MAS.160 / MAS.510 / MAS.511 Signals, Systems and Information for Media Technology
Fall 2007

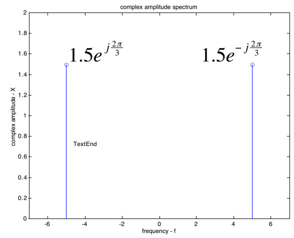
What do you do with negative amplitudes?

Amplitudes always positive

$$-C\cos(2\pi ft) = C\cos(2\pi ft \pm \pi)$$

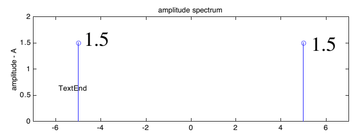
Absorb the minus sign into the angle

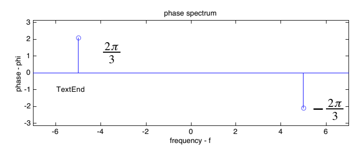
$$-3\cos(2\pi \cdot 5t + \tfrac{\pi}{3}) = 3\cos(2\pi \cdot 5t + \tfrac{\pi}{3} - \pi)$$
$$= 3\cos(2\pi \cdot 5t - \tfrac{2\pi}{3})$$



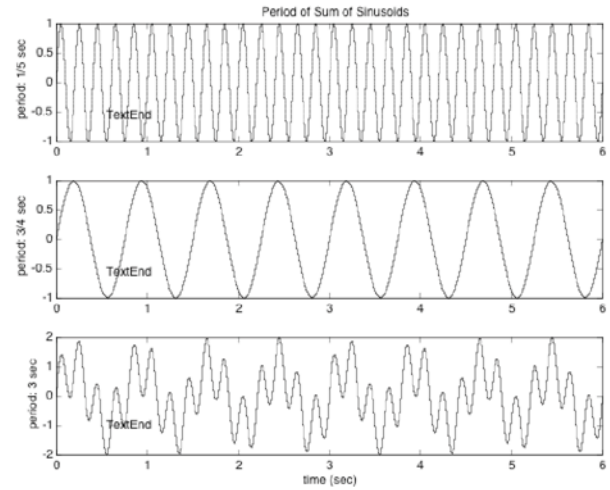
$$\cos(2\pi ft + \phi) = \frac{e^{j2\pi ft + \phi} + e^{-j2\pi ft + \phi}}{2}$$
$$= \frac{e^{j\phi}e^{j2\pi ft} + e^{-j\phi}e^{-j2\pi ft}}{2}$$
$$= X_{km}e^{j2\pi ft} + X_{kp}e^{-j2\pi ft} \quad X_{km} = \tfrac{1}{2}e^{j\phi}$$
$$= \Sigma X_k e^{j2\pi ft}$$
$$X = \begin{cases} \tfrac{1}{2}e^{j\phi} & k = f \\ \tfrac{1}{2}e^{-j\phi} & k = -f \\ 0 & otherwise \end{cases}$$

ϕ *always a odd pair*





Period of Sum of Sinusoids



$x(t) = \cos(2\pi 5t)$

$y(t) = \cos(2\pi(4/3)t)$

$z(t) = x(t) + y(t)$

$z(t) = z(t+T), T = ?$

Least common multiple

$x(t) = \cos(2\pi 5t)$

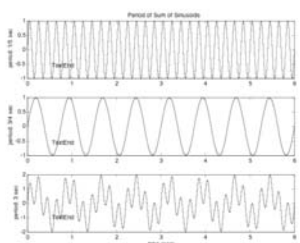
seconds to complete cycle

T_x=1/5 seconds

1/5s, 2/5s, 3/5s …

4/20s, 8/20s, 12/20s, 16/20s, 20/20s, 24/20s, 28/20s, 32/20s, 36/20s, 40/20s, 44/20s, 48/20s, 52/20s, 56/20s, 60/20s

15 cycles



$y(t) = \cos(2\pi(4/3)t)$

seconds to complete cycle

T_y=3/4 seconds

3/4s, 6/4s, …

15/20s. 30/20s, 45/20s, 60/20s

4 cycles

1/5*k=3/4*l

k/l=15/4 rational number

$z(t) = x(t) + y(t)$

T_z=15*T_x=15/5=3 seconds

T_z=4*T_y=3/4*4=3 seconds

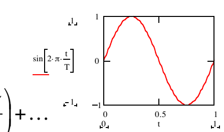
$z(t) = z(t + T_z)$

T_z=3 seconds

Fourier Series

$x(t) = \sin(2\pi t) \quad 0 \le t < 1$

$$= 0 + 1\cdot\cos\left(2\pi t - \frac{\pi}{2}\right) + 0\cdot\cos\left(2\pi 2t - \frac{\pi}{2}\right) + 0\cdot\cos\left(2\pi 3t - \frac{\pi}{2}\right) + \ldots$$



$X_0 = 0$

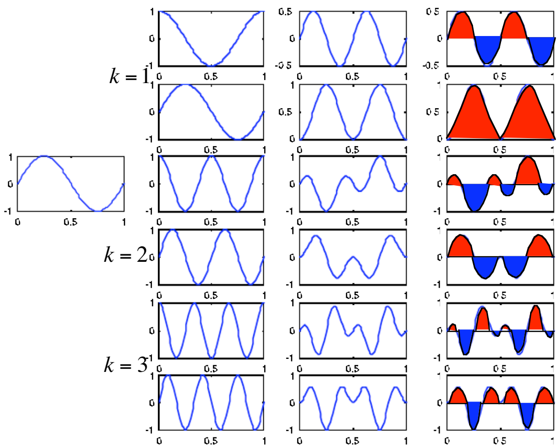
$$X_k = \begin{cases} 1e^{-j\pi/2} & k = 1 \\ 0 & k \ne 1 \end{cases}$$

$$X_0 = \frac{1}{T_0}\int_0^{T_0} x(t)dt$$
$$= \int_0^1 \sin(2\pi t)dt = 0$$

$$X_k = \frac{2}{T_0}\int_0^{T_0} x(t)e^{-j2\pi\frac{k}{T_0}t}dt$$
$$= 2\int_0^1 \sin(2\pi t)e^{-j2\pi kt}dt$$
$$= 2\int_0^1 \sin(2\pi t)[\cos(-2\pi kt) + j\sin(-2\pi kt)]dt$$

