

*Background: Simple ratios of frequencies*

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Within the range of human hearing, there exist infinitely many possible standing waves, yet in our musical practice we restrict ourselves to a limited repertoire of pitches. The selection is based on our brains' preference for simple frequency relationships, expressible as ratios of small integers (e.g., 1/1, 2/1, 3/2).

Working with these ratios is awkward for newcomers, because we're accustomed to thinking of musical tones in arithmetic rather than geometric terms. So we're prone to think "a major third plus a minor third plus a perfect fourth equals an octave," rather than "5/4 times 6/5 times 4/3 equals 2/1," even though the latter description more directly describes the relationships among frequencies (and makes much better mathematical sense than "3 + 3 + 4 = 8"). This problem becomes especially acute when we "subtract" intervals, that is to say, divide ratios: compare "a perfect fifth minus a perfect fourth equals a major second," vs. "3/2 divided by 4/3 equals 9/8."

*Converting ratios to distances*

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As the above examples illustrate, it will be useful to convert each geometric ratio into an equivalent arithmetic distance, the better to visualize the comparative sizes of intervals. Let  $c$  stand for one arithmetic unit, such that a sum of 1200 units<sup>1</sup> is equivalent to the identity ratio of 2/1.

$$1) \quad c^{1200} = 2 \quad (\text{i.e., the arithmetic unit } c \text{ corresponds to the ratio } 1.00057779/1)$$

In logarithmic terms,

$$2) \quad 1200 \log c = \log 2$$

To convert the ratio 3/2, we need to find  $x$  such that

$$3) \quad c^x = 3/2.$$

So taking logarithms as in lines 1 and 2 above,

$$4) \quad x \log c = \log (3/2)$$

and if we then divide this expression by the expression in line 2, we get:

$$5) \quad x / 1200 = \log (3/2) / \log 2$$

[continued on reverse]

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<sup>1</sup> Why 1200? The answer will reveal itself. The use of the letter "c" for our arithmetic unit provides a clue.

$$5) \quad x/1200 = \log(3/2) / \log 2$$

which we then manipulate to isolate the unknown  $x$ :

$$6) \quad x = 1200 \log(3/2) / \log 2 = 1200 (.176) / .301 = 701.955$$

[if desired, each log above can be replaced by ln:  $1200 \ln(3/2)/\ln 2 = 1200 (.405) / .693 = 701.955$  ]

From line 6, we abstract the general formula:

$$\text{value in } c = 1200 \log(\text{ratio})/\log 2 \quad [\text{or: } c = 1200 \ln(\text{ratio})/\ln 2]$$

### Assignment no. 1: SCALE CONSTRUCTION

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Attached is a graph showing the arithmetic distances among frequencies related by the ratios  $1/1$ ,  $2/1$ , and  $3/2$ .

1) Convert the following simple-integer ratios, and add them to the graph.

$$4/3 \quad 5/4 \quad 6/5 \quad 8/5 \quad 5/3$$

**Label each frequency with both its ratio and its  $c$  value**, to one decimal place (e.g., it will suffice to label  $3/2$  as  $702.0c$ ). Make sure that each ratio falls between  $1/1$  and  $2/1$ ; if a ratio doesn't fit that range, multiply or divide the denominator by two (or four, or eight) to obtain an appropriate ratio.

(e.g.  $27/8 > 2/1$ , so we would multiply the denominator by two to get  $27/16$ )

2a) What's the interval between  $3/2$  and  $4/3$ ?

2b) What's  $3/2$  of  $3/2$ ?

2c) What's  $3/2$  of that?

2d) What's  $3/2$  of that?

Add the above ratios to your graph.

2e) How many recursive applications of  $3/2$  will get you back to  $1/1$  (or  $2/1$ ,  $4/1$ , etc.)?

(Supply your answer in the blank provided below the graph.)

3a) What value falls  $6/5$  below  $4/3$ ? Add the resulting ratio to your graph.

3b) What's the difference between that value and  $9/8$ ?

(Supply your answer in the blank provided below the graph.)

4a) How would you characterize the spacing of the values you've mapped?

4b) How does this correspond to your previous knowledge (whether formal or intuitive) of scales and intervals?

4c) What ratios would you propose to fill in the gaps? Add them to your graph.

**5) Be prepared to discuss in our next class meeting:** What are the properties of the scale you've created? How many unique distances exist among its components? What problems do your proposed interpolations solve or diminish? What problems do they create or magnify?