

## 7. Radioactive decay

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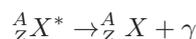
Radioactive decay is the process in which an unstable nucleus spontaneously loses energy by emitting ionizing particles and radiation. This decay, or loss of energy, results in an atom of one type, called the **parent** nuclide, transforming to an atom of a different type, named the **daughter** nuclide.

The three principal modes of decay are called the alpha, beta and gamma decays. We already introduced the general principles of radioactive decay in Section 1.3 and we studied more in depth alpha decay in Section 3.3. In this chapter we consider the other two type of radioactive decay, beta and gamma decay, making use of our knowledge of quantum mechanics and nuclear structure.

### 7.1 Gamma decay

Gamma decay is the third type of radioactive decay. Unlike the two other types of decay, it does not involve a change in the element. It is just a simple decay from an excited to a lower (ground) state. In the process of course some energy is released that is carried away by a photon. Similar processes occur in atomic physics, however there the energy changes are usually much smaller, and photons that emerge are in the visible spectrum or x-rays.

The nuclear reaction describing gamma decay can be written as



where \* indicates an excited state.

We have said that the photon carries away some energy. It also carries away momentum, angular momentum and parity (but no mass or charge) and all these quantities need to be conserved. We can thus write an equation for the energy and momentum carried away by the gamma-photon.

From special relativity we know that the energy of the photon (a massless particle) is

$$E = \sqrt{m^2 c^4 + p^2 c^2} \quad \rightarrow \quad E = pc$$

(while for massive particles in the non-relativistic limit  $v \ll c$  we have  $E \approx mc^2 + \frac{p^2}{2m}$ .) In quantum mechanics we have seen that the momentum of a wave (and a photon is well described by a wave) is  $p = \hbar k$  with  $k$  the wave number. Then we have

$$E = \hbar k c = \hbar \omega_k$$

This is the energy for photons which also defines the frequency  $\omega_k = kc$  (compare this to the energy for massive particles,  $E = \frac{\hbar^2 k^2}{2m}$ ).

Gamma photons are particularly energetic because they derive from nuclear transitions (that have much higher energies than e.g. atomic transitions involving electronic levels). The energies involved range from  $E \sim .1 \div 10\text{MeV}$ , giving  $k \sim 10^{-1} \div 10^{-3}\text{fm}^{-1}$ . Then the wavelengths are  $\lambda = \frac{2\pi}{k} \sim 100 \div 10^4\text{fm}$ , much longer than the typical nuclear dimensions.

Gamma ray spectroscopy is a basic tool of nuclear physics, for its ease of observation (since it's not absorbed in air), accurate energy determination and information on the spin and parity of the excited states. Also, it is the most important radiation used in nuclear medicine.

### 7.1.1 Classical theory of radiation

From the theory of electrodynamics it is known that an accelerating charge radiates. The power radiated is given by the integral of the energy flux (as given by the Poynting vector) over all solid angles. This gives the radiated power as:

$$P = \frac{2}{3} \frac{e^2 |a|^2}{c^3}$$

where  $a$  is the acceleration. This is the so-called Larmor formula for a non-relativistic accelerated charge.

**Example.** As an important example we consider an electric dipole. An electric dipole can be considered as an oscillating charge, over a range  $r_0$ , such that the electric dipole is given by  $d(t) = qr(t)$ . Then the equation of motion is

$$r(t) = r_0 \cos(\omega t)$$

and the acceleration

$$a = \ddot{r} = -r_0 \omega^2 \cos(\omega t)$$

Averaged over a period  $T = 2\pi/\omega$ , this is

$$\langle a^2 \rangle = \frac{\omega}{2\pi} \int_0^T dt a(t) = \frac{1}{2} r_0^2 \omega^4$$

Finally we obtain the radiative power for an electric dipole:

$$P_{E1} = \frac{1}{3} \frac{e^2 \omega^4}{c^3} |\vec{r}_0|^2$$

#### A. Electromagnetic multipoles

In order to determine the classical e.m. radiation we need to evaluate the charge distribution that gives rise to it. The electrostatic potential of a charge distribution  $\rho_e(r)$  is given by the integral:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{V_{ol'}} \frac{\rho_e(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

When treating radiation we are only interested in the potential outside the charge and we can assume the charge (e.g. a particle!) to be well localized ( $r' \ll r$ ). Then we can expand  $\frac{1}{|\vec{r} - \vec{r}'|}$  in power series. First, we express explicitly

the norm  $|\vec{r} - \vec{r}'| = \sqrt{r^2 + r'^2 - 2rr' \cos \vartheta} = r \sqrt{1 + \left(\frac{r'}{r}\right)^2 - 2\frac{r'}{r} \cos \vartheta}$ . We set  $R = \frac{r'}{r}$  and  $\epsilon = R^2 - 2R \cos \vartheta$ : this is a small quantity, given the assumption  $r' \ll r$ . Then we can expand:

$$\frac{1}{|\vec{r} - \vec{r}'|} = \frac{1}{r} \frac{1}{\sqrt{1 + \epsilon}} = \frac{1}{r} \left( 1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 + \dots \right)$$

