22.02 – Introduction to Applied Nuclear Physics

Problem set # 5

Issued on Monday March. 26, 2012. Due on Wednesday April 4, 2012

Problem 1: Coupled representation (Solved Problem)

A nucleus consists of two spin 1/2 nucleons, $s_1 = \frac{1}{2}$, and, $s_2 = \frac{1}{2}$. Both nucleons are in the orbital angular momentum $l = 0$.

a) How many spin states are there for each nucleon?

Solution:

Each nucleon can be in two states $\left|\frac{1}{2}, +\frac{1}{2}\right\rangle = |\uparrow\rangle$ and $\left|\frac{1}{2}, -\frac{1}{2}\right\rangle = |\downarrow\rangle$, since $-s \leq m_s \leq s$.

b) How many spin states does the system have (based on the uncoupled representation)?

Solution:

In the uncoupled representation good quantum numbers correspond to the eigenvalues of the operators $\hat{S}_1^2$, $\hat{S}_2^2$, $\hat{S}_{1,z}$, $\hat{S}_{2,z}$. Since $s_{1,2} = \frac{1}{2}$ while m_s for each particle can take two values, we can list four possible states: $|\uparrow\uparrow\rangle$, $|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$, $|\downarrow\downarrow\rangle$.

c) Which quantum numbers would you use to label the coupled representation states?

Solution:

In the coupled representation a complete set of commuting observables is given by S^2 S_z, S_1^2 and S_2^2, thus good quantum numbers are s, m_s, s_1, s_2 (in this case, we can omit s_1 and s_2 since they're always $\frac{1}{2}$ and write a state as $|s, m_s\rangle$.)

Problem 2: Commutation of angular momentum

a) **(Solved question)** Prove the commutation relation,

$$\left[\hat{L}_1^2, \hat{L}^2\right] = 0$$

and

$$\left[\hat{L}_{1,z}, \hat{L}^2\right] \neq 0$$

where $\hat{\vec{L}} = \hat{\vec{L}}_1 + \hat{\vec{L}}_2$.

Solution:

From the definition of $\hat{\vec{L}}$, we have that the norm of the total angular momentum is $\hat{L}^2 = \hat{L}_1^2 + \hat{L}_2^2 + 2\hat{\vec{L}}_1 \cdot \hat{\vec{L}}_2$. Also, operators acting on different particles always commute, e.g., $[\hat{L}_1^2, \hat{L}_2^2] = 0$, $[\hat{L}_{1,x}, \hat{L}_{2,y}] = 0$, since they are functions of different variables.

Then the first commutator is very easily evaluated:

$$\left[\hat{L}_1^2, \hat{L}^2\right] = \left[\hat{L}_1^2, \hat{L}_1^2 + 2\hat{\vec{L}}_1 \cdot \hat{\vec{L}}_2\right] = 0$$

For the second commutator, we use the fact that $[\hat{L}^2, \hat{L}_z] = 0$ to simplify the result:

$$\left[\hat{L}_{1,z}, \hat{L}^2\right] = \left[\hat{L}_{1,z}, \hat{L}_1^2 + \hat{L}_2^2 + 2\hat{\vec{L}}_1 \cdot \hat{\vec{L}}_2\right] = \left[\hat{L}_{1,z}, \hat{L}_1^2 + 2\hat{\vec{L}}_1 \cdot \hat{\vec{L}}_2\right]$$

$$= \left[\hat{L}_{1,z}, 2\hat{\vec{L}}_1 \cdot \hat{\vec{L}}_2\right] = 2\left[\hat{L}_{1,z}, \hat{L}_{1,x}\hat{L}_{2x} + \hat{L}_{1,y}\hat{L}_{2y} + \hat{L}_{1,z}\hat{L}_{2z}\right] = 2\left[\hat{L}_{1,z}, \hat{L}_{1,x}\hat{L}_{2x}\right] + 2\left[\hat{L}_{1,z}, \hat{L}_{1,y}\hat{L}_{2y}\right]$$

We finally also use the formula $[A, BC] = B[A, C] + [A, B]C$ to find the final result:

$$\left[\widehat{L}_{1,z}, \widehat{L}^2\right] = 2\left[\widehat{L}_{1,z}, \hat{L}_{1,x}\right]\hat{L}_{2x} + 2\left[\widehat{L}_{1,z}, \hat{L}_{1,y}\right]\hat{L}_{2y} = 2i\hbar(\hat{L}_{1y}\hat{L}_{2x} - \hat{L}_{1x}\hat{L}_{2y})$$

b) Prove the commutation relation,

$$\left[\widehat{L}_x, \widehat{L}^2\right] = 0$$

and discuss why the same relation holds for the other, (y, z), components of the angular momentum.

Problem 3: Angular momentum operator

Suppose a system is in the angular momentum state $|7, 4\rangle$, with $l = 7$ and $m_x = 4$.

a) What are the possible measurement results for the x component of angular momentum?

b) What are the possible measurement values for the y component of the angular momentum?

c) Given the uncertainty relationship for angular momentum, $\Delta L_a \Delta L_b \leq \frac{\hbar}{2}\langle L_c\rangle$ (with a, b, c permutations of x, y, z), what are $\langle L_y\rangle$ and $\langle L_z\rangle$ for the state $|7, 4\rangle$? Is this consistent with the result you found above?

d) What is $\Delta L_y = \sqrt{\langle L_y^2\rangle - \langle L_y\rangle^2}$ for the state $|7, 4\rangle$ if we assume $\Delta L_y = \Delta L_z$?

