22.02 – Introduction to Applied Nuclear Physics

Problem set # 8
Issued on Monday Apr. 23, 2012. Due on Monday Apr. 30, 2012

Problem 1: Wavefunction evolution

Consider a particle in 3D in a potential given by $V(r < R_0) = 0$, $V(r > R_0) = \infty$. At time $t = 0$, the particle is found in the state

$$\psi(\vec{r}; t = 0) = \psi(r, \vartheta, \varphi; t = 0) = \frac{1}{\sqrt{2}} |n = 1, l = 0, m_z = 0\rangle + \frac{1}{2} e^{i\pi/3} |1, 1, -1\rangle + \frac{1}{2} |2, 1, 0\rangle$$

$$= \frac{1}{\sqrt{2}} R_{1,0}(r) Y_0^0(\vartheta, \varphi) + \frac{1}{2} e^{i\pi/3} R_{1,1}(r) Y_{-1}^1(\vartheta, \varphi) + \frac{1}{2} R_{2,0}(r) Y_0^1(\vartheta, \varphi),$$

where $R_{n,l} = u(r)_{n,l}/r$ is the radial eigenfunction (for the infinite well) and $Y_m^l(\vartheta, \varphi)$ the usual spherical harmonics (which are the eigenfunctions of L^2 and L_z).

a) At a time t, when the particle has evolved to the state $\psi(\vec{r}; t)$, we measure the angular momentum L^2. What is the probability of obtaining an outcome $2\hbar^2$ ($l = 1$)? What is the probability of obtaining $6\hbar^2$ ($l = 2$)?

b) What is the expectation value of the angular momentum along the z-axis, $\langle L_z(t) \rangle$ as a function of time?

c) We now assume that the particle initial state is

$$\psi(\vec{r}; t = 0) = \psi(r, \vartheta, \varphi; t = 0) = \frac{1}{\sqrt{2}} |1, 0, 0\rangle + \frac{1}{2} e^{i\pi/3} |3, 0, 0\rangle + \frac{1}{2} |2, 0, 0\rangle$$

$$= \frac{1}{\sqrt{2}} R_{1,0}(r) Y_0^0(\vartheta, \varphi) + \frac{1}{2} e^{i\pi/3} R_{3,0}(r) Y_0^0(\vartheta, \varphi) + \frac{1}{2} R_{2,0}(r) Y_0^0(\vartheta, \varphi),$$

Evolve the system according to the Schrödinger equation to derive $\psi(\vec{r}, t)$ and compute the probability $p(\vec{r}, t) d\vec{r}$ that a measurement at time t of the particle's position finds the particle between $r = \frac{R_0}{2}$ and $r = \frac{R_0}{2} + dr$.

d) What is the probability that a measurement of the energy of the system finds: 1) $E_{n,l} = E_{1,0}$, 2) $E = E_{2,1}$, 3) $E = E_{3,0}$? What is the time evolution of the average energy $\left\langle \hat{E}(t) \right\rangle$?

e) What is the time evolution of the average radial position $\langle r(t) \rangle$?

Problem 2: Gamma Decay: permitted and dominant multipoles

For the following γ transitions give all the permitted multipoles and indicate which multipole might be the most intense in the emitted radiation:

a) $\frac{5}{2}^- \rightarrow \frac{7}{2}^+$ **b)** $4^+ \rightarrow 2^+$ **c)** $\frac{9}{2}^+ \rightarrow \frac{3}{2}^-$

d) $\frac{3}{2}^- \rightarrow \frac{7}{2}^-$ **e)** $3^+ \rightarrow 2^-$ **f)** $2^+ \rightarrow 2^+$

Problem 3: Gamma decay and shell structure

a) Bromine-81 is one of the products of fission reactions. Because of the energy released in the fission, Bromine-81 is usually left in an excited state, from which it can decay by emitting gamma radiation. The observed gamma radiations are as follow:

- M2 with energy $E_\gamma = 260$keV, from the second excited level to the first excited level.
- M1, E2 with energy $E_\gamma = 276$keV, from the first excited level to the ground state.

The ground state and first two excited states of $^{81}_{33}$Br can be determined by considering an unpaired proton in the 5th nuclear shell, between the magic number 28 and 50 (see Krane, Fig. 5.6, also reproduced in the Lecture notes.)

Taking into account the selection rules of gamma decay, determine the spin parity assignment for the ground state and the first two excited states of Bromine-81.

[Hints: i) the three levels are among the 4 possible levels inside the 5th nuclear shell, since jumping to the next shell will imply a much larger energy change. ii) Remember that lower electric multipoles would be favorable, unless they are forbidden because we must have $|I_i - I_f| \le L_\gamma \le I_f + I_i$. iii) The pairing force plays a role in determining which configuration has a lower energy.]