Replacing  $\epsilon$  with its expression we have:

$$\begin{aligned} \frac{1}{r} \frac{1}{\sqrt{1 + \epsilon}} &= \frac{1}{r} \left( 1 - \frac{1}{2}(R^2 - 2R \cos \vartheta) + \frac{3}{8}(R^2 - 2R \cos \vartheta)^2 - \frac{5}{16}(R^2 - 2R \cos \vartheta)^3 + \dots \right) \\ &= \frac{1}{r} \left( 1 + \left[-\frac{1}{2}R^2 + R \cos \vartheta\right] + \left[\frac{3}{8}R^4 - \frac{3}{2}R^3 \cos \vartheta + \frac{3}{2}R^2 \cos^2 \vartheta\right] + \left[-\frac{5R^6}{16} + \frac{15}{8}R^5 \cos(\vartheta) - \frac{15}{4}R^4 \cos^2(\vartheta) + \frac{5}{2}R^3 \cos^3(\vartheta)\right] + \dots \right) \\ &= \frac{1}{r} \left( 1 + R \cos \vartheta + R^2 \left( \frac{3 \cos^2 \vartheta}{2} - \frac{1}{2} \right) + R^3 \left( \frac{5 \cos^3(\vartheta)}{2} - \frac{3 \cos(\vartheta)}{2} \right) + \dots \right) \end{aligned}$$

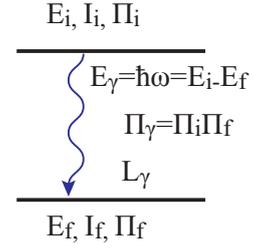


Fig. 42: Schematics of gamma decay

We recognized in the coefficients to the powers of  $R$  the Legendre Polynomials  $P_l(\cos \vartheta)$  (with  $l$  the power of  $R^l$ , and note that for powers  $> 3$  we should have included higher terms in the original  $\epsilon$  expansion):

$$\frac{1}{r} \frac{1}{\sqrt{1+\epsilon}} = \frac{1}{r} \sum_{l=0}^{\infty} R^l P_l(\cos \vartheta) = \frac{1}{r} \sum_{l=0}^{\infty} \left(\frac{r'}{r}\right)^l P_l(\cos \vartheta)$$

With this result we can as well calculate the potential:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{1}{r} \int_{Vol'} \rho(\vec{r}') \frac{1}{r} \sum_{l=0}^{\infty} \left(\frac{r'}{r}\right)^l P_l(\cos \vartheta) d\vec{r}'$$

The various terms in the expansion are the multipoles. The few lowest ones are :

$$\begin{aligned} \frac{1}{4\pi\epsilon_0} \frac{1}{r} \int_{Vol'} \rho(\vec{r}') d\vec{r}' &= \frac{Q}{4\pi\epsilon_0 r} && \text{Monopole} \\ \frac{1}{4\pi\epsilon_0} \frac{1}{r^2} \int_{Vol'} \rho(\vec{r}') r' P_1(\cos \vartheta) d\vec{r}' &= \frac{1}{4\pi\epsilon_0} \frac{1}{r^2} \int_{Vol'} \rho(\vec{r}') r' \cos \vartheta d\vec{r}' = \frac{\hat{r} \cdot \vec{d}}{4\pi\epsilon_0 r^2} && \text{Dipole} \\ \frac{1}{4\pi\epsilon_0} \frac{1}{r^3} \int_{Vol'} \rho(\vec{r}') r'^2 P_2(\cos \vartheta) d\vec{r}' &= \frac{1}{4\pi\epsilon_0} \frac{1}{r^3} \int_{Vol'} \rho(\vec{r}') r'^2 \left(\frac{3}{2} \cos^2 \vartheta - \frac{1}{2}\right) d\vec{r}' && \text{Quadrupole} \end{aligned}$$

This type of expansion can be carried out as well for the magnetostatic potential and for the electromagnetic, time-dependent field.

At large distances, the lowest orders in this expansion are the only important ones. Thus, instead of considering the total radiation from a charge distribution, we can approximate it by considering the radiation arising from the first few multipoles: i.e. radiation from the electric dipole, the magnetic dipole, the electric quadrupole etc.

Each of these radiation terms have a peculiar angular dependence. This will be reflected in the quantum mechanical treatment by a specific angular momentum value of the radiation field associated with the multipole. In turns, this will give rise to selection rules determined by the addition rules of angular momentum of the particles and radiation involved in the radiative process.

### 7.1.2 Quantum mechanical theory

In quantum mechanics, gamma decay is expressed as a transition from an excited to a ground state of a nucleus. Then we can study the transition rate of such a decay via Fermi's Golden rule

$$W = \frac{2\pi}{\hbar} |\langle \psi_f | \hat{V} | \psi_i \rangle|^2 \rho(E_f)$$

There are two important ingredients in this formula, the density of states  $\rho(E_f)$  and the interaction potential  $\hat{V}$ .

#### A. Density of states

The density of states is defined as the number of available states per energy:  $\rho(E_f) = \frac{dN_s}{dE_f}$ , where  $N_s$  is the number of states. We have seen at various time the concept of degeneracy: as eigenvalues of an operator can be degenerate, there might be more than one eigenfunction sharing the same eigenvalues. In the case of the Hamiltonian, when there are degeneracies it means that more than one state share the same energy.

By considering the nucleus+radiation to be enclosed in a cavity of volume  $L^3$ , we have for the emitted photon a wavefunction represented by the solution of a particle in a 3D box that we saw in a Problem Set.

