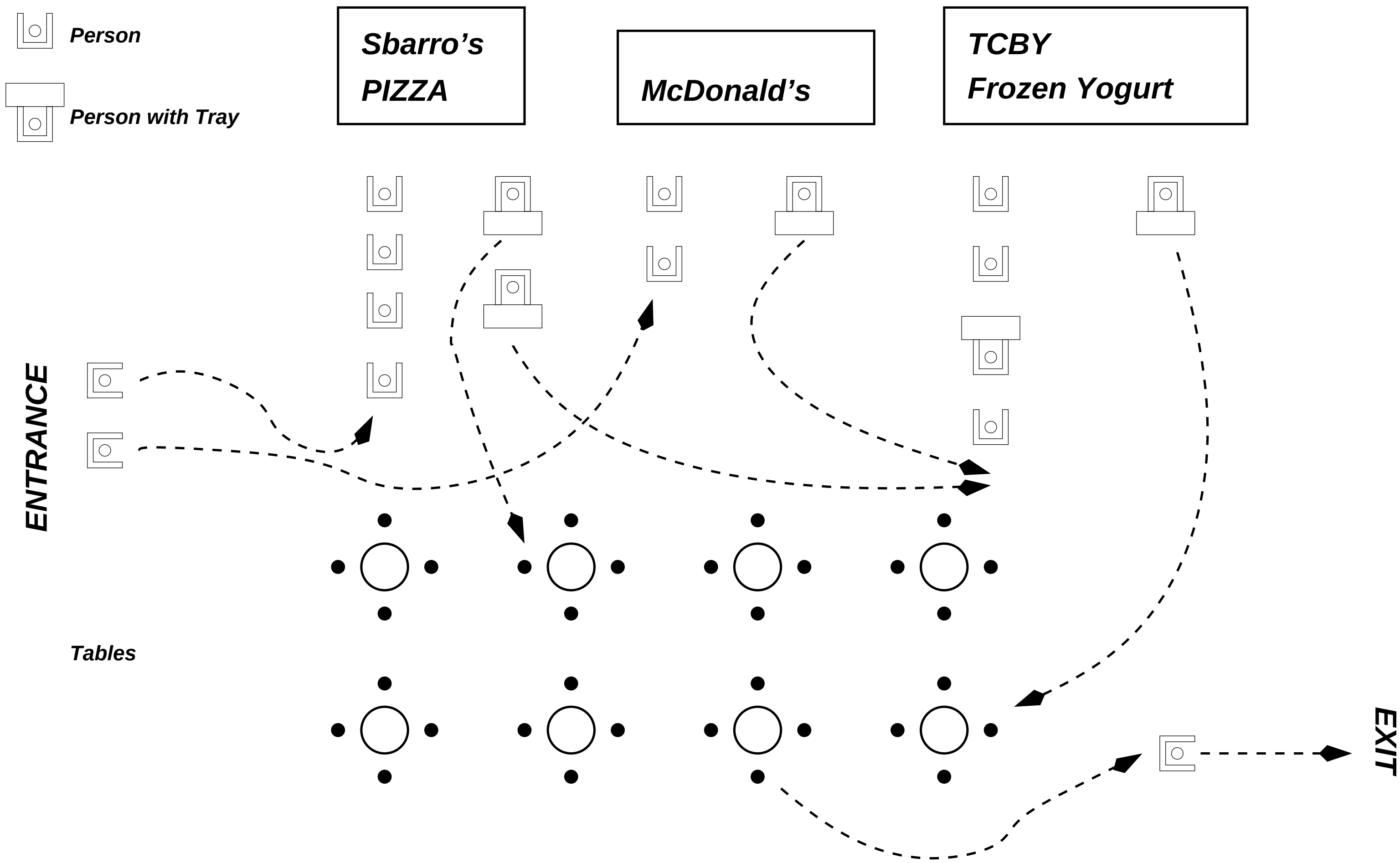
Open networks

- internet traffic
- emergency room
- food court
- airport (*arrive*, ticket counter, security, passport control, gate, *board plane*)
- factory with serial production system and no material control after it enters

Networks of Queues

Examples

Food Court



Closed Networks

- factory with material controlled by keeping the number of items constant (CONWIP)
- factory with limited fixtures or pallets

Jackson Networks

Benefits

Queueing networks are often modeled as *Jackson networks.*

- Easy to compute performance measures (capacity, average time in system, average queue lengths).
- Easily gives intuition.
- Easy to optimize and to use for design.
- Valid (or good approximation) for a large class of systems ...