Fourier Series

$\sin(2\pi t)$	$\cos(2k\pi t)$ $\sin(2k\pi t)$	$\sin(2\pi t)\cos(2k\pi t)$ $\sin(2\pi t)\sin(2k\pi t)$	$\int_0^1 \sin(2\pi t)\cos(2\pi kt)dt$ $\int_0^1 \sin(2\pi t)\sin(2\pi kt)dt$	X_k



$= 1e^{-j\pi/2}$

$= 0$

$= 0$

Fourier Series (frequency space)

$$X_0 = \frac{1}{T}\cdot\int_0^T z(t)\,dt \qquad X_k = \frac{2}{T}\cdot\int_0^T z(t)\cdot e^{-j\cdot 2\cdot\pi\cdot k\cdot\frac{t}{T}}\,dt$$

$$X_0 = \frac{1}{3}\cdot\int_0^3 \cos(2\cdot\pi\cdot 5\cdot t) + \cos\left[2\cdot\pi\cdot\left(\frac{4}{3}\right)\cdot t\right]dt \qquad X_k = \frac{2}{3}\cdot\int_0^3 \left(\cos(2\cdot\pi\cdot 5\cdot t) + \cos\left(2\cdot\pi\cdot\frac{4}{3}\cdot t\right)\right)\cdot e^{-j\cdot 2\cdot\pi\cdot k\cdot\frac{t}{3}}\,dt$$

$X_0 = 0$

$$X_k = i\cdot\frac{\left(2\cdot\exp(-2\cdot i\cdot\pi\cdot k)\cdot k^3 - 2\cdot k^3 - 241\cdot\exp(-2\cdot i\cdot\pi\cdot k)\cdot k + 241\cdot k\right)}{\left(\pi\cdot k^4 - 241\cdot\pi\cdot k^2 + 3600\cdot\pi\right)}$$

$k = 4,\ k = 15$

$X_{4,15} = \frac{0}{0}$

So, use L'Hopitals Rule

$k \neq 4,\ 15$

$$X_k = 1i\cdot\frac{2\cdot 1\cdot k^3 - 2\cdot k^3 - 241\cdot 1\cdot k + 241\cdot k}{[\pi\cdot(k+15)\cdot(k-4)\cdot(k+4)\cdot(k-15)]}$$

$X_k = 0$

$$X_k = \frac{1}{4}\cdot 1i\cdot\frac{-4\cdot 1i\cdot\pi\cdot\exp(-2\cdot 1i\cdot\pi\cdot k)\cdot k^3 + 6\cdot\exp(-2\cdot 1i\cdot\pi\cdot k)\cdot k^2 - 6\cdot k^2 + 482i\cdot\pi\cdot\exp(-2\cdot 1i\cdot\pi\cdot k)\cdot k - 241\cdot\exp(-2\cdot 1i\cdot\pi\cdot k) + 241}{\pi\cdot(k-4)\cdot(k+4)\cdot(k-15) + \pi\cdot(k+15)\cdot(k+4)\cdot(k-15) + \pi\cdot(k+15)\cdot(k-4)\cdot(k-15) + \pi\cdot(k+15)\cdot(k-4)\cdot(k+4)}$$

$X_4 = 1$

$X_{15} = 1$

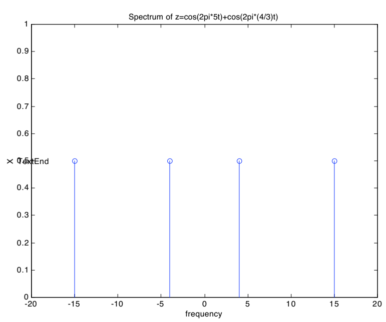
Fourier Series (frequency space)