As for the 1D case, we have a quantization of the momentum (and hence of the wave-number  $k$ ) in order to fit the wavefunction in the box. Here we just have a quantization in all 3 directions:

$$k_x = \frac{2\pi}{L} n_x, \quad k_y = \frac{2\pi}{L} n_y, \quad k_z = \frac{2\pi}{L} n_z,$$

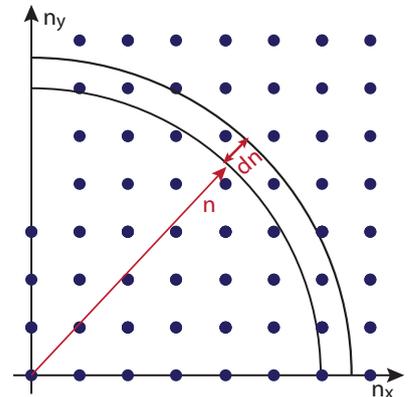


Fig. 43: Density of states: counting the states (2D)

(with  $n$  integers). Then, going to spherical coordinates, we can count the number of states in a spherical shell between  $n$  and  $n + dn$  to be  $dN_s = 4\pi n^2 dn$ . Expressing this in terms of  $k$ , we have  $dN_s = 4\pi k^2 dk \frac{L^3}{(2\pi)^3}$ . If we consider just a small solid angle  $d\Omega$  instead of  $4\pi$  we have then the number of state  $dN_s = \frac{L^3}{(2\pi)^3} k^2 dk d\Omega$ . Since  $E = \hbar kc = \hbar\omega$ , we finally obtain the density of states:

$$\rho(E) = \frac{dN_s}{dE} = \frac{L^3}{(2\pi)^3} k^2 \frac{dk}{dE} d\Omega = \frac{L^3}{(2\pi)^3} \frac{k^2}{\hbar c} d\Omega = \frac{\omega^2}{\hbar c^3} \frac{L^3}{(2\pi)^3} d\Omega$$

## B. The vector potential

Next we consider the potential causing the transition. The interaction of a particle with the e.m. field can be expressed in terms of the vector potential  $\hat{A}$  of the e.m. field as:

$$\hat{V} = \frac{e}{mc} \hat{A} \cdot \hat{p}$$

where  $\hat{p}$  is the particle's momentum. The vector potential  $\hat{A}$  in QM is an operator that can *create* or *annihilate* photons,

$$\hat{A} = \sum_k \sqrt{\frac{2\pi\hbar c^2}{V\omega_k}} (\hat{a}_k e^{i\vec{k}\cdot\vec{r}} + \hat{a}_k^\dagger e^{-i\vec{k}\cdot\vec{r}}) \vec{\epsilon}_k$$

where  $\hat{a}_k$  ( $\hat{a}_k^\dagger$ ) annihilates (creates) one photon of momentum  $\vec{k}$ . Also,  $\vec{\epsilon}_k$  is the polarization of the e.m. field. Since gamma decay (and many other atomic and nuclear processes) is able to create photons (or absorb them) it makes sense that the operator describing the e.m. field would be able to describe the creation and annihilation of photons. The second characteristic of this operator are the terms  $\propto e^{-i\vec{k}\cdot\vec{r}}$  which describe a plane wave, as expected for e.m. waves, with momentum  $\hbar k$  and frequency  $ck$ .

## C. Dipole transition for gamma decay

To calculate the transition rate from the Fermi's Golden rule,

$$W = \frac{2\pi}{\hbar} |\langle \psi_f | \hat{V} | \psi_i \rangle|^2 \rho(E_f),$$

we are really only interested in the matrix element  $\langle \psi_f | \hat{V} | \psi_i \rangle$ , where the initial state does not have any photon, and the final has one photon of momentum  $\hbar k$  and energy  $\hbar\omega = \hbar kc$ . Then, the only element in the sum above for the vector potential that gives a non-zero contribution will be the term  $\propto \hat{a}_k^\dagger$ , with the appropriate  $\vec{k}$  momentum:

$$V_{if} = \frac{e}{mc} \sqrt{\frac{2\pi\hbar c^2}{V\omega_k}} \vec{\epsilon}_k \cdot \langle \hat{p} e^{-i\vec{k}\cdot\vec{r}} \rangle$$

This can be simplified as follow. Remember that  $[\hat{p}^2, \hat{r}] = -2i\hbar\hat{p}$ . Thus we can write,  $\hat{p} = \frac{i}{2\hbar} [\hat{p}^2, \hat{r}] = \frac{im}{\hbar} [\frac{\hat{p}^2}{2m}, \hat{r}] = \frac{im}{\hbar} [\frac{\hat{p}^2}{2m} + V_{nuc}(\hat{r}), \hat{r}]$ . We introduced the nuclear Hamiltonian  $\mathcal{H}_{nuc} = \frac{\hat{p}^2}{2m} + V_{nuc}(\hat{r})$ : thus we have  $\hat{p} = \frac{im}{\hbar} [\mathcal{H}_{nuc}, \hat{r}]$ . Taking the expectation value

$$\langle \psi_f | \hat{p} | \psi_i \rangle = \frac{im}{\hbar} \left( \langle \psi_f | \mathcal{H}_{nuc} \hat{r} | \psi_i \rangle - \langle \psi_f | \hat{r} \mathcal{H}_{nuc} | \psi_i \rangle \right)$$

and remembering that  $|\psi_{i,f}\rangle$  are eigenstates of the Hamiltonian, we have

$$\langle \psi_f | \hat{p} | \psi_i \rangle = \frac{im}{\hbar} (E_f - E_i) \langle \psi_f | \hat{r} | \psi_i \rangle = im\omega_k \langle \psi_f | \hat{r} | \psi_i \rangle,$$

where we used the fact that  $(E_f - E_i) = \hbar\omega_k$  by conservation of energy. Thus we obtain

$$V_{if} = \frac{e}{mc} \sqrt{\frac{2\pi\hbar c^2}{V\omega_k}} im\omega_k \vec{\epsilon}_k \cdot \langle \hat{r} e^{-i\vec{k}\cdot\vec{r}} \rangle = i \sqrt{\frac{2\pi\hbar e^2 \omega_k}{V}} \vec{\epsilon}_k \cdot \langle \hat{r} e^{-i\vec{k}\cdot\vec{r}} \rangle$$

We have seen that the wavelengths of gamma photons are much larger than the nuclear size. Then  $\vec{k} \cdot \vec{r} \ll 1$  and we can make an expansion in series :  $e^{-\vec{k} \cdot \vec{r}} \sim \sum_l \frac{1}{l!} (-i\vec{k} \cdot \vec{r})^l = \sum_l \frac{1}{l!} (-ikr \cos \vartheta)^l$ . This series is very similar in meaning to the multipole series we saw for the classical case.