Problem 4: Gamma decay and shell structure (solved)

a) $^{211}_{84}$Po decays by alpha decay. The product of the alpha decay is Lead-207 ($^{207}_{82}$Pb). Based only on filling the levels of the shell model, what is the spin/parity assignment of this nuclide?

Solution:

We have 125 neutrons, with the last unpaired neutron in the i$\frac{13}{2}$ level. Thus the spin should be $\frac{13}{2}$ and the parity $\Pi = (-1)^l$ with $l = 6$, then even parity.

b) Another characteristics of the nuclear force makes the naive spin/parity assignment you gave in question a) improbable. A rearrangement of the nucleons in the shell gives a lower energy: what is the resulting in which spin/parity assignment?

Solution:

The pairing force tends to pair up nucleons of the same type. This pairing lowers the energy and the energy gain is much more pronounced the larger the angular momentum. Thus, we expect that a neutron will jump to the i$\frac{13}{2}$ level to leave only pairs in that level. The remaining unpaired neutron is likely to be in the 3p$\frac{1}{2}$ level (which has very low j and is still close in energy to i$\frac{13}{2}$.) giving a spin parity: $\frac{1}{2}^-$.

c) Following the alpha decay from ^{211}Po, ^{207}Pb is left in an excited state with spin/parity $\frac{3}{2}^-$ and energy 897keV above the ground state. The nucleus decays from this state by gamma emission to the ground state. What are the permitted multipoles and which multipoles are most likely to be observed? Give the approximate ratio of the two most probable decay mode rates.

Solution:

Considering the transition from $\frac{3}{2}^-$ to $\frac{1}{2}^-$, we have no change in parity and possible angular momentum given by $\frac{3}{2} \pm \frac{1}{2} = \{2, 1\}$. Thus we have an E2 or M1 transition. It is known that the ratio between the rates of these two transitions is usually about 10^{-3}. This is confirmed by using the rates given in the lecture notes (and Krane):

$$\frac{\lambda(E2)}{\lambda(M1)} = \frac{7.3 \times 10^7 A^{4/3} E^5}{5.6 \times 10^{13} E^3} \approx 10^{-4}(207)^{4/3} 0.9^2 \approx 1.6 \times 10^{-3}$$

Considering the transition from $\frac{3}{2}^-$ to $\frac{13}{2}^+$, we have change in parity and possible angular momentum given by $\frac{13}{2} - \frac{3}{2} \le l \le \frac{13}{2} + \frac{3}{2} = \{5, 6, 7, 8\}$. Thus we have E5, M6, E7,M8 transitions, with E5 the most probable. It is known that the ratio between the rates of the two strongest transitions (E5 and M6) is usually about 10^{-7}.

d) Detection of gamma rays with lower energies (328keV and 569keV) indicates that there is an intermediate level between the $\frac{3}{2}^-$ excited state and the ground state. The observed multipoles are M1 (associated with the energy 328keV) and E2. With the help of the selection rules and of the shell structure, find the spin/parity of this intermediate level and its energy above the ground state.

Solution:

Because the two transitions have different energies, they must arise from the decay between different energy levels. Thus one will be associated with the decay from the excited to the intermediate level and the second from the intermediate level (first excited level) to the ground state (see also figure 1). Also, you must ensure that both these transitions respect the selection rules in a consistent way when finding the spin-parity of the intermediate level. The transitions