$X_0 = 0$

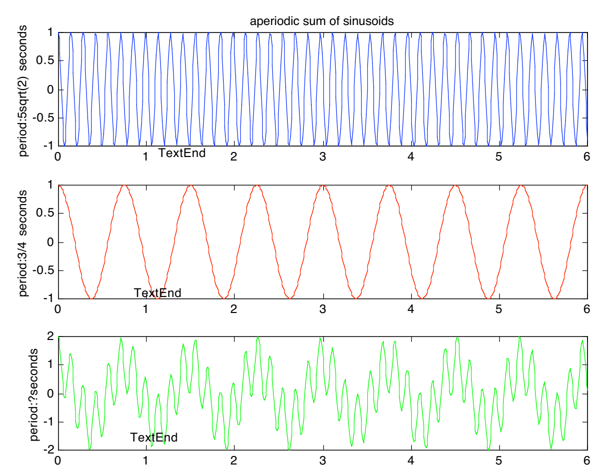
$X_k = 0 \quad k \neq 4,\ 15$

$X_4 = 1$

$X_{15} = 1$



Aperiodic Sum of Sinusoids w/ an Irrational Frequency



$x(t) = \cos\left(2\pi 5\sqrt{2}t\right)$

$y(t) = \cos\left(2\pi\left(4/3\right)t\right)$

$z(t) = x(t) + y(t)$

$z(t) = z(t+T), T = ?$

$x(t) = \cos\left(2\pi 5\sqrt{2}t\right)$

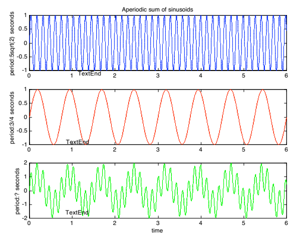
Least common multiple

$y(t) = \cos\left(2\pi(4/3)t\right)$

seconds to complete cycle

$T = \frac{1}{5\sqrt{2}}$ seconds

$\frac{1}{\sqrt{}}s, \frac{2}{5\sqrt{2}}s, \frac{3}{5\sqrt{2}}s\ldots$



seconds to complete cycle

$T_y = \frac{3}{4}$ seconds

$\frac{3}{4}s, \frac{6}{4}s, \frac{9}{4}s\ldots$

$\frac{1}{5\sqrt{2}}k = \frac{3}{4}l$

$\frac{k}{l} = \frac{15\sqrt{2}}{4}$ irrational number

$z(t) = x(t) + y(t)$

$z(t) = z(t + T_z)$

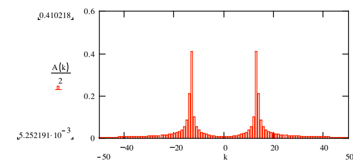
$T_z = \infty$ seconds $\quad z(t)$ *aperiodic*

Fourier Series for Irrational Frequency
“What’s the frequency, Kenneth?”

In 1986, CBS Anchorman Rather was confronted about 11 p.m. while walking on Park Avenue, when he was punched from behind and knocked to the ground then chased into a building and kicked him several times in the back while the assailant demanded to know 'Kenneth, what is the frequency?' The assailant was convinced the media had him under surveillance and were beaming hostile messages into his head, and he demanded that Rather tell him the frequency being used.

In the Fourier Series for an aperiodic signal “what’s the period, Quinn?”

$X_0 = \frac{1}{T}\cdot\int_0^T z(t)\,dt \qquad X_k = \frac{2}{T}\cdot\int_0^T z(t)\cdot e^{-1j\cdot 2\cdot\pi\cdot k\cdot\frac{t}{T}}\,dt$



PS3-5

Pick a T, plot the spectrum, then repeat with larger T’s.

```
QUAD8  Numerically evaluate integral, higher order method.
  Q = QUAD8('F',A,B) approximates the integral of F(X) from A to B
  'F' is a string containing the name of the function.
   The function must return a vector of output values given a vector of input values.
 Q = QUAD8('F',A,B,TOL,TRACE,P1,P2,...) allows coefficients P1, P2, ...
   to be passed directly to function F:   G = F(X,P1,P2,...).
   To use default values for TOL or TRACE, you may pass in the empty
    matrix ([]).
```

$x(t) = \int_0^1 \cos^2\left(2\pi f t\right)dt$

```
function y =  myintegrand(t,f)
y=cos(2*pi*f*t).^2;

save as myintegrand.m

»f=1;
»quad8('myintegrand',0,1,[],[],1)
ans =
   0.50000000000000
```

$X_0 = \frac{1}{T}\cdot\int_0^T z(t)\,dt \qquad X_k = \frac{2}{T}\cdot\int_0^T z(t)\cdot e^{-1j\cdot 2\cdot\pi\cdot k\cdot\frac{t}{T}}\,dt$

Pick a T, compute X0, loop over Xk’s, plot the spectrum, then repeat with larger T’s.

Bases are building blocks to form more complex things

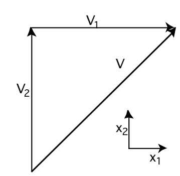
$x = \sum_{k=1}^{m} A_k\phi_k$

Synthesize x from a weighted sum of basis elements, ϕ

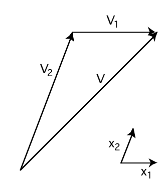
or

Decompose x into a weighted sum of basis elements, ϕ

Basis Vector

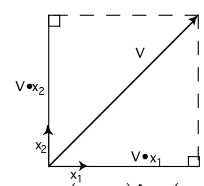


$V = a\hat{x}_1 + b\hat{x}_2$

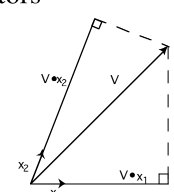


$V = a\hat{x}_1 + b\hat{x}_2$

Prefer orthonormal basis vectors



$V = (V \bullet x_1)\hat{x}_1 + (V \bullet x_2)\hat{x}_2$



$V \neq (V \bullet x_1)\hat{x}_1 + (V \bullet x_2)\hat{x}_2$

projections

Basis Functions

$x(t) = \sum_k A_k \phi_k(t)$ $\qquad$ $x(t), \phi_k(t)$ are functions

When are functions orthogonal to each other?

$$\int_a^b \phi_j \phi_k^* dt = \begin{cases} 1 & if \quad j = k \\ 0 & if \quad j \neq k \end{cases}$$

normalized orthogonality condition

over an interval $a \le t \le b$

How do you "project" functions onto each other?

$A_k = \int_a^b x(t)\phi_k^* dt$ $\qquad$ How much of $\phi_k(t)$ is in $x(t)$?