For example, for  $l = 0$  we obtain:

$$V_{if} = \sqrt{\frac{2\pi\hbar e^2 \omega_k}{V}} \langle \hat{r} \rangle \cdot \vec{\epsilon}_k$$

which is the dipolar approximation, since it can be written also using the electric dipole operator  $e\hat{r}$ .

The angle between the polarization of the e.m. field and the position  $\hat{r}$  is  $\langle \hat{r} \rangle \cdot \vec{\epsilon} = \langle \hat{r} \rangle \sin \vartheta$

The transition rate for the dipole radiation,  $W \equiv \lambda(E1)$  is then:

$$\lambda(E1) = \frac{2\pi}{\hbar} |\langle \psi_f | \hat{V} | \psi_i \rangle|^2 \rho(E_f) = \frac{\omega^3}{2\pi c^3 \hbar} |\langle \hat{r} \rangle|^2 \sin^2 \vartheta d\Omega$$

and integrating over all possible direction of emission ( $\int_0^{2\pi} d\varphi \int_0^\pi (\sin^2 \vartheta) \sin \vartheta d\vartheta = 2\pi \frac{4}{3}$ ):

$$\lambda(E1) = \frac{4}{3} \frac{e^2 \omega^3}{\hbar c^3} |\langle \hat{r} \rangle|^2$$

Multiplying the transition rate (or photons emitted per unit time) by the energy of the photons emitted we obtain the radiated power,  $P = W\hbar\omega$ :

$$P = \frac{4}{3} \frac{e^2 \omega^4}{c^3} |\langle \hat{r} \rangle|^2$$

Notice the similarity of this formula with the classical case:

$$P_{E1} = \frac{1}{3} \frac{e^2 \omega^4}{c^3} |\vec{r}_0|^2$$

We can estimate the transition rate by using a typical energy  $E = \hbar\omega$  for the photon emitted (equal to a typical energy difference between excited and ground state nuclear levels) and the expectation value for the dipole ( $|\langle \hat{r} \rangle| \sim R_{nuc} \approx r_0 A^{1/3}$ ). Then, the transition rate is evaluated to be

$$\lambda(E1) = \frac{e^2}{\hbar c} \frac{E^3}{(\hbar c)^3} r_0^2 A^{2/3} = 1.0 \times 10^{14} A^{2/3} E^3$$

(with  $E$  in MeV). For example, for  $A = 64$  and  $E = 1\text{MeV}$  the rate is  $\lambda \approx 1.6 \times 10^{15} s^{-1}$  or  $\tau = 10^{-15}$  (femtoseconds!) for  $E = 0.1\text{MeV}$   $\tau$  is on the order of picoseconds.

**Obs.** Because of the large energies involved, very fast processes are expected in the nuclear decay from excited states, in accordance with Fermi's Golden rule and the energy/time uncertainty relation.

### 7.1.3 Extension to Multipoles

We obtained above the transition rate for the electric dipole, i.e. when the interaction between the nucleus and the e.m. field is described by an electric dipole and the emitted radiation has the character of electric dipole radiation. This type of radiation can only carry out of the nucleus one quantum of angular momentum (i.e.  $\Delta l = \pm 1$ , between excited and ground state). In general, excited levels differ by more than 1  $l$ , thus the radiation emitted need to be a higher multipole radiation in order to conserve angular momentum.

#### A. Electric Multipoles

We can go back to the expansion of the radiation interaction in multipoles:

$$\hat{V} \sim \sum_l \frac{1}{l!} (i\vec{k} \cdot \hat{r})^l$$

Then the transition rate becomes:

$$\lambda(E1) = \frac{8\pi(l+1)}{l[(2l+1)!!]^2} \frac{e^2}{\hbar c} \left(\frac{E}{\hbar c}\right)^{2l+1} \left(\frac{3}{l+3}\right)^2 c \langle |\hat{r}| \rangle^{2l}$$

Notice the strong dependence on the  $l$  quantum number. Setting again  $|\langle \hat{r} \rangle| \sim r_0 A^{1/3}$  we also have a strong dependence on the mass number.

Thus, we have the following estimates for the rates of different electric multipoles:

- $\lambda(E1) = 1.0 \times 10^{14} A^{2/3} E^3$
- $\lambda(E2) = 7.3 \times 10^7 A^{4/3} E^5$
- $\lambda(E3) = 34 A^2 E^7$
- $\lambda(E4) = 1.1 \times 10^{-5} A^{8/3} E^9$

## B. Magnetic Multipoles

The e.m. potential can also contain magnetic interactions, leading to magnetic transitions. The transition rates can be calculated from a similar formula:

$$\lambda(Ml) = \frac{8\pi(l+1)}{l[(2l+1)!!]^2} \frac{e^2}{\hbar c} \frac{E^{2l+1}}{\hbar c} \left(\frac{3}{l+3}\right)^2 c \langle |\hat{r}| \rangle^{2l-2} \left[ \frac{\hbar}{m_p c} \left( \mu_p - \frac{1}{l+1} \right) \right]$$

where  $\mu_p$  is the magnetic moment of the proton (and  $m_p$  its mass).