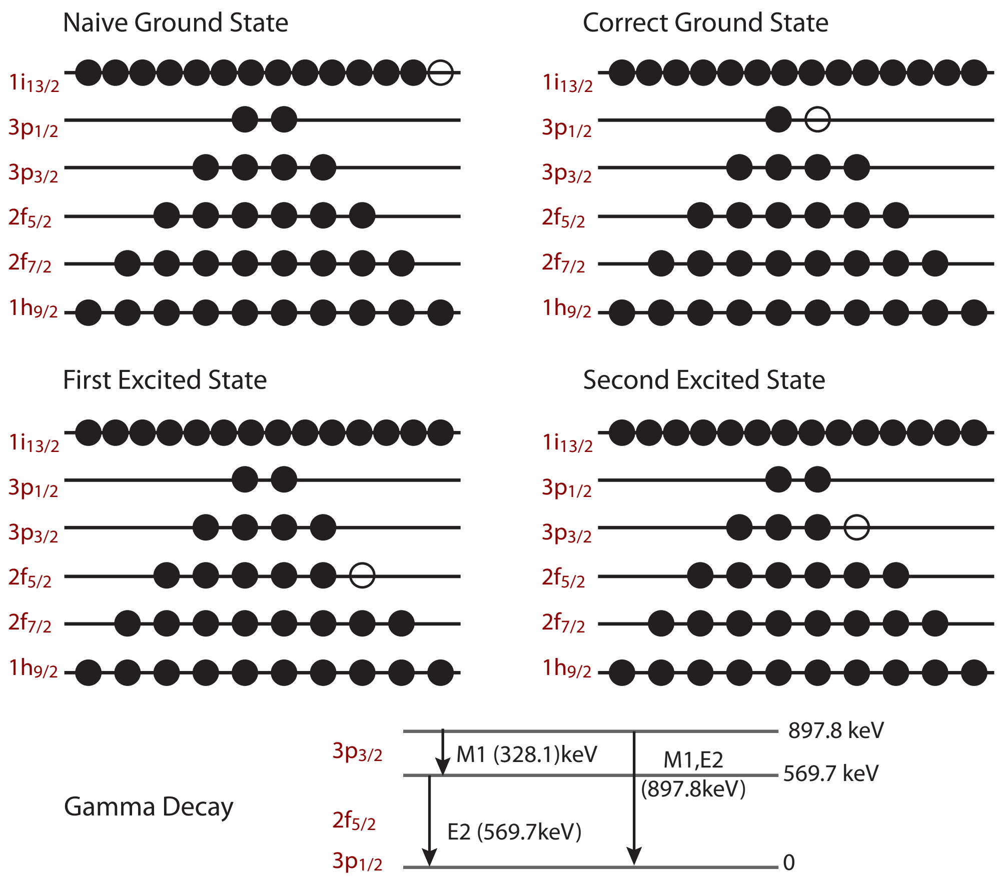

Figure 1: Unclosed shell filling for ground and excited states (open circle indicate a missing neutron). Related gamma decay levels.

observed, E2 and M1, imply no change in parity, thus the parity of the intermediate level must be odd. Thus we are looking for a state with l odd, and we can then focus on the $N = 5$ shell levels, which have l odd, without needing to look in the higher shell. If the M1 transition were from the intermediate level to the ground state $\frac{1}{2}^-$, the level spin parity could only be $\frac{3}{2}^-$, but this is not possible (since we already know that that is the second excited state). Thus, the order of the transitions is M1 from $\frac{3}{2}^-$ to the intermediate level and E2 from that level to the ground state. Finally to have a $\Delta l = 1$ from $\frac{3}{2}^-$ and $\Delta l = 2$ to $\frac{1}{2}^-$ the only possible intermediate level is $2f\frac{5}{2}$. From the gamma energy, this level is predicted to be 569keV above the ground state. *Notice that considering the ground state to be* $\frac{13}{2}^+$ *would have led to a contradictory result here.*

e) Based on your previous answers and what you found in Problem 3, draw a scheme of the unclosed shell for ^{207}Pb showing the level occupancy for the ground state and the two first excited states (involved in the gamma decay) that explain the various spins and parities you found.

Solution:

See Figure 1

Problem 5: Gaussian wavepacket evolution (Solved Problem)

This problem will be solved in recitation.

Consider a free particle in 1D. At time t=0 the particle is well localized, thus it is described by a Gaussian wavepacket:

$$\psi(x, t = 0) = \frac{1}{(2\pi a^2)^{1/4}} e^{-x^2/(4a^2)}$$

a) What are the eigenvalues and eigenfunctions of the Hamiltonian for this free particle?

Solution:

Eigenvalues and eigenfunctions of the system only depend on the Hamiltonian, and not on the particular state of the system.

The free particle Hamiltonian is $\mathcal{H} = \hat{p}^2/2m$. The eigenfunctions of this Hamiltonian are $\psi_k(x) = Ae^{ikx}$ or $\psi_k(x) = A'e^{-ikx}$ with eigenvalues $E_k = \hbar^2 k^2/2m$.

b) Express the wavefunction $\psi(x, t = 0)$ in terms of the energy (Hamiltonian) eigenfunctions.

Use the properties of the Fourier transform of a Gaussian function to find an explicit expression.

Solution:

We can always express the wavefunction as a linear combination of energy eigenfunctions, since they form a basis. The variable k is continuous, so we can express $\psi(x, 0)$ as an integral over the $\psi_k(x)$ instead of a discrete summation,

$$\psi(x,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} c(k) e^{ikx} dk \tag{1}$$

where

$$c(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(0) e^{-ikx} dx$$

since the coefficient are calculated from the inner product of the basis eigenfunction with the wavefunction, $c(k) = \langle e^{ikx} | \psi(x, 0)\rangle$. Note that the expression above is just the inverse Fourier transform of $c(k)$. We can evaluate explicitly $c(k)$ using properties of the Fourier transform of Gaussian functions:

$$\begin{aligned}
c(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{(2\pi a^2)^{1/4}} e^{-x^2/4a^2} e^{-ikx} dx \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{(2\pi a^2)^{1/4}} \sqrt{\pi 4a^2} e^{-a^2k^2} \\
&= 2\left(\frac{a^2}{2\pi}\right)^{1/4} e^{-a^2k^2} \\
\psi(x, t=0) &= \frac{\sqrt{a}}{(2\pi^3)^{1/4}} \int_{-\infty}^{\infty} e^{-a^2k^2} e^{-ikx} dk
\end{aligned}$$

c) Evolve the system according to the Schrödinger equation to derive $\psi(x, t)$ using the results in a) and b) [Notice that this is just the continuous version of the problem above]