If $\phi_k(t)$ are orthonormal:

$x(t) = \sum_k A_k \phi_k(t)$ where $A_k = \int_a^b x(t)\phi_k^* dt$

Fourier Series

$\phi_k = \frac{1}{\sqrt{T_0}} e^{j2\pi kt/T_0}$

Complex exponentials

Basis Functions

$$\int_a^b \phi_j \phi_k^* dt = \begin{cases} 1 & if \quad j = k \\ 0 & if \quad j \neq k \end{cases}$$

normalized orthogonality condition

over an interval $a \le t \le b$

Ex:

$\phi_1 = \frac{\sqrt{6}}{3}\cos(2\pi t)$

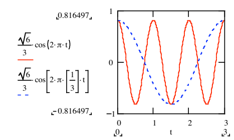
$\phi_2 = \frac{\sqrt{6}}{3}\cos\left(2\pi\left(\frac{1}{3}\right)t\right)$

over an interval $0 \le t \le 3$

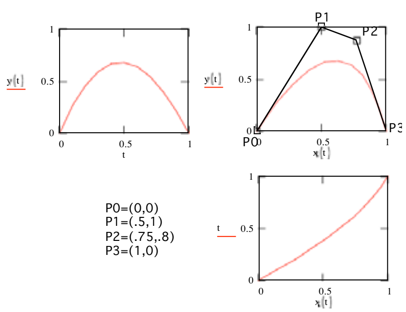
$\int_a^b \phi_1\phi_2 dt = \int_0^3 \frac{\sqrt{6}}{3}\cos(2\pi t)\frac{\sqrt{6}}{3}\cos\left(2\pi\left(\frac{1}{3}\right)t\right)dt = 0$ $\qquad$ orthogonal

$\int_a^b \phi_1\phi_1 dt = \int_0^3 \frac{\sqrt{6}}{3}\cos(2\pi t)\frac{\sqrt{6}}{3}\cos(2\pi t)dt = 1$ $\qquad$ normalized

$\int_a^b \phi_2\phi_2 dt = \frac{1}{3}\int_0^3 \frac{\sqrt{6}}{3}\cos\left(2\pi\left(\frac{1}{3}\right)t\right)\frac{\sqrt{6}}{3}\cos\left(2\pi\left(\frac{1}{3}\right)t\right)dt = 1$ $\qquad$ normalized



Bezier Curves



P0=(0,0)
P1=(.5,1)
P2=(.75,.8)
P3=(1,0)

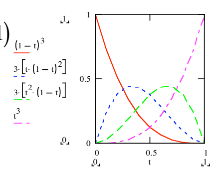
cubic spline basis function

$\phi_1 = (t-1)^3$ $\qquad 0 \le t \le 1$

$\phi_2 = 3t(t-1)^2$

$\phi_3 = 3t^2(t-1)$

$\phi_4 = t^3$



$x(t) = P0_x(1-t)^3 + 3P1_x(1-t)^2t + 3P2_x(1-t)t^2 + P3_x(1-t)t^3$

$y(t) = P0_y(1-t)^3 + 3P1_y(1-t)^2t + 3P2_y(1-t)t^2 + P3_y(1-t)t^3$

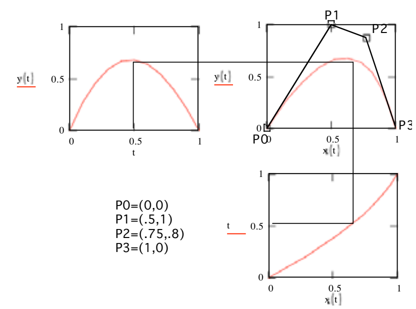
parametric curve

$\phi_1 = (t-1)^3$

$\phi_2 = 3t(t-1)^2$

$\int_a^b \phi_1\phi_2 dt = \int_0^1 (t-1)^3 3t(t-1)^2 dt = -\frac{1}{14}$ $\qquad$ not orthogonal

Bezier Curves



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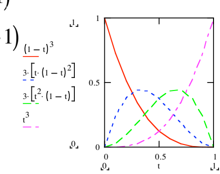
$\phi_4 = t^3$



$x(t) = P0_x(1-t)^3 + 3P1_x(1-t)^2t + 3P2_x(1-t)t^2 + P3_x(1-t)t^3$

$y(t) = P0_y(1-t)^3 + 3P1_y(1-t)^2t + 3P2_y(1-t)t^2 + P3_y(1-t)t^3$

parametric curve

$\phi_1 = (t-1)^3$

$\phi_2 = 3t(t-1)^2$

$\int_a^b \phi_1\phi_2 dt = \int_0^1 (t-1)^3 3t(t-1)^2 dt = -\frac{1}{14}$ $\qquad$ not orthogonal

Make your own set of basis functions

1. Pick an initial basis function ϕ_1 and an interval $a \le t \le b$
2. Normalize ϕ_1

$$\int_a^b \phi_1 \phi_1 dt = 1$$ 1 eqn

3. Pick a general formula for a second function ϕ_2
4. “Orthonormalize”
   a. make ϕ_2 orthogonal to ϕ_1 b. normalize ϕ_2

$$\int_a^b \phi_1 \phi_2 dt = 0 \qquad \int_a^b \phi_2 \phi_2 dt = 1$$ 2 eqns

5. Pick a general formula for a second function ϕ_3
6. “Orthonormalize”
   a. make ϕ_3 orthogonal to ϕ_1 & ϕ_2 b. normalize ϕ_3

$$\int_a^b \phi_3 \phi_1 dt = 0 \qquad \int_a^b \phi_3 \phi_2 dt = 0 \qquad \int_a^b \phi_3 \phi_3 dt = 1$$ 3 eqns

