

# 22.01 Fall 2015, Quiz 1 Solutions

November 17, 2015

Answers should be given symbolically or graphically, no calculation should be necessary. Define any intermediate variables which you need to complete the problems. Partial credit will be given for methodology.

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## 1 (40 points) Short Answers, 10 points each

### 1.1 Graph the mass per nucleon of a nucleus as a function of nucleons ( $\frac{M(A)}{A}$ vs. $A$ ), for the full range of $A=1$ to $A=250$ .

The graph for binding energy per nucleon ( $\frac{BE(A,Z)}{A}$ ) is well known from the liquid drop model, described as follows:

$$\frac{BE(A,Z)}{A} = \frac{a_v A - a_s A^{2/3} - a_c \frac{Z(Z-1)}{A^{1/3}} - a_a \frac{(N-Z)^2}{A} + a_p \delta}{A} \quad (1)$$

This function graphed looks as follows:



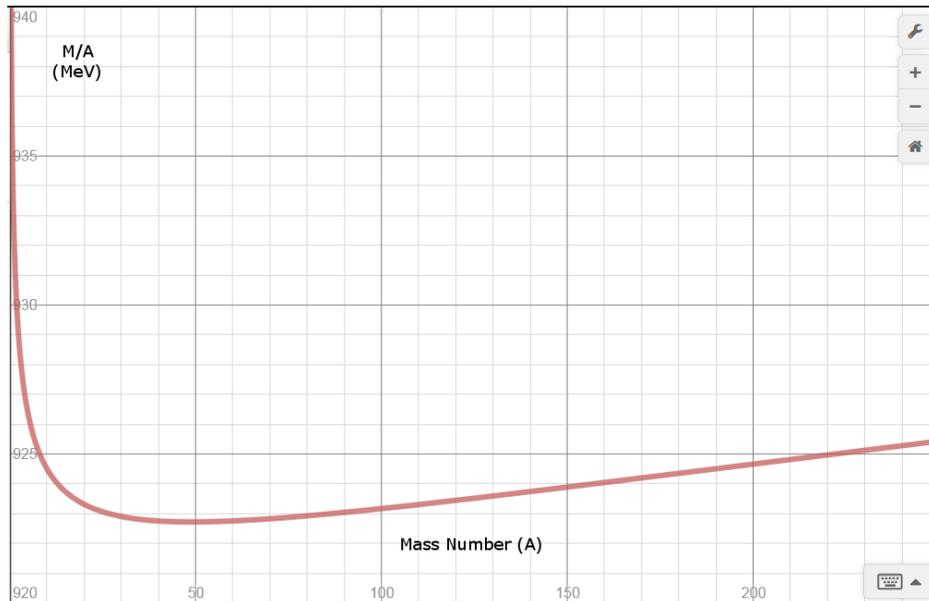
The binding energy and the mass of a nucleus are directly related as follows:

$$BE(A,Z) = ZM_p + (A - Z)M_n - M(A,Z) \quad (2)$$

Therefore, the graph of  $\frac{M(A,Z)}{A}$  should just be the binding energy curve, flipped on the y-axis and shifted up by one, because  $ZM_p + (A - Z)M_n \approx A \left( \frac{931.49 \text{ MeV}}{\text{amu} \cdot c^2} \right)$ , turning Equation 2 into:

$$\frac{BE(A,Z)}{A} = \frac{ZM_p + (A - Z)M_n}{A} - \frac{M(A,Z)}{A} \approx 931.49 - \frac{M(A,Z)}{A}; \quad \frac{M(A,Z)}{A} = 931.49 - \frac{BE(A,Z)}{A} \quad (3)$$