Estimates for the transition rates can be found by setting  $\mu_p - \frac{1}{l+1} \approx 10$ :

- $\lambda(M1) = 5.6 \times 10^{13} E^3$
- $\lambda(M2) = 3.5 \times 10^7 A^{2/3} E^5$
- $\lambda(M3) = 16 A^{4/3} E^7$
- $\lambda(M4) = 4.5 \times 10^{-6} A^2 E^9$

### 7.1.4 Selection Rules

The angular momentum must be conserved during the decay. Thus the difference in angular momentum between the initial (excited) state and the final state is carried away by the photon emitted. Another conserved quantity is the total parity of the system.

#### A. Parity change

The parity of the gamma photon is determined by its character, either magnetic or electric multipole. We have

$$\Pi_\gamma(El) = (-1)^l \quad \text{Electric multipole}$$

$$\Pi_\gamma(Ml) = (-1)^{l-1} \quad \text{Magnetic multipole}$$

Then if we have a parity change from the initial to the final state  $\Pi_i \rightarrow \Pi_f$  this is accounted for by the emitted photon as:

$$\Pi_\gamma = \Pi_i \Pi_f$$

This of course limits the type of multipole transitions that are allowed given an initial and final state.

$$\Delta\Pi = \text{no} \rightarrow \text{Even Electric, Odd Magnetic}$$

$$\Delta\Pi = \text{yes} \rightarrow \text{Odd Electric, Even Magnetic}$$

#### B. Angular momentum

From the conservation of the angular momentum:

$$\hat{I}_i = \hat{I}_f + \hat{L}_\gamma$$

the allowed values for the angular momentum quantum number of the photon,  $l$ , are restricted to

$$l_\gamma = |I_i - I_f|, \dots, I_i + I_f$$

Once the allowed  $l$  have been found from the above relationship, the character (magnetic or electric) of the multipole is found by looking at the parity.

In general then, the most important transition will be the one with the lowest allowed  $l$ ,  $\Pi$ . Higher multipoles are also possible, but they are going to lead to much slower processes.

Multipolarity	Angular Momentum $l$	Parity $\Pi$	Multipolarity	Angular Momentum $l$	Parity $\Pi$
M1	1	+	E1	1	-
M2	2	-	E2	2	+
M3	3	+	E3	3	-
M4	4	-	E4	4	+
M5	5	+	E5	5	-

Table 3: Angular momentum and parity of the gamma multipoles

### C. Dominant Decay Modes

In general we have the following predictions of which transitions will happen:

1. The lowest permitted multipole dominates
2. Electric multipoles are more probable than the same magnetic multipole by a factor  $\sim 10^2$  (however, which one is going to happen depends on the parity)

$$\frac{\lambda(El)}{\lambda(Ml)} \approx 10^2$$

3. Emission from the multipole  $l + 1$  is  $10^{-5}$  times less probable than the  $l$ -multipole emission.

$$\frac{\lambda(E, l+1)}{\lambda(El)} \approx 10^{-5}, \quad \frac{\lambda(M, l+1)}{\lambda(Ml)} \approx 10^{-5}$$

4. Combining 2 and 3, we have:

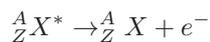
$$\frac{\lambda(E, l+1)}{\lambda(Ml)} \approx 10^{-3}, \quad \frac{\lambda(M, l+1)}{\lambda(El)} \approx 10^{-7}$$

Thus  $E2$  competes with  $M1$  while that's not the case for  $M2$  vs.  $E1$

### D. Internal conversion

What happens if no allowed transitions can be found? This is the case for even-even nuclides, where the decay from the  $0^+$  excited state must happen without a change in angular momentum. However, the photon always carries some angular momentum, thus gamma emission is impossible.

Then another process happens, called **internal conversion**:

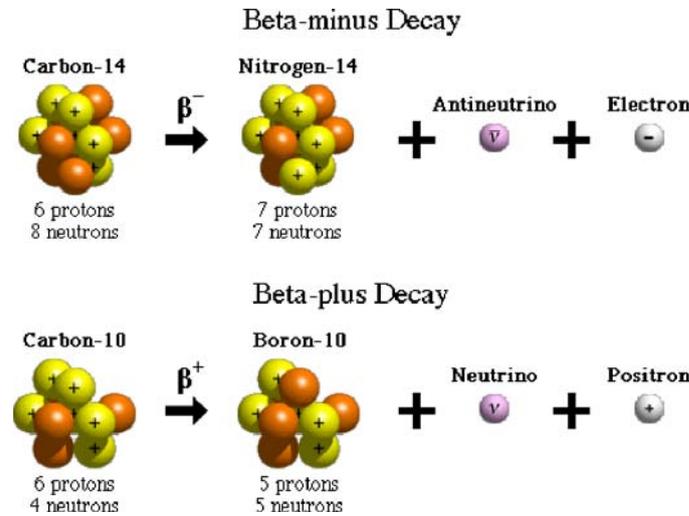


where  ${}^A_Z X$  is a ionized state and  $e^-$  is one of the atomic electrons.

Besides the case of even-even nuclei, internal conversion is in general a competing process of gamma decay (see Krane for more details).

## 7.2 Beta decay

The beta decay is a radioactive decay in which a proton in a nucleus is converted into a neutron (or vice-versa). In the process the nucleus emits a beta particle (either an electron or a positron) and quasi-massless particle, the **neutrino**.



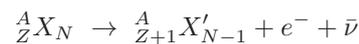
Courtesy of Thomas Jefferson National Accelerator Facility - Office of Science Education. Used with permission.