7. Continue for ϕ_k k eqns

Make your own set of basis functions

1. Pick an initial basis function ϕ_1 with one parameter, & interval $a \le t \le b$
2. Normalize ϕ_1

$$\int_a^b \phi_1 \phi_1 dt = 1$$ 1 eqn

3. Pick a general formula for a second function ϕ_2 with 2 parameters & 1 zero-crossings
4. “Orthonormalize”
   a. make ϕ_2 orthogonal to ϕ_1 b. normalize ϕ_2

$$\int_a^b \phi_1 \phi_2 dt = 0 \qquad \int_a^b \phi_2 \phi_2 dt = 1$$ 2 eqns

5. Pick a general formula for a second function ϕ_3 with 3 parameters & 2 zero-crossings
6. “Orthonormalize”
   a. make ϕ_3 orthogonal to ϕ_1 & ϕ_2 b. normalize ϕ_3

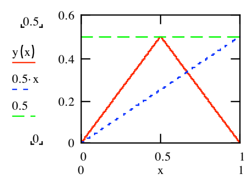
$$\int_a^b \phi_3 \phi_1 dt = 0 \qquad \int_a^b \phi_3 \phi_2 dt = 0 \qquad \int_a^b \phi_3 \phi_3 dt = 1$$ 3 eqns

7. Continue for ϕ_k with k parameters k eqns

Make your own set of basis functions

1. Pick an initial basis function ϕ_1 with one parameter & an interval

$\phi_1 = A$ $\qquad 0 \le t \le 1$



Linear segments

2. Normalize $\int_{t1}^{t2} \phi_1 \phi_1 dt = 1$

$$\int_0^1 AAdt = 1$$
$$A^2 t\Big|_0^1 = 1$$
$$A^2 = 1$$
$$A = 1$$
$$\phi_1 = 1$$

3. Pick an second basis function ϕ_2 with two parameters

$\phi_2 = Bt + C$

4. Orthonormalize

$$\int_{t1}^{t2} \phi_2 \phi_2 dt = 1 \qquad \int_{t1}^{t2} \phi_2 \phi_1 dt = 0 \qquad \frac{B^2}{3} - B\frac{B^2}{2} + \left(-\frac{B}{2}\right)^2 = 1$$

$$\int_0^1 (Bt + C)(Bt + C)dt = 1 \qquad \int_0^1 (Bt + C)1dt = 0 \qquad B = \pm 2\sqrt{3}$$

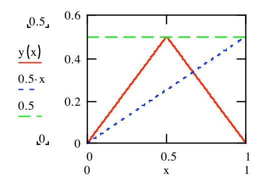
$$B^2 t^3/3 + 2CBt^2/2 + C^2 t\Big|_0^1 = 1 \qquad \frac{B}{2} + C = 0 \qquad C = \mp\sqrt{3}$$

$$\frac{B^2}{3} + BC + C^2 = 1 \qquad C = -\frac{B}{2} \qquad \phi_2 = 2\sqrt{3}t - \sqrt{3}$$

$$\phi_1 = 1$$
$$\phi_2 = 2\sqrt{3}t - \sqrt{3}$$

5. Pick an third basis function ϕ_3 with three parameters

$$\phi_3 = \begin{vmatrix} bt + a & t \le c \\ (bc + a) - b(t - c) & t > c \end{vmatrix}$$



Linear segments

6. Orthonormalize

$$\int_{t1}^{t2} \phi_3 \phi_3 dt = 1 \qquad \int_{t1}^{t2} \phi_3 \phi_1 dt = 0 \qquad \int_{t1}^{t2} \phi_3 \phi_2 dt = 0$$

$$\int_0^c (bt + a)(bt + a)dt + \int_c^1 (bc + a - b(t - c))\phi_3 dt = 1$$

$$\int_0^c (bx + a)1dt + \int_c^1 (bc + a - b(x - c))1dt = 0$$

$$\int_0^c (bx + a)(2\sqrt{3}t - \sqrt{3})dt + \int_c^1 (bc + a - b(x - c))(2\sqrt{3}t - \sqrt{3})dt = 0$$

6. Orthonormalize

$$-2\cdot b^2\cdot c^3 + -2\cdot b\cdot(-2\cdot b + a)\cdot c^2 + 2\cdot b\cdot(-b + 2\cdot a)\cdot c + a^2 - a\cdot b + \frac{1}{3}\cdot b^2 = 1$$

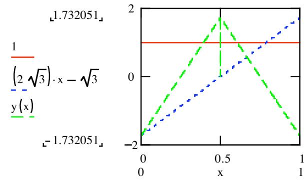
$$\left(-c^2 + 2\cdot c - \frac{1}{2}\right)\cdot b + a = 0$$

$$\frac{-1}{12}\cdot\sqrt{3}\cdot b\cdot\left(2\cdot c - 1 - \sqrt{3}\right)\cdot\left(2\cdot c - 1 + \sqrt{3}\right)\cdot(2\cdot c - 1) = 0$$

$$c = \frac{1}{2} \qquad \frac{1}{4}b + a = 0 \qquad a = \frac{-1}{4}b$$

$$a = -\sqrt{3} \qquad b = 4\sqrt{3} \qquad c = \frac{1}{2}$$

$$\phi_3 = \begin{cases} 4\sqrt{3}t - \sqrt{3} & t \le \frac{1}{2} \\ (2\sqrt{3}t + \sqrt{3}) - 4\sqrt{3}(t - \frac{1}{2}) & t > \frac{1}{2} \end{cases}$$

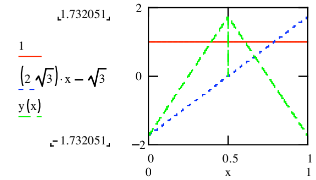


Linear segments

$$\phi_1 = 1$$
$$\phi_2 = 2\sqrt{3}t - \sqrt{3}$$
$$\phi_3 = \begin{cases} 4\sqrt{3}t - \sqrt{3} & t \le \frac{1}{2} \\ (2\sqrt{3}t + \sqrt{3}) - 4\sqrt{3}(t - \frac{1}{2}) & t > \frac{1}{2} \end{cases}$$