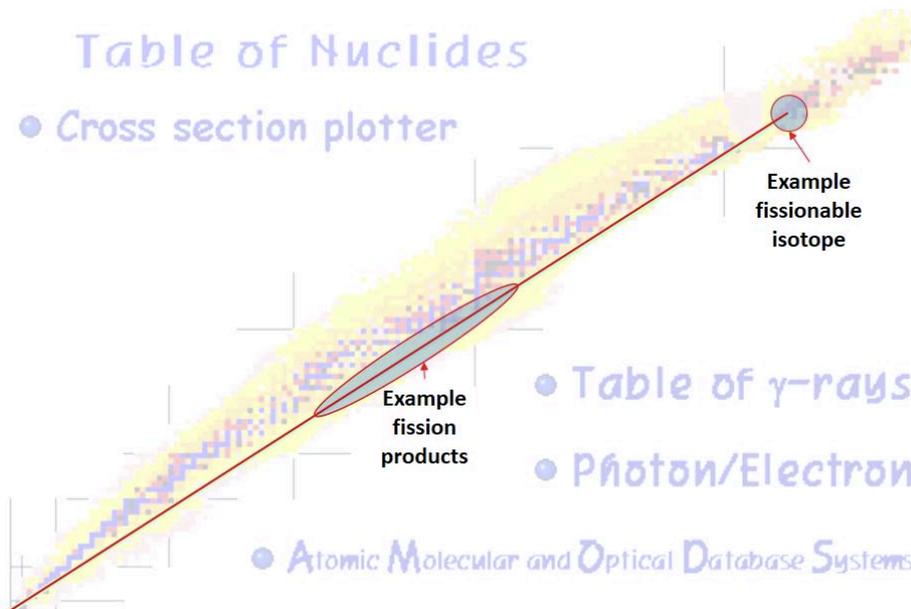
When graphed, it looks as follows:



Points: 9 points for correct shape, 1 point for noting correct-ish y-axis. 5 points partial credit for correct equation relating mass & binding energy with wrong graph.

## 1.2 Explain why most heavy element fission products undergo $\beta^-$ decay.

Heavy isotopes which undergo fission, and their fission products will also be neutron-rich. The only way to “remove” neutrons and “create” protons is to undergo  $\beta^-$  decay. Mathematically, it can be explained by reducing the asymmetry term in the liquid drop model of the nucleus  $\left(\frac{(N-Z)^2}{A}\right)$ . Graphically, one can look at or draw the table of nuclides, noting that the stability line shifts towards neutron-rich isotopes for heavier elements, then draw a straight line between any given fissionable isotope and the origin to plot where the fission products would lie:



Courtesy of Korea Atomic Energy Research Institute. Used with permission.

Clearly these all lie within the unstable, neutron-rich region.

Points: 10 points for the right reason, 5 points partial credit for mentioning the assymetry term or attempting to reason out where fission products would lie with incorrect results.

**1.3 Write a necessary and sufficient condition for a nuclear reaction to occur in terms of its Q-value and the incoming particle’s kinetic energy (KE). What does this say about the combined KEs of the outgoing particles?**

A necessary and sufficient condition would be that the sum of kinetic energy of the incoming particle and the Q-value have to be positive:

$$T_i + Q \geq 0 \tag{4}$$

Otherwise the reaction cannot proceed. This means that the sum of the combined kinetic energies of the outgoing particles must also be positive. In other words, the outgoing particles cannot have negative kinetic energies, as that would require an imaginary velocity, which doesn’t make physical sense.

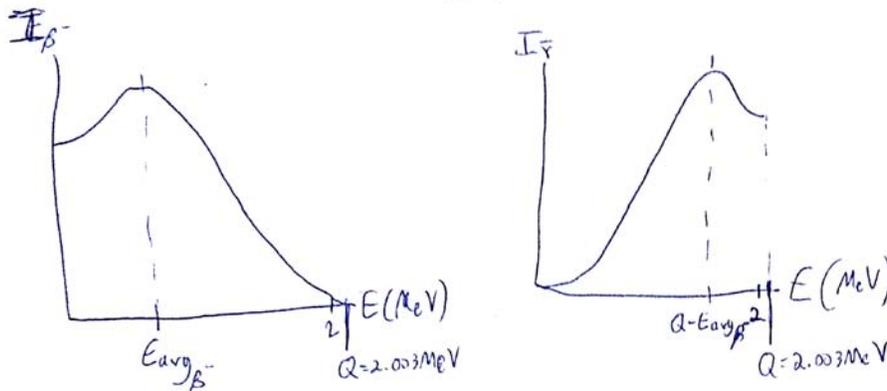
Points: 8 points for the correct condition, 2 points for answering the second question correctly. Partial credit: up to 5 points for a sensible, yet incorrect, condition, including a necessary but insufficient condition, like just saying  $Q > 0$ .