Fig. 44: Beta decay schematics

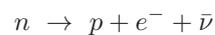
Recall the mass chain and Beta decay plots of Fig. 7. When studying the binding energy from the SEMF we saw that at fixed  $A$  there was a minimum in the nuclear mass for a particular value of  $Z$ . In order to reach that minimum, unstable nuclides undergo beta decay to transform excess protons in neutrons (and vice-versa).

### 7.2.1 Reactions and phenomenology

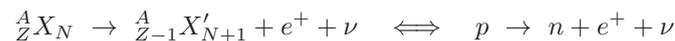
The beta-decay reaction is written as:



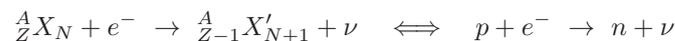
This is the  $\beta^-$  decay. (or negative beta decay) The underlying reaction is:



which converts a proton into a neutron with the emission of an electron and an anti-neutrino. There are two other types of reactions, the  $\beta^+$  reaction,

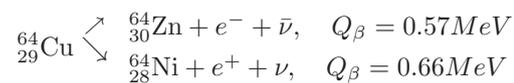


which sees the emission of a positron (the electron anti-particle) and a neutrino; and the electron capture:



a process that competes with, or substitutes, the positron emission.

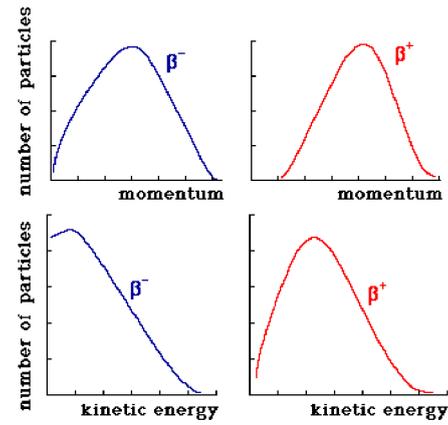
### Examples



The neutrino and beta particle ( $\beta^\pm$ ) share the energy.

Since the neutrinos are very difficult to detect (as we will see they are almost massless and interact very *weakly* with matter), the electrons/positrons are the particles detected in beta-decay and they present a characteristic energy spectrum (see Fig. 45). The difference between the spectrum of the  $\beta^\pm$  particles is due to the Coulomb repulsion or attraction from the nucleus.

Notice that the neutrinos also carry away angular momentum. They are spin-1/2 particles, with no charge (hence the name) and very small mass. For many years it was actually believed to have zero mass. However it has been confirmed that it does have a mass in 1998.



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Fig. 45: Beta decay spectra: Distribution of momentum (top plots) and kinetic energy (bottom) for  $\beta^-$  (left) and  $\beta^+$  (right) decay.

## 7.2.2 Conservation laws

As the neutrino is hard to detect, initially the beta decay seemed to violate energy conservation. Introducing an extra particle in the process allows one to respect conservation of energy. Besides energy, there are other conserved quantities:

- **Energy:** The  $Q$  value of a beta decay is given by the usual formula:

$$Q_{\beta^-} = [m_N(^A X) - m_N(^A_{Z+1} X') - m_e]c^2.$$

Using the atomic masses and neglecting the electron's binding energies as usual we have

$$Q_{\beta^-} = \{[m_A(^A X) - Zm_e] - [m_A(^A_{Z+1} X') - (Z+1)m_e] - m_e\}c^2 = [m_A(^A X) - m_A(^A_{Z+1} X')]c^2.$$

The kinetic energy (equal to the  $Q$ ) is shared by the neutrino and the electron (we neglect any recoil of the massive nucleus). Then, the emerging electron (remember, the only particle that we can really observe) does not have a fixed energy, as it was for example for the gamma photon. But it will exhibit a spectrum of energy (which is the number of electron at a given energy) as well as a distribution of momenta. We will see how we can reproduce these plots by analyzing the QM theory of beta decay.

- **Momentum:** The momentum is also shared between the electron and the neutrino. Thus the observed electron momentum ranges from zero to a maximum possible momentum transfer.

- **Angular momentum** (both the electron and the neutrino have spin 1/2)

- **Parity?** It turns out that parity is not conserved in this decay. This hints to the fact that the interaction responsible violates parity conservation (so it cannot be the same interactions we already studied, e.m. and strong interactions)

- **Charge** (thus the creation of a proton is for example always accompanied by the creation of an electron)

- **Lepton number:** we do not conserve the total number of particles (we create beta and neutrinos). However the number of massive, heavy particles (or baryons, composed of 3 quarks) is conserved. Also the lepton number is conserved. Leptons are fundamental particles (including the electron, muon and tau, as well as the three types of neutrinos associated with these 3). The lepton number is +1 for these particles and -1 for their antiparticles. Then an electron is always accompanied by the creation of an antineutrino, e.g., to conserve the lepton number (initially zero).

## 7.2.3 Fermi's Theory of Beta Decay

The properties of beta decay can be understood by studying its quantum-mechanical description via Fermi's Golden rule, as done for gamma decay.

$$W = \frac{2\pi}{\hbar} |\langle \psi_f | \hat{V} | \psi_i \rangle|^2 \rho(E_f)$$

In gamma decay process we have seen how the e.m. field is described as an operator that can create (or destroy) photons. Nobody objected to the fact that we can create these massless particles. After all, we are familiar with charged particles that produce (create) an e.m. field. However in QM photons are also particles, and by analogy we can have also creation of other types of particles, such as the electron and the neutrino.

For the beta decay we need another type of interaction that is able to create massive particles (the electron and neutrino). The interaction cannot be given by the e.m. field; moreover, in the light of the possibilities of creating and annihilating particles, we also need to find a new description for the particles themselves that allows these processes. All of this is obtained by **quantum field theory** and the second quantization. Quantum field theory gives a unification of e.m. and weak force (electro-weak interaction) with one coupling constant  $e$ .