Decomposition via the "Q" basis

$$\phi_1 = 1 \qquad 0 \le t \le 1$$
$$\phi_2 = 2\sqrt{3}t - \sqrt{3}$$
$$\phi_3 = \begin{cases} 4\sqrt{3}t - \sqrt{3} & t \le \frac{1}{2} \\ (2\sqrt{3}t + \sqrt{3}) - 4\sqrt{3}(t - \frac{1}{2}) & t > \frac{1}{2} \end{cases}$$



Linear segments

$$x(t) = \cos(\pi t)$$

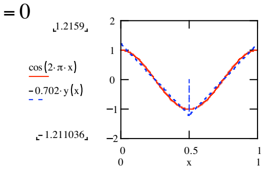
$$x(t) = \sum_k A_k\phi_k(t) \quad \text{where} \quad A_k = \int_a^b x(t)\phi_k^* dt \qquad a \le t \le b$$

$$A_1 = \int_a^b x(t)\phi_1\, dt \qquad A_2 = \int_a^b x(t)\phi_2 dt \qquad A_3 = \int_a^b x(t)\phi_3 dt$$

$$A_1 = \int_0^1 \cos(2\pi t)1dt \qquad A_2 = \int_0^1 \cos(2\pi t)\left(2\sqrt{3}t - \sqrt{3}\right)dt$$

$$A_3 = \int_0^{1/2} \cos(\pi t)\left(4\sqrt{3}t - \sqrt{3}\right)dt + \int_{1/2}^1 \cos(\pi t)\left(2\sqrt{3}t + \sqrt{3} - 4\sqrt{3}\left(t - \frac{1}{2}\right)\right)dt$$

$$A_1 = 0 \qquad A_2 = 0 \qquad A_3 = -0.702$$



$$x(t) = -0.702\phi_3(t)$$

Decomposition via the "Q" basis

$$x(t) = \sin\left(\frac{\pi}{2}t\right) \qquad 0 \le t \le 1$$

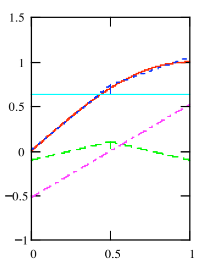
$$x(t) = \sum_k A_k\phi_k(t) \quad \text{where} \quad A_k = \int_0^T x(t)\phi_k^* dt$$

$$A_1 = \int_0^T x(t)\phi_1\, dt \qquad A_2 = \int_0^T x(t)\phi_2 dt \qquad A_3 = \int_0^T x(t)\phi_3 dt$$

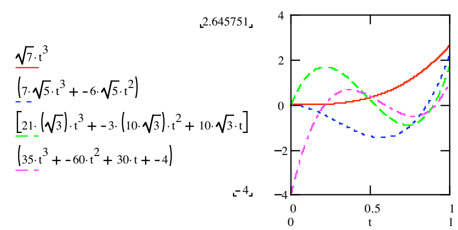
$$A_1 = \int_0^1 \sin\left(\frac{\pi}{2}t\right)1dt \qquad A_2 = \int_0^1 \sin\left(\frac{\pi}{2}t\right)\left(2\sqrt{3}t - \sqrt{3}\right)dt$$

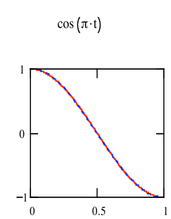
$$A_3 = \int_0^{1/2} \sin\left(\frac{\pi}{2}t\right)\left(4\sqrt{3}t - \sqrt{3}\right)dt + \int_{1/2}^1 \sin\left(\frac{\pi}{2}t\right)\left(2\sqrt{3}t + \sqrt{3} - 4\sqrt{3}\left(t - \frac{1}{2}\right)\right)dt$$

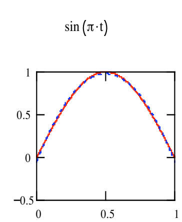
$$A_1 = 0.673 \qquad A_2 = 0.301 \qquad A_3 = 0.06$$

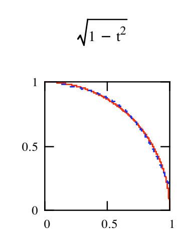


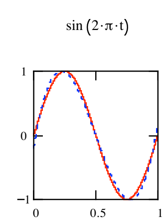
Orthonormal Cubic Splines





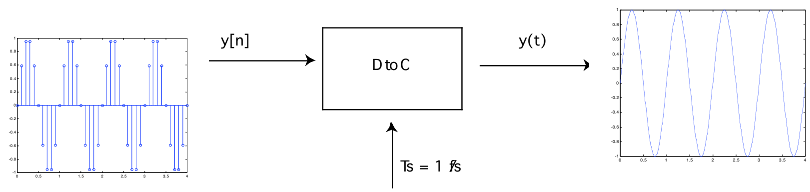






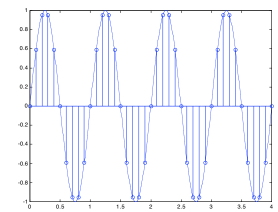
Sampling and Aliasing

D/A Conversion



Ideal discrete to continuous converter for sample spacing Ts

Interpolates discrete samples to form a continuous signal

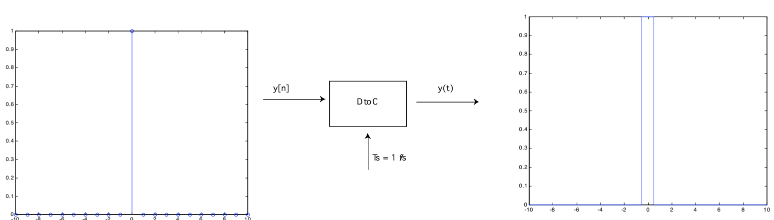


D/A Conversion

Pulse

For each sample y[n], a pulse p(t) is produced

$$y(t) = \sum_{n=-\infty}^{\infty} y[n]p(t - nT_s)$$



$$y(t) = p(t) = \begin{cases} 1 & -T_s/2 < t \le T_s/2 \\ 0 & otherwise \end{cases}$$