Jackson Networks

Limitations

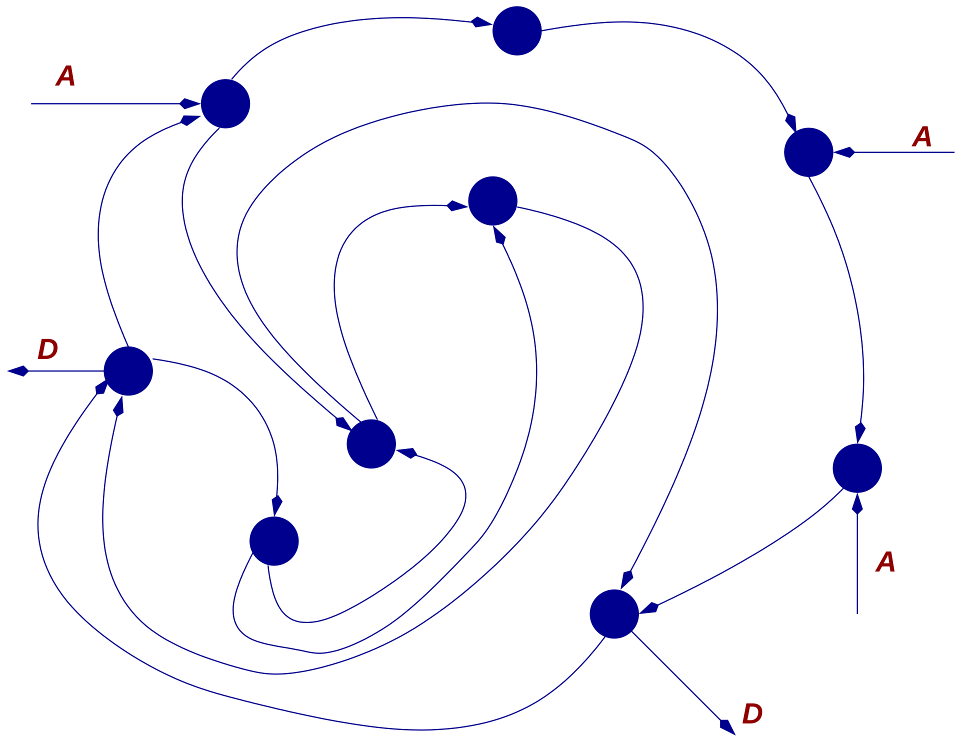
- ... but not everything. Storage areas must be infinite (i.e., blocking never occurs).
  - ★ This assumption *fails* for systems with bottlenecks.

- In Jackson networks, there is only one class. That is, all items are interchangeable. However, this restriction can be relaxed.

Jackson Networks

Open Jackson Networks

Assumptions



Goal of analysis: say something about how much inventory there is in this system and how it is distributed.

Assumptions

- Items *arrive* from outside the system to node i according to a Poisson process with rate α_i.
- $\alpha_i > 0$ for at least one i.
- When an item's service at node i is finished, it goes to node j next with probability p_{ij}.
- If $p_{i0} = 1 - \sum_j p_{ij} > 0$, then items *depart* from the network from node i.
- $p_{i0} > 0$ for at least one i.
- We will focus on the special case in which each node has a single server with exponential processing time. The service rate of node i is μ_i.

Jackson Networks

Open Jackson Networks

- Define $\boldsymbol{\lambda_i}$ as the total arrival rate of items to node $\boldsymbol{i}$. This includes items entering the network at $\boldsymbol{i}$ and items coming from all other nodes.
- Then $\boldsymbol{\lambda_i = \alpha_i + \sum_j p_{ji}\lambda_j}$
- In matrix form, let $\boldsymbol{\lambda}$ be the vector of $\boldsymbol{\lambda_i}$, $\boldsymbol{\alpha}$ be the vector of $\boldsymbol{\alpha_i}$, and P be the matrix of $\boldsymbol{p_{ij}}$. Then

$$\boldsymbol{\lambda = \alpha + \mathsf{P}^T \lambda}$$

or

$$\boldsymbol{\lambda = (I - \mathsf{P}^T)^{-1} \alpha}$$

Jackson Networks

Open Jackson Networks

Product Form Solution

- Define $\pi(n_1, n_2, \ldots, n_k)$ to be the steady-state probability that there are n_i items at node i, $i = 1, \ldots, k$.
- Define $\rho_i = \lambda_i / \mu_i$; $\quad \pi_i(n_i) = (1-\rho_i)\rho_i^{n_i}$.
- Then

$$\pi(n_1, n_2, \ldots, n_k) = \prod_i \pi_i(n_i)$$

$$\bar{n}_i = E n_i = \frac{\rho_i}{1-\rho_i}$$

Does this look familiar?

Product Form Solution

- This looks as though each station is an $M/M/1$ queue. But even though this is *NOT* in general true, the formula holds.