**1.4 Draw a graph showing the intensities of antineutrinos vs. energy from a given  $\beta^-$  decay reaction. Comment on the absolute probabilities of finding  $\beta^-$ s vs. antineutrinos.**

The probability of finding a  $\beta^-$  particle with a given energy for a fixed Q starts non-zero, peaks at some average energy, and goes towards zero as  $E_{\beta^-} \rightarrow Q$ . Due to energy conservation, the antineutrino must carry away all the remaining energy in the equation, assuming we neglect the recoil of the daughter nucleus:

$$E_{\beta^-} + E_{\bar{\nu}} = Q \tag{5}$$

Therefore, the probability of finding a  $\beta^-$  with an energy  $E_{\beta^-}$  is equal to the probability of finding an antineutrino with energy  $Q - E_{\beta^-}$ . This just means that the graph for antineutrinos is the x-mirror image of the beta particle one, as follows:



Points: 9 points for the correct graph, 1 point for saying something like “one will never find an antineutrino with zero energy,” or “antineutrino energies are the mirror image of beta energies.” Up to 5 points for a well-reasoned, but incorrect, graph.

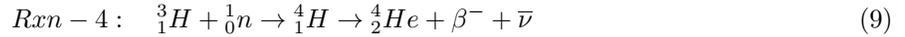
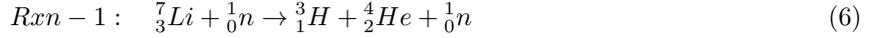
**2 (30 points) Fusion Reactor End Products**

Fusion reactors can be used to generate tritium ( $^3_1H$ ) in their first walls by bombarding lithium-7 ( $^7_3Li$ ) with 14 MeV neutrons produced from fusion [*Rxn 1*], generating their own fuel. However,  $^3_1H$  also decays to

${}^3_2\text{He}$  [Rxn 2], a very useful and very expensive gaseous neutron detector (currently valued at \$53,000/gram). However,  ${}^3_2\text{He}$  and  ${}^3_1\text{H}$  also absorb neutrons [Rxn 3 – 4], so they get destroyed as they are created. Key data for this problem include:

$$\sigma_{c,7-Li} = 10^{-4}b \quad \sigma_{c,3-H} = 10^{-10}b \quad \sigma_{c,3-He} = 10^{-5}b \quad \lambda_{3H} = 1.8 \cdot 10^{-9} \left[\frac{1}{s}\right]$$

**2.1 (8 points) Write the complete nuclear reactions for each step [Rxn 1 – 4] above.**



Points: 2 points per reaction. Rxn-4 should be recognized as producing a remarkably unstable isotope (due to extreme asymmetry) and should undergo beta decay to fix itself.

**2.2 (22 points) Draw a graph of the concentration of  ${}^3_2\text{He}$  in the fusion reactor, assuming it turns on at time  $t = 0$  and shuts off 50 years later.**

First, we set up the equations for the concentration of each isotope, assuming we have some amount of  ${}^7_3\text{Li}$  to begin with:

$$\frac{dLi}{dt} = -N_{Li}\sigma_{c,7-Li}\Phi_{reactor} \quad (\text{Rxn} - 1) \quad (10)$$

$$\frac{dH}{dt} = N_{Li}\sigma_{c,7-Li}\Phi_{reactor} - N_H\sigma_{c,H}\Phi_{reactor} - \lambda_H N_H \quad (\text{Rxn} - 1, 2, 4) \quad (11)$$

$$\frac{dHe-3}{dt} = \lambda_H N_H - N_{He-3}\sigma_{c,He-3}\Phi_{reactor} \quad (\text{Rxn} - 2, 3) \quad (12)$$

$$\frac{dHe-4}{dt} = N_{He-3}\sigma_{c,He-3}\Phi_{reactor} + N_H\sigma_{c,H}\Phi_{reactor} \quad (\text{Rxn} - 3, 4) \quad (13)$$