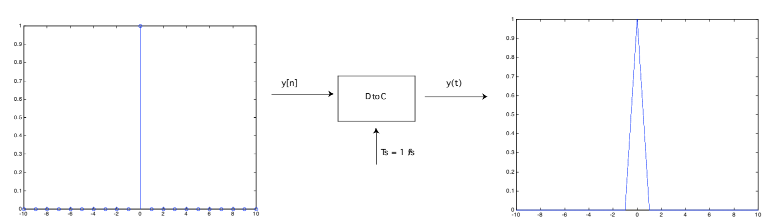
Different pulse shapes produce different D-to-C interpolations

D/A Conversion

Pulse

For each sample y[n], a pulse p(t) is produced

$$y(t) = \sum_{n=-\infty}^{\infty} y[n]p(t - nT_s)$$



$$y(t) = p(t) = \begin{cases} 1 - |t|/T_s & -T_s \le t \le T_s \\ 0 & otherwise \end{cases}$$

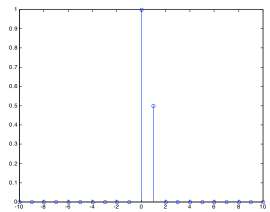
Different pulse shapes produce different D-to-C interpolations

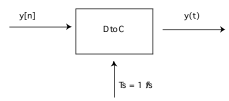
D/A Conversion

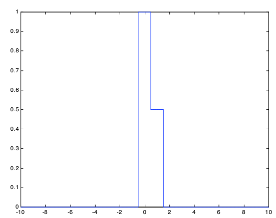
Pulse

For each sample y[n], a pulse p(t) is produced

$$y(t) = \sum_{n=-\infty}^{\infty} y[n]p(t-nT_s)$$





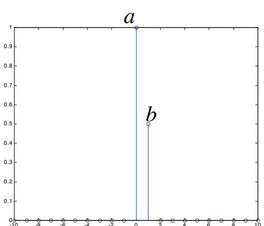


$$y(t) = y[0]p(t) + y[1]p(t-Ts)$$

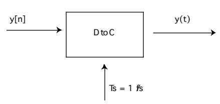
$$y(t) = 1p(t) + 0.5p(t-1)$$

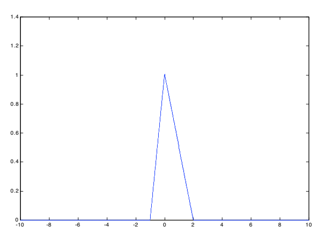
$$p(t) = \begin{cases} 1 & -T_s/2 < t \le T_s/2 \\ 0 & otherwise \end{cases}$$

$$y(t) = \begin{cases} 1(0)+0.5(0)=0 & t < -1/2 \\ 1(1)+0.5(0)=1 & -1/2 < t < 1/2 \\ 1(0)+0.5(1)=0.5 & 1/2 < t < 3/2 \\ 1(0)+0.5(0)=0 & 3/2 < t \end{cases}$$



$T_s = 1$





$$y(t) = \begin{cases} 0 & t < -T_s \\ 1+t/T_s & -T_s \le t < 0 \\ a(1-t/T_s)+b(1+(t-T_s)/T_s) & 0 \le t < T_s \\ 1-(t-T_s)/T_s & T_s \le t < 2T_s \\ 0 & t > 2T_s \end{cases}$$

$$y(t) = \begin{cases} 0 & t < -T_s \\ 1+t/T_s & -T_s \le t < 0 \\ a+(b-a)t/T_s & 0 \le t < T_s \\ 1-t/T_s & T_s \le t < 2T_s \\ 0 & t > 2T_s \end{cases}$$

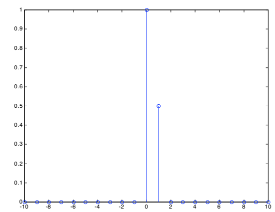
linear interpolation

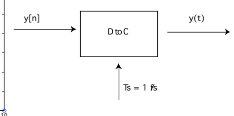
D/A Conversion

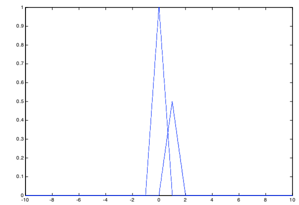
Pulse

For each sample y[n], a pulse p(t) is produced

$$= \sum_{n=-\infty}^{\infty} y[n]p(t-nT_s)$$







$$y(t) = y[0]p(t) + y[1]p(t-Ts)$$

$$y(t) = 1p(t) + \tfrac{1}{2}p(t-Ts)$$

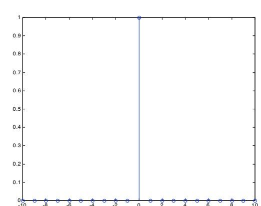
$$y(t) = ap(t) + bp(t-Ts)$$

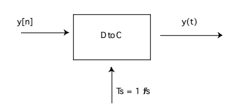
$$p(t) = \begin{cases} 1-|t|/T_s & -T_s \le t \le T_s \\ 0 & otherwise \end{cases}$$