- The product form solution holds for some more general cases.

- This exact analytic formula is the reason that the Jackson network model is very widely used — *sometimes where it does not belong!*

Closed Jackson Networks

- Consider an extension in which
  - ⋆ $\alpha_i = 0$ for all nodes i.
  - ⋆ $p_{i0} = 1 - \sum_j p_{ij} = 0$ for all nodes i.
- Then
  - ⋆ Since nothing is entering and nothing is departing from the network, the number of items in the network is *constant* .

    That is, $\sum_i n_i(t) = N$ for all t.
  - ⋆ $\lambda_i = \sum_j p_{ji}\lambda_j$ does not have a unique solution:

    If $\{\lambda_1^*, \lambda_2^*, ..., \lambda_k^*\}$ is a solution, then $\{s\lambda_1^*, s\lambda_2^*, ..., s\lambda_k^*\}$ is also a solution for any $s \geq 0$.

Closed Jackson Networks

For some s, define

$$\pi^o(n_1, n_2, \ldots, n_k) = \prod_i \left[(1-\rho_i)\rho_i^{n_i}\right] = \left[\prod_i (1-\rho_i)\right]\left[\prod_i \rho_i^{n_i}\right]$$

where

$$\rho_i = \frac{s\lambda_i^*}{\mu_i}$$

This looks like the open network probability distribution, but it is a function of s.

Closed Jackson Networks

Consider a closed network with a population of N. Then if $\sum_i n_i = N$,

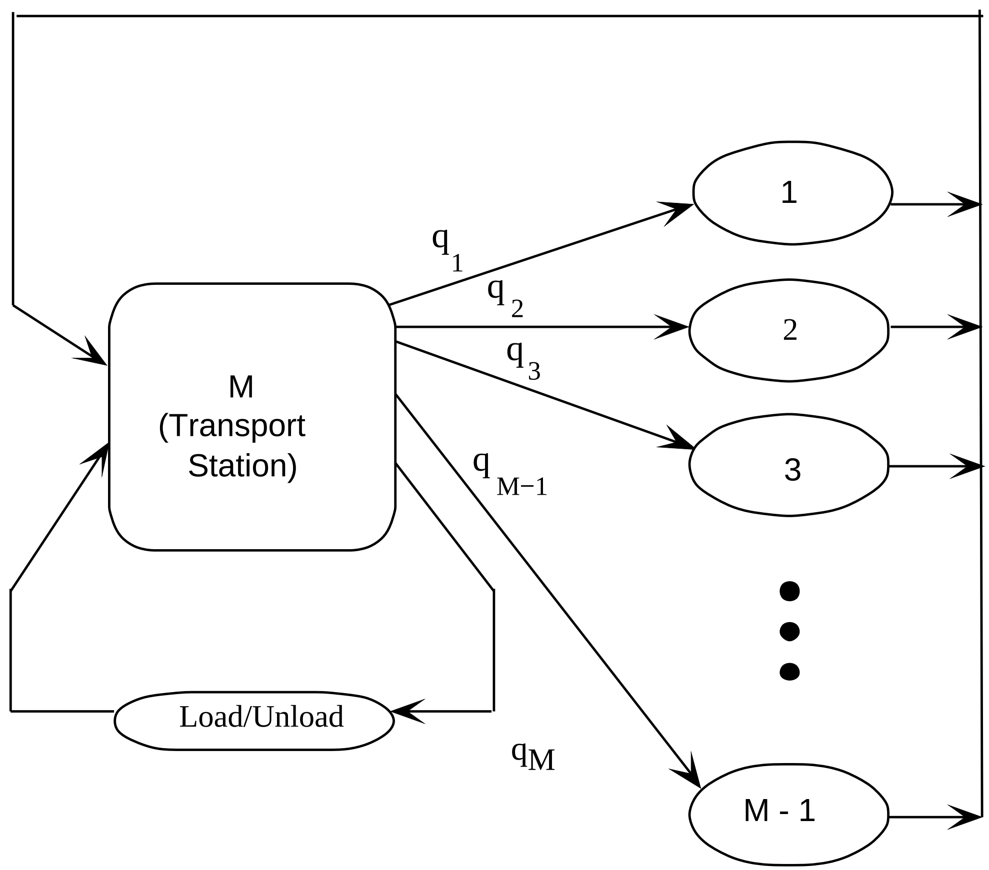
$$\pi(n_1, n_2, ..., n_k) = \frac{\pi^o(n_1, n_2, ..., n_k)}{\sum_{m_1+m_2+...+m_k=N} \pi^o(m_1, m_2, ..., m_k)}$$

Since π^o is a function of s, it looks like π is a function of s. *But it is not because all the s's cancel!* There are nice ways of calculating

$$C(k, N) = \sum_{m_1+m_2+...+m_k=N} \pi^o(m_1, m_2, ..., m_k)$$

Closed Jackson Networks

Application — Simple FMS model



Solberg's "CANQ" model.