The interaction responsible for the creation of the electron and neutrino in the beta decay is called the **weak interaction** and its one of the four fundamental interactions (together with gravitation, electromagnetism and the strong interaction that keeps nucleons and quarks together). One characteristic of this interaction is parity violation.

### A. Matrix element

The weak interaction can be written in terms of the particle field wavefunctions:

$$V_{int} = g\Psi_e^\dagger\Psi_\nu^\dagger$$

where  $\Psi_a$  ( $\Psi_a^\dagger$ ) annihilates (creates) the particle  $a$ , and  $g$  is the coupling constant that determines how strong the interaction is. Remember that the analogous operator for the e.m. field was  $\propto a_k^\dagger$  (creating one photon of momentum  $k$ ).

Then the matrix element

$$V_{if} = \langle\psi_f|\mathcal{H}_{int}|\psi_i\rangle$$

can be written as:

$$V_{if} = g \int d^3\vec{x} \Psi_p^*(\vec{x}) [\Psi_e^*(\vec{x})\Psi_\nu^*(\vec{x})] \Psi_n(\vec{x})$$

(Here  $\dagger \rightarrow *$  since we have scalar operators).

To first approximation the electron and neutrino can be taken as plane waves:

$$V_{if} = g \int d^3\vec{x} \Psi_p^*(\vec{x}) \frac{e^{i\vec{k}_e\cdot\vec{x}}}{\sqrt{V}} \frac{e^{i\vec{k}_\nu\cdot\vec{x}}}{\sqrt{V}} \Psi_n(\vec{x})$$

and since  $kR \ll 1$  we can approximate this with

$$V_{if} = \frac{g}{V} \int d^3\vec{x} \Psi_p^*(\vec{x})\Psi_n(\vec{x})$$

We then write this matrix element as

$$V_{if} = \frac{g}{V} M_{np}$$

where  $M_{np}$  is a very complicated function of the nuclear spin and angular momentum states. In addition, we will use in the Fermi's Golden Rule the expression

$$|M_{np}|^2 \rightarrow |M_{np}|^2 F(Z_0, Q_\beta)$$

where the Fermi function  $F(Z_0, Q_\beta)$  accounts for the Coulomb interaction between the nucleus and the electron that we had neglected in the previous expression (where we only considered the weak interaction).

### B. Density of states

In studying the gamma decay we calculated the density of states, as required by the Fermi's Golden Rule. Here we need to do the same, but the problem is complicated by the fact that there are two types of particles (electron and neutrino) as products of the reaction and both can be in a continuum of possible states.

Then the number of states in a small *energy volume* is the product of the electron and neutrino's states:

$$d^2 N_s = dN_e dN_\nu.$$

The two particles share the  $Q$  energy:

$$Q_\beta = T_e + T_\nu.$$

For simplicity we assume that the mass of the neutrino is zero (it's much smaller than the electron mass and of the kinetic mass of the neutrino itself). Then we can take the relativistic expression

$$T_\nu = cp_\nu,$$

while for the electron

$$E^2 = p^2 c^2 + m^2 c^4 \quad \rightarrow \quad E = T_e + m_e c^2 \quad \text{with} \quad T_e = \sqrt{p_e^2 c^2 + m_e^2 c^4} - m_e c^2$$

and we then write the kinetic energy of the neutrino as a function of the electron's,

$$T_\nu = Q_\beta - T_e.$$

The number of states for the electron can be calculated from the quantized momentum, under the assumption that the electron state is a free particle ( $\psi \sim e^{i\vec{k}\cdot\vec{r}}$ ) in a region of volume  $V = L^3$ :

$$dN_e = \left(\frac{L}{2\pi}\right)^3 4\pi k_e^2 dk_e = \frac{4\pi V}{(2\pi\hbar)^3} p_e^2 dp_e$$

and the same for the neutrino,

$$dN_\nu = \frac{4\pi V}{(2\pi\hbar)^3} p_\nu^2 dp_\nu,$$

where we used the relationship between momentum and wavenumber:  $\vec{p} = \hbar\vec{k}$ .

At a given momentum/energy value for the electron, we can write the density of states as

$$\rho(p_e) dp_e = dN_e \frac{dN_\nu}{dT_\nu} = 16\pi^2 \frac{V^2}{(2\pi\hbar)^6} p_e^2 dp_e p_\nu^2 \frac{dp_\nu}{dT_\nu} = \frac{V^2}{4\pi^4 \hbar^6 c^3} [Q - T_e]^2 p_e^2 dp_e$$

where we used :  $\frac{dT_\nu}{dp_\nu} = c$  and  $p_\nu = (Q_\beta - T_e)/c$ .

The density of states is then

$$\rho(p_e) dp_e = \frac{V^2}{4\pi^4 \hbar^6 c^3} [Q - T_e]^2 p_e^2 dp_e = \frac{V^2}{4\pi^4 \hbar^6 c^3} [Q - (\sqrt{p_e^2 c^2 + m_e^2 c^4} - m_e c^2)]^2 p_e^2 dp_e$$

or rewriting this expression in terms of the electron kinetic energy:

$$\rho(T_e) = \frac{V^2}{4\pi^4 \hbar^6 c^3} [Q - T_e]^2 p_e^2 \frac{dp_e}{dT_e} = \frac{V^2}{4c^6 \pi^4 \hbar^6} [Q - T_e]^2 \sqrt{T_e^2 + 2T_e m_e c^2} (T_e + m_e c^2)$$

(as  $p_e dp_e = (T_e + m_e c^2)/c^2 dT_e$ )

