Fourier Transform

The Fourier transform is a generalization of the complex Fourier series. The complex Fourier Series is an expansion of a periodic function (periodic in the interval $[-L/2, L/2]$) in terms of an infinite sum of complex exponential:

$$\sum_{n=-\infty}^{\infty} A_n e^{2i\pi nx/L} \tag{1}$$

where the coefficients A_n are:

$$A_n = \frac{1}{L}\int_{-L/2}^{L/2} f(x) e^{-2i\pi nx/L} dx. \tag{2}$$

Note that this expansion of a periodic function is equivalent to using the exponential functions $u_n(x) = e^{2i\pi nx/L}$ as a *basis* for the function vector space of periodic functions. The coefficient of each "vector" in the basis are given by the coefficient A_n. Accordingly, we can interpret equation (2) as the *inner product* $\langle u_n(x)|f(x)\rangle$.

In the limit as $L \to \infty$ the sum over n becomes an integral. The discrete coefficients A_n are replaced by the continuous function $F(k)dk$ where $k = n/L$. Then in the limit ($L \to \infty$) the equations defining the Fourier series become :

$$f(x) = \int_{-\infty}^{\infty} F(k) e^{2\pi ikx} dk \tag{3}$$

$$F(k) = \int_{-\infty}^{\infty} f(x) e^{-2\pi ikx} dx. \tag{4}$$

Here,

$$F(k) = F_x[f(x)](k) = \int_{-\infty}^{\infty} f(x) e^{-2\pi ikx} dx$$

is called the *forward* Fourier transform, and

$$f(x) = F_k^{-1}[F(k)](x) = \int_{-\infty}^{\infty} F(k) e^{2\pi ikx} dk$$

is called the *inverse* Fourier transform.
The notation $F_x[f(x)](k)$ is common but $\hat{f}(k)$ and $\tilde{f}(x)$ are sometimes also used to denote the Fourier transform.

In physics we often write the transform in terms of angular frequency $\omega = 2\pi\nu$ instead of the oscillation frequency ν (thus for example we replace $2\pi k \to k$). To maintain the symmetry between the forward and inverse transforms, we will then adopt the convention

$$F(k) = F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

$$f(x) = F^{-1}[F(k)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k) e^{ikx} dk.$$

Sine-Cosine Fourier Transform

Since any function can be split up into even and odd portions E(x) and O(x),

$$f(x) = \frac{1}{2}[f(x) + f(-x)] + \frac{1}{2}[f(x) - f(-x)] = E(x) + O(x),$$

a Fourier transform can always be expressed in terms of the Fourier cosine transform and Fourier sine transform as

$$F_x[f(x)](k) = \int_{-\infty}^{\infty} E(x)\cos(2\pi kx)dx - i\int_{-\infty}^{\infty} O(x)\sin(2\pi kx)dx.$$

Properties of the Fourier Transform

• The smoother a function (i.e., the larger the number of continuous derivatives), the more compact its Fourier transform.
• The Fourier transform is linear, since if f(x) and g(x) have Fourier transforms F(k) and G(k) , then

$$\int [af(x)+bg(x)]e^{-2\pi ikx}dx = a\int_{-\infty}^{\infty} f(x)e^{-2\pi ikx}dx + b\int_{-\infty}^{\infty} g(x)e^{-2\pi ikx}dx = aF(k)+bG(k).$$

Therefore,

$$F[af(x)+bg(x)] = aF[f(x)] + bF[g(x)] = aF(k)+bG(k).$$

• The Fourier transform is also symmetric since $F(k) = F_x[f(x)](k)$ implies $F(-k) = F_x[f(-x)](k)$.
• The Fourier transform of a derivative f'(x) of a function f(x) is simply related to the transform of the function f(x) itself. Consider

$$F_x[f'(x)](k) = \int_{-\infty}^{\infty} f'(x)e^{-2\pi ikx}dx.$$

Now use integration by parts

$$\int vdu = [uv] - \int udv$$

with

$$du = f'(x)dx \qquad v = e^{-2\pi ikx}$$

and

$$u = f(x) \qquad dv = -2\pi ike^{-2\pi ikx}dx,$$

then

$$F_x[f'(x)](k) = \left[f(x)e^{-2\pi ikx}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)\left(-2\pi ike^{-2\pi ikx}\right)dx$$