Let $\{p_{ij}\}$ be the set of routing probabilities, as defined on slide 67.

$p_{iM} = 1$ if $i \neq M$

$p_{Mj} = q_j$ if $j \neq M$

$p_{ij} = 0$ otherwise

Service rate at Station i is μ_i.

Application — Simple FMS model

Let N be the number of pallets.

The production rate is

$$P = \frac{C(M, N-1)}{C(M, N)} \mu_m$$

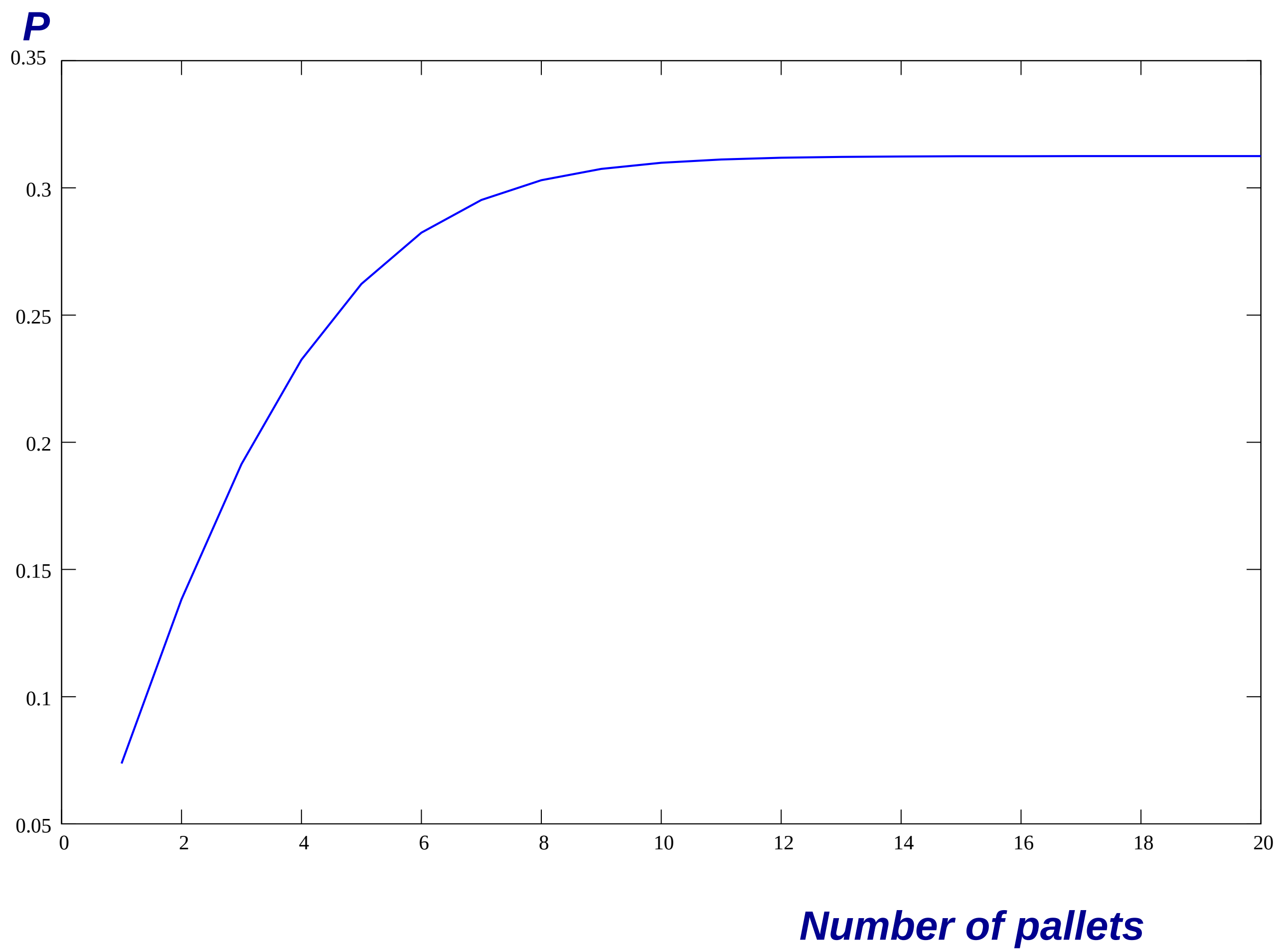
and $C(M, N)$ is easy to calculate in this case.