Problem 4: Ladder operators

Consider a system in the state $|l, m_z\rangle$, that is, in an eigenstate of the L_z angular momentum with eigenvalue $\hbar m_z$ and with total angular momentum quantum number is l [i.e. the state is also an eigenstate of L^2 with eigenvalue $\hbar^2 l(l + 1)$].

a) Consider the **Ladder operators** $L_+ = L_x + iL_y$ and $L_- = L_x - iL_y$. What is $L_+ |l, m_z\rangle$? (see lecture notes and Griffiths).
What is then the expectation value of L_+ and L_- for the state considered ($|l, m_z\rangle$)?

b) Using the result you found above, prove that the result you found in Problem 3:c (the value of $\langle L_x\rangle$) is in general true for any eigenstate of L_z, L^2.

Problem 5: Sum of angular momenta

The electron in an hydrogen atom is in the state $\psi(r, \vartheta, \varphi) = R_{21}(r)\left(\frac{1}{\sqrt{3}}Y_1^0(\vartheta, \varphi)|\downarrow\rangle + \sqrt{\frac{2}{3}}Y_1^{-1}(\vartheta, \varphi)|\uparrow\rangle\right)$, where $|\uparrow\rangle = \left|m_s = \frac{1}{2}\right\rangle$ and $|\downarrow\rangle = \left|m_s = -\frac{1}{2}\right\rangle$ are eigenstates of the intrinsic spin with eigenvalues $+\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$ respectively, $Y_l^m(\vartheta, \varphi) = |l, m\rangle$ are eigenfunctions of L^2 and L_z, with quantum numbers l and m and $R_{21}(r)$ is the radial part of the wavefunction .

a) **(Solved question)** Using the sum rules, find the possible values of the quantum number j (which sets the eigenvalue of $\hat{J}^2$ to $\hbar^2 j(j + 1)$), where $\hat{\vec{J}} = \hat{\vec{L}} + \hat{\vec{S}}$ is the total angular momentum.

Solution:

The sum rules state that in the addition of two angular momentum operators, we have that $|l_1 - l_2| \leq l \leq l_1 + l_2$. In this case we have:

$$l - s \leq j \leq l + s \quad \rightarrow \quad 1 - \frac{1}{2} \leq j \leq 1 + \frac{1}{2} \quad \rightarrow \quad \frac{1}{2} \leq j \leq \frac{3}{2}$$

Since the angular momentum quantum number can only increase by integers between the minimum and the maximum value, we have that there are only two possible values for j, $j = \frac{1}{2}$ and $j = \frac{3}{2}$.

b) The wavefunction

$$\psi(r, \vartheta, \varphi) = R_{21}(r)\left(\frac{1}{\sqrt{3}}Y_1^0(\vartheta, \varphi)|\downarrow\rangle + \sqrt{\frac{2}{3}}Y_1^{-1}(\vartheta, \varphi)|\uparrow\rangle\right)$$

can also be written as:

$$\psi(r,\vartheta,\varphi) = R_{21}(r)\left(\frac{2}{3}\sqrt{2}\left|j=\frac{3}{2}, m_j=-\frac{1}{2}, s=\frac{1}{2}, l=1\right\rangle - \frac{1}{3}\left|j=\frac{1}{2}, m_j=-\frac{1}{2}, s=\frac{1}{2}, l=1\right\rangle\right)$$

Which one of these two expression is the coupled representation? Is the second expression consistent with what found in the previous question? How would you find the second expression for ψ from the first one?

c) What are the possible outcomes and probabilities of a measurement of L^2, L_z, S_z, J^2 and J_z ?

d) **(Solved question)** Two p electrons ($l_1 = l_2 = 1$) are in a state with angular momentum $|l, m, l_1, l_2\rangle = |2, -1, 1, 1\rangle$. What are the possible values of m_{1z} and m_{2z}?

Solution:

From the state $|2, -1, 1, 1\rangle$ we know that $l_1 = 1$ and $l_2 = 1$. Thus $m_{1z} = \{-1, 0, 1\}$ and $m_{2z} = \{-1, 0, 1\}$. The values of m_{1z} and m_{2z} must add up to give $m = -1$. We can obtain this result in two ways: either $m_{1z} = -1$ and $m_{2z} = 0$ or vice-versa, $m_{1z} = 0$ and $m_{2z} = -1$. Thus notice that the eigenvalues of L_{1z} and L_{2z} are not known from the coupled representation state: indeed these two operators do not commute with the total angular momentum L^2 so **in general** we cannot know the eigenvalue of L^2 and of L_{1z} and L_{2z} with certainty at the same time.

e) Two p electrons ($l_1 = l_2 = 1$) are in a state with angular momentum $|l, m, l_1, l_2\rangle = |2, -2, 1, 1\rangle$. What are the possible outcomes of a measurement of L_z^1? What are the probabilities of each of these outcomes? What is the joint probability of measuring for both electrons $L_z^1 = L_z^2 = -\hbar$?