Hint:

$$\frac{1}{(2\pi^3)^{1/4}} \int_{-\infty}^{\infty} \sqrt{a} e^{-a^2k^2} e^{i\left(kx - t\frac{k^2\hbar}{2m}\right)} dk = \left(\frac{2a^2}{\pi\left(2a^2 + \frac{it\hbar}{m}\right)^2}\right)^{1/4} e^{-\frac{x^2}{4a^2+2it\hbar/m}}$$

Solution:

We know the evolution of each energy eigenfunction: $e^{ikx} \to \exp(-iE_k t/\hbar)e^{ikx}$. Thus in the linear superposition that represents the wavefunction we simply need to multiply $c(k)$ by $\exp(-iE_k t/\hbar) = \exp(-i\hbar k^2 t/2m)$,

$$\begin{aligned}
\psi(x,t) &= \frac{\sqrt{a}}{(2\pi^3)^{1/4}} \int_{-\infty}^{\infty} e^{-a^2k^2} e^{-i\hbar k^2 t/2m} e^{-ikx} dk \\
&= \left(\frac{2a^2}{\pi\left(2a^2 + \frac{it\hbar}{m}\right)^2}\right)^{1/4} \exp\left(\frac{-x^2}{4a^2 + 2it\hbar/m}\right)
\end{aligned}$$

d) Consider the probability of finding the particle at a position x at time t. You should find a Gaussian distribution $p(x,t) = \frac{1}{\sqrt{2\pi\alpha(t)^2}} e^{-x^2/[2\alpha(t)^2]}$, where $\alpha(t)$ is the width of the wavepacket (or the variance of the position). How does the width evolves in time? Sketch the wavepacket at time $t = 0$ and at a later time $t > 0$.

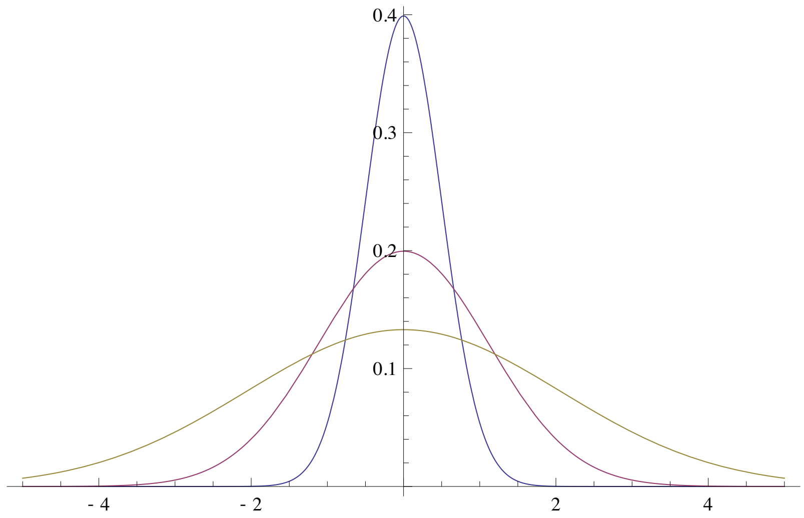

Figure 2: The blue curve is the wavefunction at $t = 0$, the purple and brown curves are at successively later times.

Solution:

$$p(x,t) = |\psi(x,t)|^2 = \left(\frac{2a^2}{\pi\left(2a^2 + \frac{it\hbar}{m}\right)^2}\right)^{1/2} \exp\left(\frac{-x^2}{4a^2 + 2it\hbar/m}\right) \exp\left(\frac{-x^2}{4a^2 - 2it\hbar/m}\right)$$

$$= \left(\frac{2a^2}{\pi\left(2a^2 + \frac{it\hbar}{m}\right)^2}\right)^{1/2} \exp\left(\frac{-2(4a^2)x^2}{(4a^2)^2 + (\frac{2\hbar t}{m})^2}\right)$$

$$= \left(\frac{2a^2}{\pi\left(2a^2 + \frac{it\hbar}{m}\right)^2}\right)^{1/2} \exp\left(-x^2/(2\alpha(t))^2\right)$$

$$\Rightarrow \alpha(t) = \left(a^2 + \frac{\hbar^2 t^2}{4a^2 m^2}\right)^{1/2}$$

The width, $\alpha(t)$, will increase over time. The figure shows the wavefunction spreading out over time.

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