First, we make an assumption that because  $\sigma_{c,H-3}$  is so small, that it gives a negligible change in concentrations. This reduces the equations to the following:

$$\frac{dLi}{dt} = -N_{Li}\sigma_{c,7-Li}\Phi_{reactor} \quad (14)$$

$$\frac{dH}{dt} = N_{Li}\sigma_{c,7-Li}\Phi_{reactor} - \lambda_H N_H \quad (15)$$

$$\frac{dHe-3}{dt} = \lambda_H N_H - N_{He-3}\sigma_{c,He-3}\Phi_{reactor} \quad (16)$$

$$\frac{dHe-4}{dt} = N_{He-3}\sigma_{c,He-3}\Phi_{reactor} \quad (17)$$

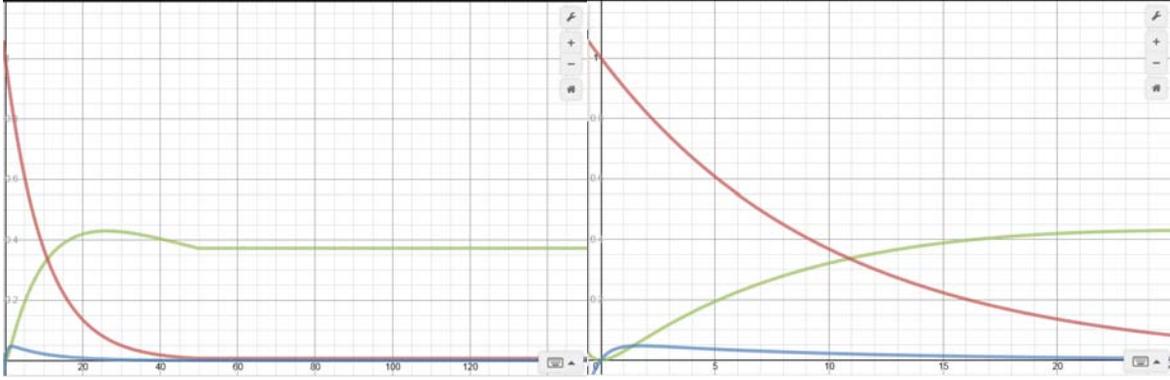
Now is where we look at the relative strengths of different terms.  $\sigma_{c,Li} > \sigma_{c,He-3}$ , so the buildup of  ${}^3\text{H}$  is faster than the burning of  ${}^3\text{He}$ . Also, if we assume some reasonable flux of neutrons in the reactor ( $\Phi_{reactor}$ ) of something like  $10^{18} \frac{\text{neutrons}}{\text{m}^2\text{-sec}}$  (anything within a few orders of magnitude is fine, it doesn't really matter), and recalling that  $1b = 10^{-28}\text{m}^2$ , that makes  $(\sigma_c\Phi)$  much smaller than  $\lambda_H$  than both cases. These equations are **functionally identical** to the Bateman equations that we studied in class, with the addition of one destruction term for  ${}^3\text{He}$ :

$$\frac{dN_1}{dt} = -\lambda_1 N_1 \quad (18)$$

$$\frac{dN_2}{dt} = \lambda_1 N_1 - \lambda_2 N_2 \quad (19)$$

$$\frac{dN_3}{dt} = \lambda_2 N_2 - \lambda_3 N_3 \quad (20)$$

where we start with  $N_{Li_0} = N_{10}$ , " $\lambda_1$ " =  $\sigma_{c,7-Li} \Phi_{\text{reactor}} \approx 10^{-14} \frac{1}{s}$  and " $\lambda_3$ " =  $\sigma_{c,He-3} \Phi_{\text{reactor}} \approx 10^{-15} \frac{1}{s}$ . Using this analogy,  $\lambda_1 \ll \lambda_2$  and  $\lambda_2 \gg \lambda_3$ . Without explicitly solving the equations, we can say that isotope  $N_2$  ( $^3\text{H}$ ) is destroyed almost as soon as it is created, because  $\lambda_1 \ll \lambda_2$ . Therefore, it's like it doesn't even exist! Below is a graph of what the solution should look like, with the  $^3\text{H}$  concentration greatly exaggerated to see its features (graphed to scale, it appears to stay on the x-axis):

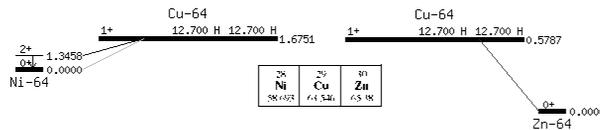


Left: Zoomed-out, full solution. Only the green curve ( $^3\text{He}$ ) is required for full credit, with reasoning. Right: Zoomed-in version, showing that the  $^3\text{He}$  curve starts with a low slope, because it does take a little bit of time for the amount of  $^3\text{H}$  (blue curve) to increase. The red curve is  $^7\text{Li}$ .