- Input data: $M, N, q_j, \mu_j (j = 1, ..., M)$
- Output data: $P, W, \rho_j (j = 1, ..., M)$

Jackson Networks

Closed Jackson Networks

Application — Simple FMS model

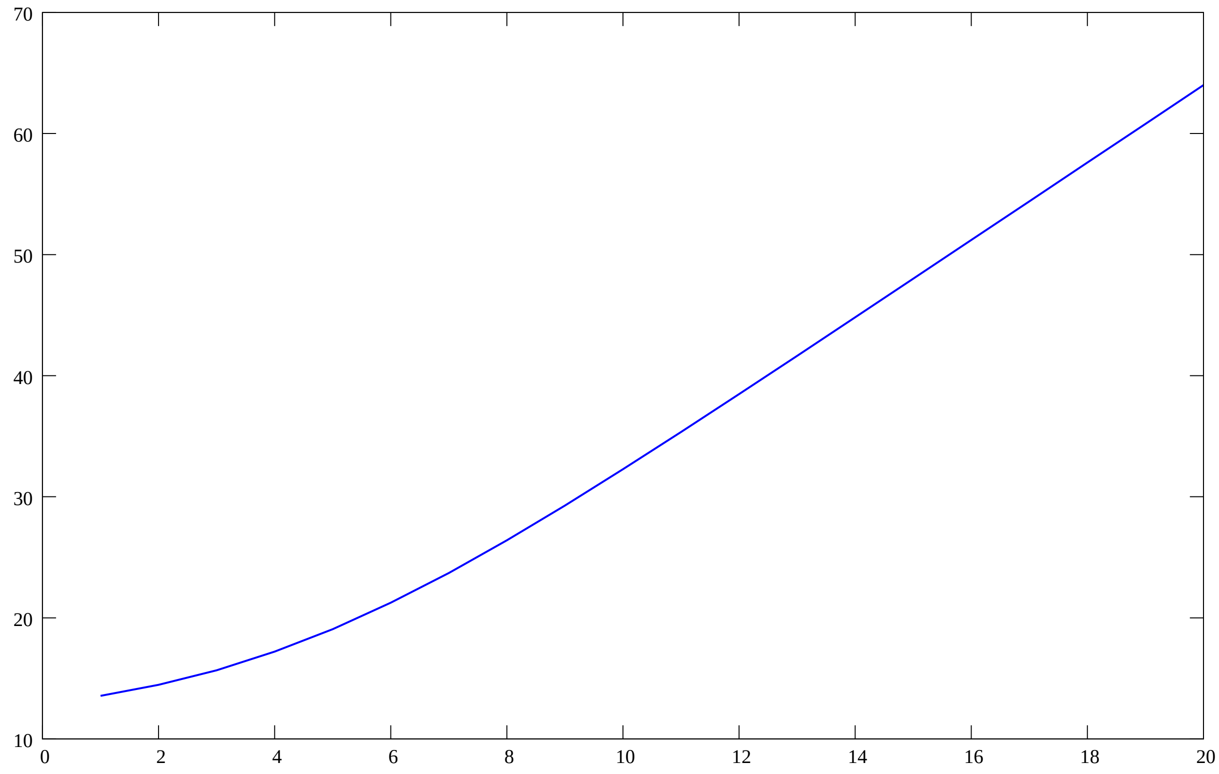


Jackson Networks