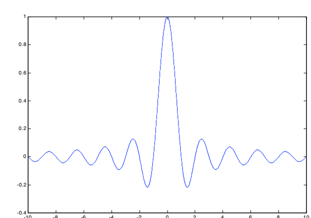
$$y(t) = a\begin{cases} 0 & t < -T_s \\ 1+t/T_s & -T_s \le t < 0 \\ 1-t/T_s & 0 \le t < T_s \\ 0 & T_s \le t < 2T \\ 0 & t > 2T \end{cases} + b\begin{cases} 0 & t < -T_s \\ 0 & -T_s \le t < 0 \\ 1+(t-T_s)/T_s & 0 \le t < T_s \\ 1-(t-T_s)/T_s & T_s \le t < 2T_s \\ 0 & t > 2T \end{cases}$$

D/A Conversion

Ideal



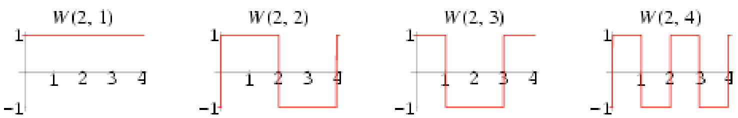




$$y(t) = p(t) = \left\{ \frac{\sin\frac{\pi t}{T_s}}{\frac{\pi t}{T_s}} \right. \quad for \quad -\infty < t < \infty$$

$$= \sin c\left(\frac{\pi t}{T_s}\right)$$

Walsh functions



$$\int_a^b \phi_j \phi_k^* dt = \begin{cases} 1 & if \quad j = k \\ 0 & if \quad j \neq k \end{cases}$$ normal, orthogonality condition

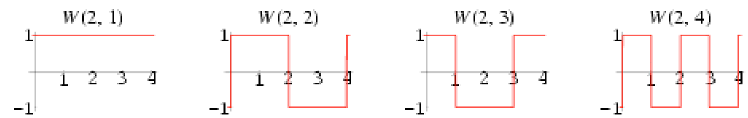
$$\int_0^4 \phi_1 \phi_1^* dt = \sum_1^4 1 \cdot 1 = 4$$ not normal

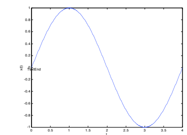
$$\int_0^4 \phi_2 \phi_2^* dt = \big(1\cdot 1 + (-1\cdot -1) + 1\cdot 1 + (-1\cdot -1)\big) = 4$$ not normal

$$\int_0^4 \phi_1 \phi_2^* dt = 1\cdot 1 + 1\cdot 1 + (1\cdot -1) + (1\cdot -1) = 1 + 1 - 1 - 1 = 0$$ orthogonal

$$\int_a^b \phi_j \phi_k^* dt = \begin{cases} \lambda_k = 4 & if \quad j = k \\ 0 & if \quad j \neq k \end{cases}$$

Walsh functions





$x(t) = t \qquad 0 \le t \le 4$

$$x(t) = \sum_k A_k \phi_k(t) \quad \text{where} \quad A_k = \frac{1}{\lambda_k}\int_0^4 x(t)\phi^*(t)dt = \frac{1}{4}\int_0^4 x(t)W(2,k+1)dt$$

$$A_0 = \tfrac14 \int_0^4 t\cdot 1dt = \tfrac{t^2}{2}\Big|_0^4 = \tfrac14 8 = 2$$

$$A_1 = \tfrac14 \int_0^2 t\cdot 1dt + \tfrac14 \int_2^4 t(-1)dt = \tfrac14 \tfrac{t^2}{2}\Big|_0^2 - \tfrac14 \tfrac{t^2}{2}\Big|_2^4 = \tfrac14 (2-6) = -4\tfrac14 = -1$$

$$A_2 = \tfrac14 \int_0^1 t\cdot 1dt + \tfrac14 \int_1^3 t\cdot(-1)dt + \tfrac14 \int_3^4 t\cdot 1dt = \tfrac14\left(\tfrac{t^2}{2}\Big|_0^1 - \tfrac{t^2}{2}\Big|_1^3 + \tfrac{t^2}{2}\Big|_3^4\right) = \tfrac14\left(\tfrac12 - \left[\tfrac92 - \tfrac12\right] + \left[\tfrac{16}{2} - \tfrac92\right]\right) = 0$$

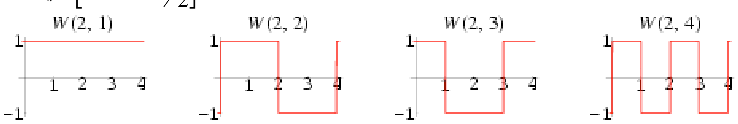
$$A_3 = \tfrac14 \int_0^1 t\cdot 1dt + \tfrac14 \int_1^2 t\cdot(-1)dt + \tfrac14 \int_2^3 t\cdot(1)dt + \tfrac14 \int_3^4 t\cdot(-1)dt = \tfrac14\left(\tfrac{t^2}{2}\Big|_0^1 - \tfrac{t^2}{2}\Big|_1^2 + \tfrac{t^2}{2}\Big|_2^3 - \tfrac{t^2}{2}\Big|_3^4\right)$$

$$= \tfrac14\left(\tfrac12 - \left[\tfrac42 - \tfrac12\right] + \left[\tfrac92 - \tfrac42\right] - \left[\tfrac{16}{2} - \tfrac92\right]\right) = -2\tfrac14 = -\tfrac12$$

Walsh functions

$$x(t) = \sum_k A_k \phi_k(t)$$

$$A_k = \left[2, -1, 0, -\tfrac{1}{2}\right]$$



[2 2 2 2]+[-1 -1 1 1] + [0 0 0 0]+[-0.5 0.5 -0.5 0.5]=[0.5 1.5 2.5 3.5]