Knowing the density of states, we can calculate how many electrons are emitted in the beta decay *with a given energy*. This will be proportional to the rate of emission calculated from the Fermi Golden Rule, times the density of states:

$$N(p) = CF(Z, Q) |V_{fi}|^2 \frac{p^2}{c^2} [Q - T]^2 = CF(Z, Q) |V_{fi}|^2 \frac{p^2}{c^2} \left[ Q - (\sqrt{p_e^2 c^2 + m_e^2 c^4} - m_e c^2) \right]^2$$

and

$$N(T_e) = \frac{C}{c^5} F(Z, Q) |V_{fi}|^2 [Q - T_e]^2 \sqrt{T_e^2 + 2T_e m_e c^2} (T_e + m_e c^2)$$

These distributions are nothing else than the *spectrum* of the emitted beta particles (electron or positron).

In these expression we collected in the constant  $C$  various parameters deriving from the Fermi Golden Rule and density of states calculations, since we want to highlight only the dependence on the energy and momentum. Also, we introduced a new function,  $F(Z, Q)$ , called the Fermi function, that takes into account the *shape* of the nuclear wavefunction and in particular it describes the Coulomb attraction or repulsion of the electron or positron from the nucleus. Thus,  $F(Z, Q)$  is different, depending on the type of decay.

These distributions were plotted in Fig. 45. Notice that these distributions (as well as the decay rate below) are the product of three terms:

- the Statistical factor (arising from the density of states calculation),  $\frac{p^2}{c^2} [Q - T]^2$
- the Fermi function (accounting for the Coulomb interaction),  $F(Z, Q)$
- and the Transition amplitude from the Fermi Golden Rule,  $|V_{fi}|^2$

These three terms reflect the three ingredients that determine the spectrum and decay rate of in beta decay processes.

### C. Decay rate

The decay rate is obtained from Fermi's Golden rule:

$$W = \frac{2\pi}{\hbar} |V_{if}|^2 \rho(E)$$

where  $\rho(E)$  is the total density of states.  $\rho(E)$  (and thus the decay rate) is obtained by summing over all possible states of the beta particle, as counted by the density of states. Thus, in practice, we need to integrate the density of states over all possible momentum of the outgoing electron/positron. Upon integration over  $p_e$  we obtain:

$$\rho(E) = \frac{V^2}{4\pi^4 \hbar^6 c^3} \int_0^{p_e^{max}} dp_e [Q - T_e]^2 p_e^2 \approx \frac{V^2}{4\pi^4 \hbar^6 c^3} \frac{(Q - mc^2)^5}{30c^3}$$

(where we took  $T_e \approx pc$  in the relativistic limit for high electron speed).

We can finally write the decay rate as:

$$\begin{aligned} W &= \frac{2\pi}{\hbar} |V_{if}|^2 \rho(E) = \frac{2\pi}{\hbar} \left(\frac{g}{V}\right)^2 |M_{np}|^2 F(Z, Q_\beta) \frac{V^2}{4\pi^4 \hbar^6 c^3} \frac{(Q - mc^2)^5}{30c^3} \\ &= G_F^2 |M_{np}|^2 F(Z, Q_\beta) \frac{(Q - mc^2)^5}{60\pi^3 \hbar (\hbar c)^6} \end{aligned}$$

where we introduced the constant

$$G_F = \frac{1}{\sqrt{2}} \frac{g m_e^2 c}{\pi^3 \hbar^3}$$

which gives the strength of the weak interaction. Comparing to the strength of the electromagnetic interaction, as given by the fine constant  $\alpha = \frac{e^2}{\hbar c} \sim \frac{1}{137}$ , the weak interaction is much smaller, with a constant  $\sim 10^{-6}$ .

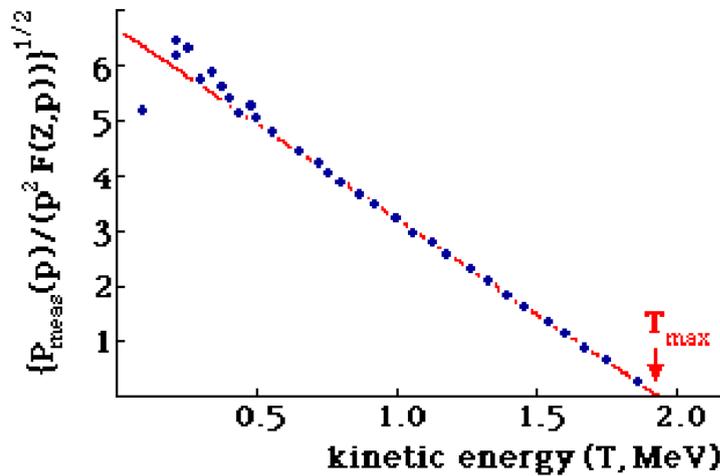
We can also write the differential decay rate  $\frac{dW}{dp_e}$ :

$$\frac{dW}{dp_e} = \frac{2\pi}{\hbar} |V_{if}|^2 \rho(p_e) \propto F(Z, Q) [Q - T_e]^2 p_e^2$$

The square root of this quantity is then a linear function in the neutrino kinetic energy,  $Q - T_e$ :

$$\sqrt{\frac{dW}{dp_e} \frac{1}{p_e^2 F(Z, Q)}} \propto Q - T_e$$

This is the Fermi-Kurie relation. Usually, the Fermi-Kurie plot is used to infer by linear regression the maximum electron energy (or  $Q$ ) by finding the straight line intercept.



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Fig. 46: Example of Fermi-Kurie plot (see also Krane, Fig. 9.4, 9.5)

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