Closed Jackson Networks

Application — Simple FMS model

Average time in system

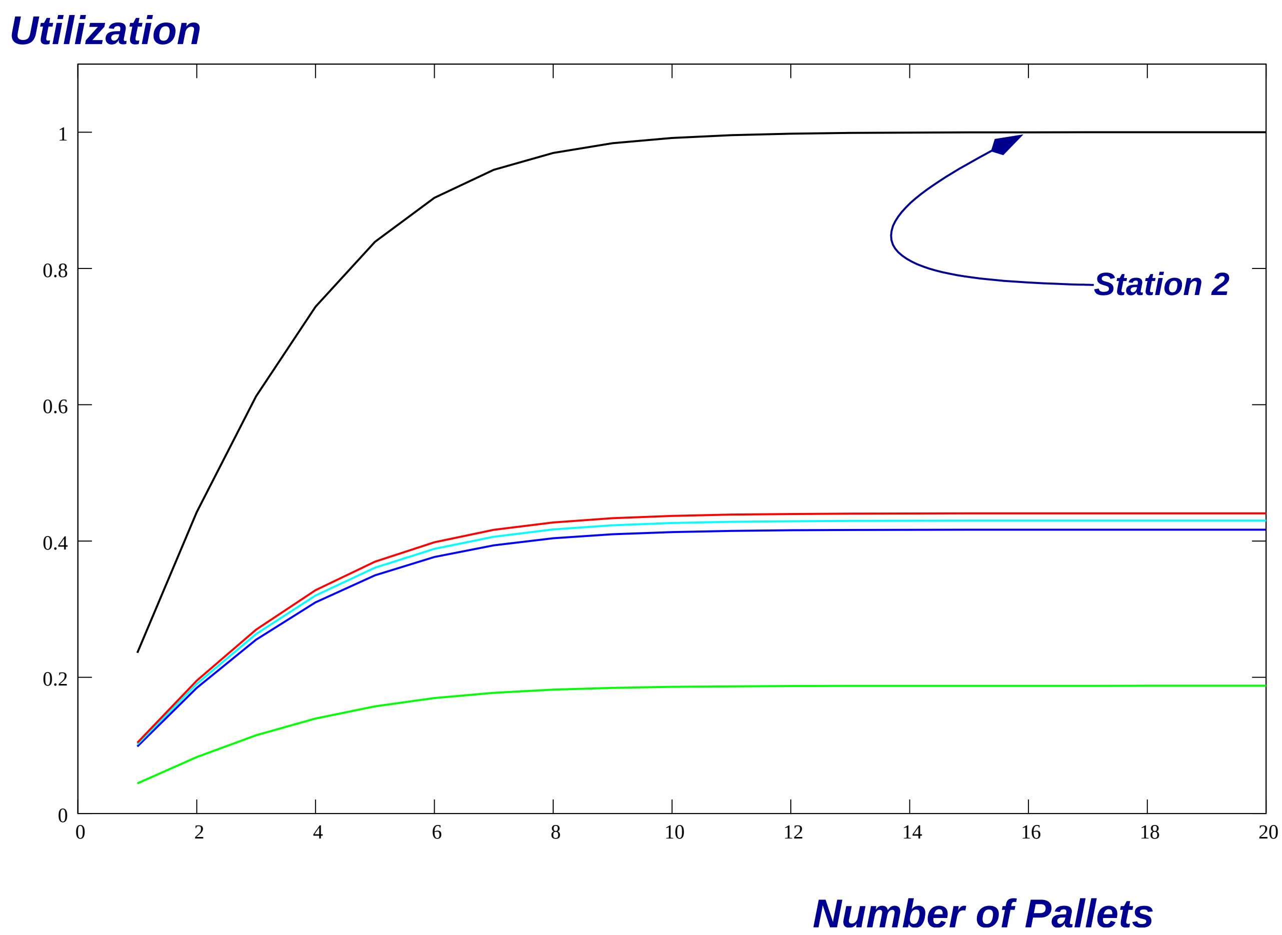


Number of Pallets

Jackson Networks

Closed Jackson Networks

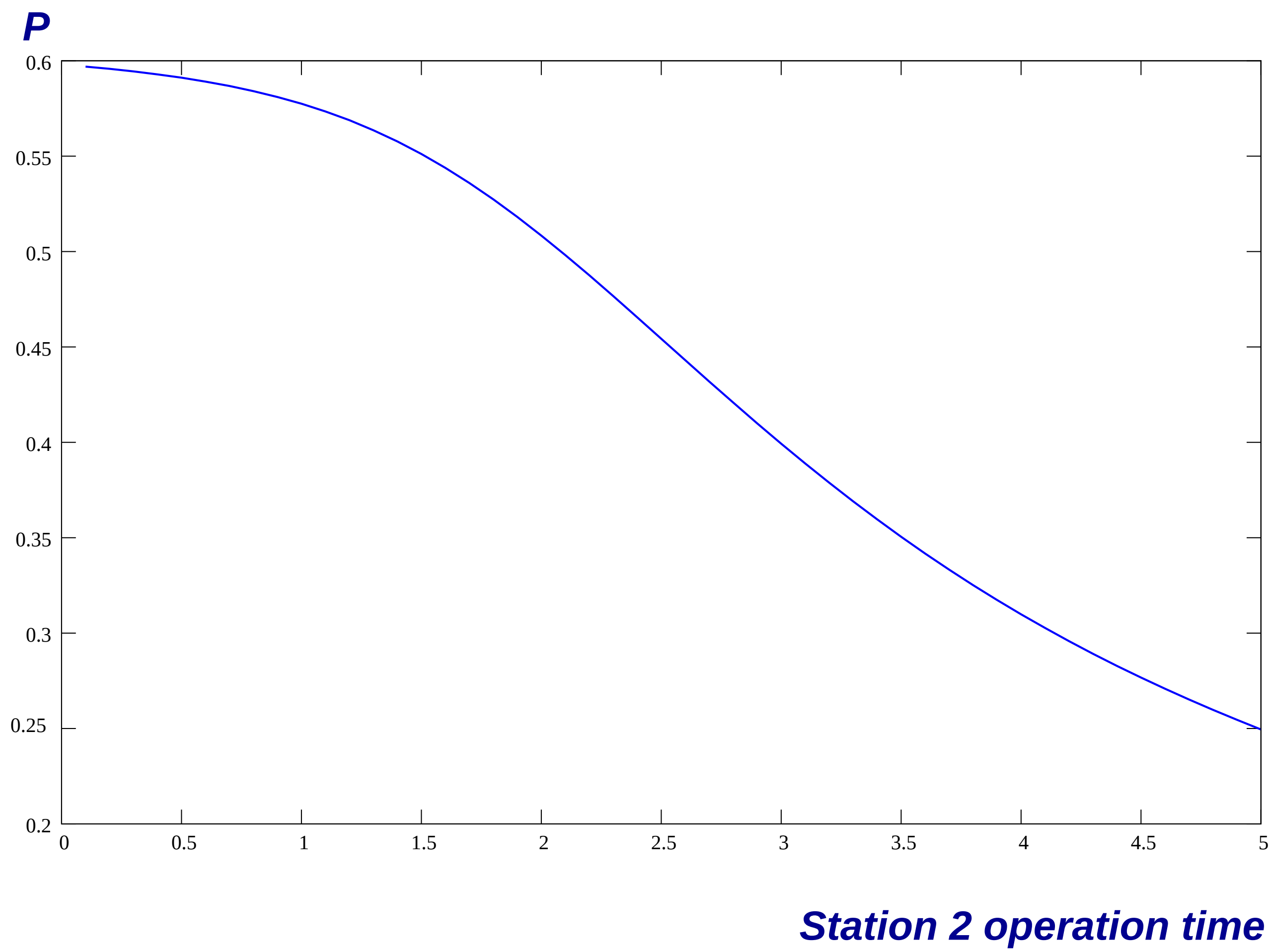