The first term consists of an oscillating function times f(x) . But if the function is bounded so that

$$\lim_{x-\rightarrow\pm\infty} f(x) = 0$$

(as any physically significant signal must be), then the term vanishes, leaving

$$F_x[f'(x)](k) = 2\pi ik\int_{-\infty}^{\infty} f(x)e^{-2\pi ikx}dx = 2\pi ikF_x[f(x)](k).$$

This process can be iterated for the n^{th} derivative to yield

$$F_x[f^{(n)}(x)](k) = (2\pi ik)^n F_x[f(x)](k).$$

• If $f(x)$ has the Fourier transform $F_x[f(x)](k) = F(k)$, then the Fourier transform has the shift property

$$\int_{-\infty}^{\infty} f(x-x_0)e^{-2\pi ikx}dx = \int_{-\infty}^{\infty} f(x-x_0)e^{-2\pi i(x-x_0)k}e^{-2\pi i(kx_0)}d(x-x_0) = e^{-2\pi ikx_0}F(k),$$

so $f(x-x_0)$ has the Fourier transform

$$F_x[f(x-x_0)](k) = e^{-2\pi ikx_0}F(k).$$

• If $f(x)$ has a Fourier transform $F_x[f(x)](k) = F(k)$, then the Fourier transform obeys a similarity theorem.

$$\int_{-\infty}^{\infty} f(ax)e^{-2\pi ikx}dx = 1/(|a|)\int_{-\infty}^{\infty} f(ax)e^{-2\pi i(ax)(k/a)}d(ax) = 1/(|a|)F(k/a),$$

so $f(ax)$ has the Fourier transform

$$F_x[f(ax)](k) = |a|^{-1}F(k/a).$$

• Any operation on $f(x)$ which leaves its area unchanged leaves $F(0)$ unchanged, since

$$\int_{-\infty}^{\infty} f(x)dx = F_x[f(x)](0) = F(0).$$

Table of common Fourier transform pairs.

Function	$f(x)$	$F(k) = F_x[f(x)](k)$
Constant	1	$\delta(k)$
Delta function	$\delta(x - x_0)$	$e^{-2\pi ikx_0}$
Cosine	$\cos(2\pi k_0 x)$	$\frac{1}{2}[\delta(k - k_0) + \delta(k + k_0)]$
Sine	$\sin(2\pi k_0 x)$	$\frac{1}{2}i[\delta(k + k_0) - \delta(k - k_0)]$
Exponential function	$e^{-2\pi k_0 \lvert x \rvert}$	$\frac{1}{\pi}\frac{k_0}{k^2 + k_0^2}$
Gaussian	e^{-ax^2}	$\sqrt{\frac{\pi}{a}}e^{-\pi^2 k^2/a}$
Heaviside step function	$H(x)$	$\frac{1}{2}[\delta(k) - i/(\pi k)]$
Lorentzian function	$\frac{1}{\pi}\frac{\Gamma/2}{(x - x_0)^2 + (\Gamma/2)^2}$	$e^{-2\pi ikx_0}e^{-\Gamma\pi\lvert k \rvert}$

In two dimensions, the Fourier transform becomes

$$F(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(k_x, k_y)e^{-2\pi i(k_x x + k_y y})dk_x dk_y f(k_x, k_y)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x, y)e^{2\pi i(k_x x + k_y y})dxdy.$$

Similarly, the n -dimensional Fourier transform can be defined for k , x in $\mathbb{R}^n$ by

$$F(x) = \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} f(k)e^{-2\pi ik\cdot x}d^n k f(k) = \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} F(x)e^{2\pi ik\cdot x}d^n x.$$