Points: Full credit for the correct  $^3\text{He}$  curve shape, including the flat line after shut-off, AND for some reasoning. This reasoning can be graphical, mathematical, or intuitive. Partial credit (up to 18 points) given for correct reasoning and equations, but wrong curve shape. Partial credit (up to 11 points) given for correct equational formulation. Partial credit (up to 6 points) given for just a simple explanation of what it should be, without math or graphing. Partial credit (up to 15 points) given for correctly shaped graph with no explanation or intermediate work. 20 points partial credit for everything correct except the flat line after shutdown.

### 3 (30 points) Decay Chain Diagrams

For these problems, consider the decay of  $^{64}_{29}\text{Cu}$ , which decays by multiple paths as shown below:



**3.1 (8 points) Write the complete nuclear reactions for all possible decays shown, assuming that Ni-64 and Zn-64 are stable isotopes.**



Note that positron decay can only proceed to the ground state, because the Q-value of the decay to the excited state is less than 1.022 MeV.

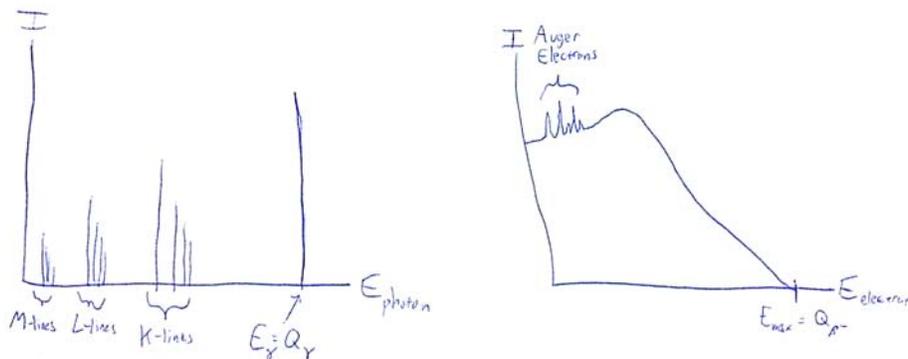
Points: Two points per equation, or one point per partially correct equation.

**3.2 (16 points) Draw complete photon and electron spectra which would be observed from the decay of Cu-64.**

The photon spectrum consists of the only gamma ray ( $E_\gamma = 1.3458 \text{ MeV}$ ) allowed, plus the x-rays released from electrons falling down to more bound states from both electron capture and internal conversion. Let's just assume that we see the K, L, and M transition lines.

The electron spectrum consists of the continuum of beta particles typically observed, with a peak energy of  $Q=578.7 \text{ keV}$ , plus any Auger electrons released as a competing process to electron transitions to more bound orbitals.

Graphs should look something like this:



Points: 3 points for each of: (1) Peak gamma energy, (2), x-ray transition lines, (3) beta spectrum and shape, (4) low-energy Auger electrons, (5) multiple transition lines. One point for knowing that K-lines are more likely than L, and L are more likely than M, because IC & EC tend to interact most strongly with the innermost shell.

**3.3 (6 points) Give an energetic argument why nuclei like Cu-64 can decay by either of the mechanisms shown here, while most isotopes only have one mode of decay.**

Cu-64 is an odd-odd isotope, meaning that it has a highly negative pairing term of its binding energy, making it less stable.  ${}^{64}_{29}\text{Cu}$  has 29 protons and 35 neutrons. It would be far more energetically favorable to become an even-even nucleus. It can do so by either gaining or losing a neutron, while simultaneously losing or gaining a proton, respectively, maintaining constant mass. This would make it either  ${}^{64}_{28}\text{Ni}$ , with 28 protons and 36 neutrons, or  ${}^{64}_{30}\text{Zn}$ , with 30 protons and 34 neutrons. Neither nucleus is that asymmetric, though we would expect that the decay to  ${}^{64}\text{Zn}$  would be more likely, as it is less asymmetric. Most isotopes only have one mode of decay, because there is only one method which can reduce the binding energy of the nucleus. Therefore, only odd-odd nuclei that are rather symmetric in N and Z should have two opposing modes of decay like this.

Points: Partial credit (up to 3 points) for partial reasoning, or a correct energetic argument applied incorrectly.

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Fall 2015

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