Application — Simple FMS model



Jackson Networks

Closed Jackson Networks

Application — Simple FMS model

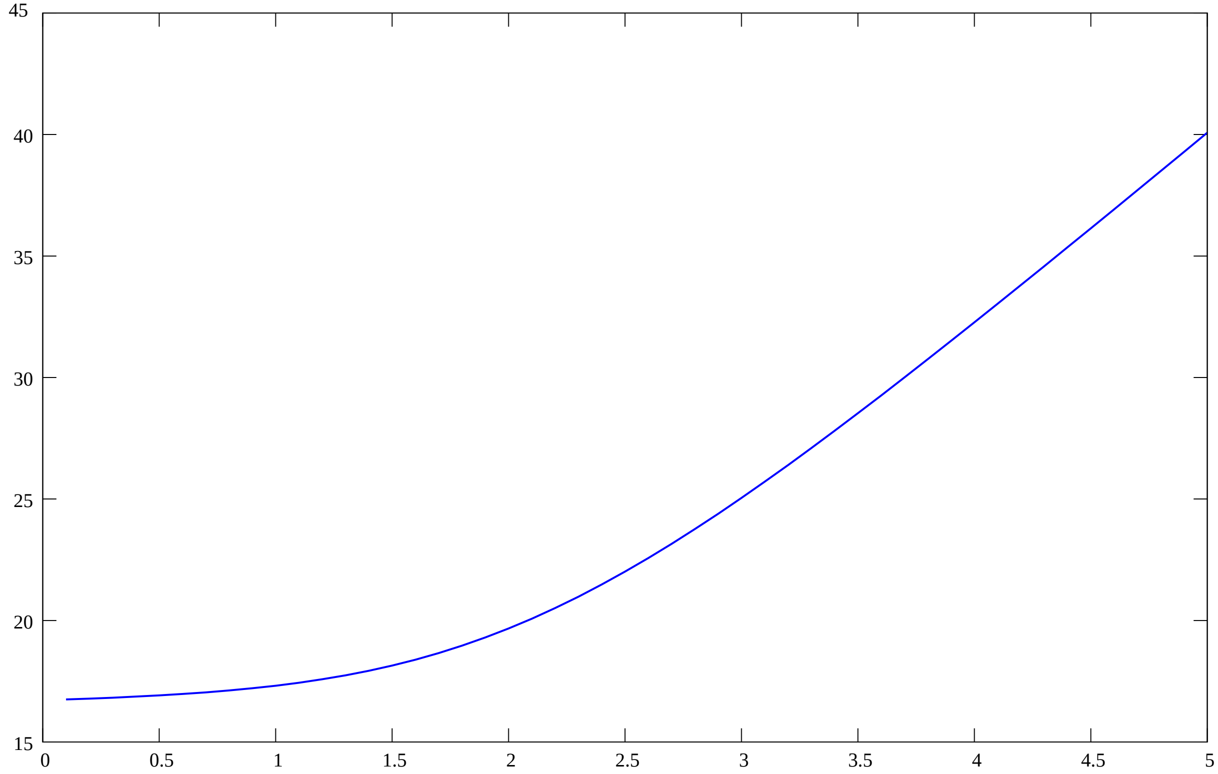


Jackson Networks

Closed Jackson Networks

Application — Simple FMS model

Average time in system

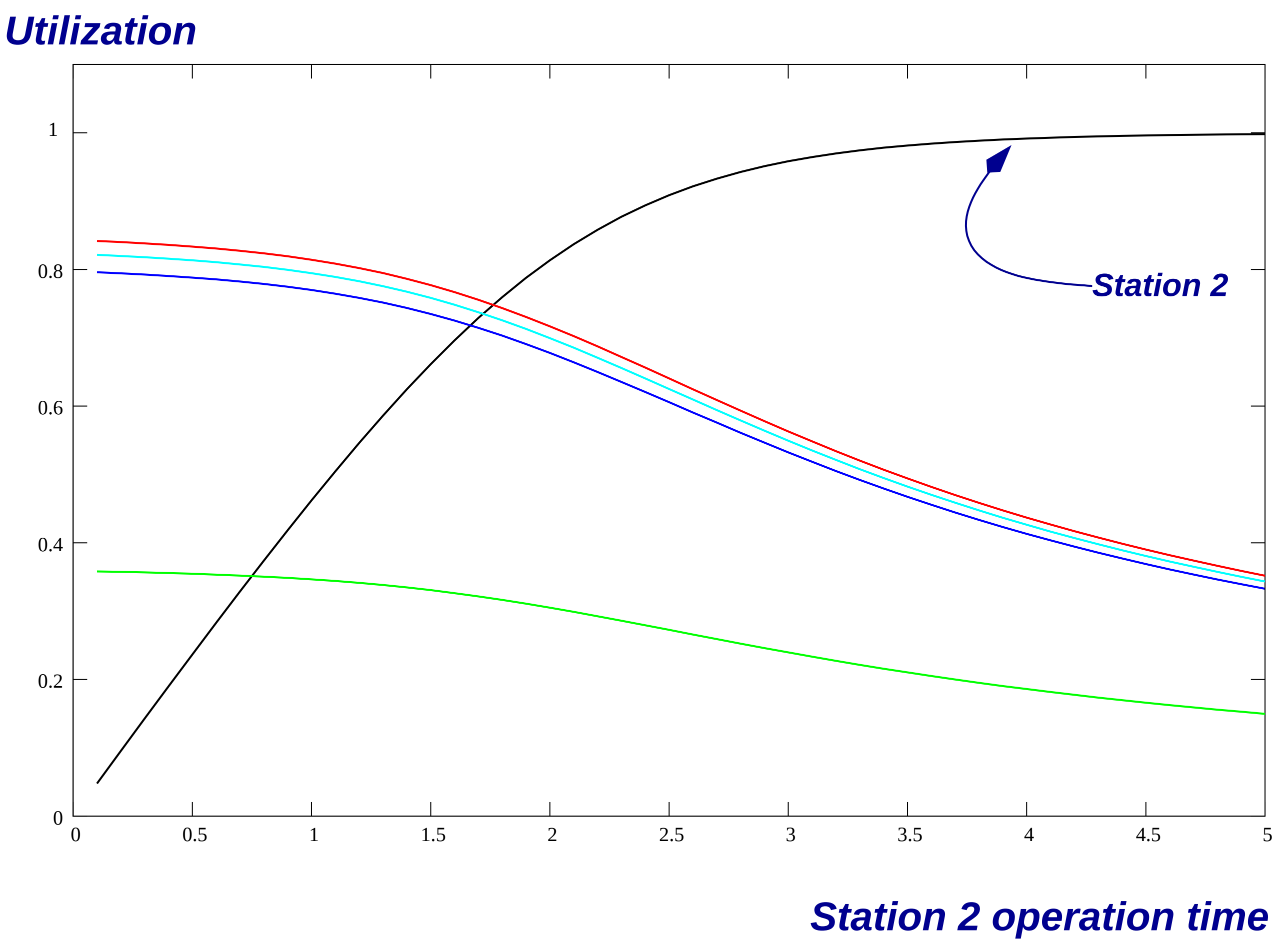


Station 2 operation time

Jackson Networks

Closed Jackson Networks

Application — Simple FMS model



2.854 / 2.853 Introduction to Manufacturing Systems
Fall 2010