22.058, Principles of Medical Imaging
Fall 2002
Homework #3

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1. Define aliasing, bandwidth limiting and the Nyquist condition.

The Nyquist condition states that for a frequency to be correctly measured it must be sampled twice per period.

Aliasing is the result of not following the Nyquist condition. An aliased frequency appears at the difference of its true frequency and half of the Nyquist frequency.

Bandwidth limiting is used to restrict the set of frequencies in a measurement to those that satisfy the Nyquist condition. Frequencies that are higher than half the Nyquist frequency are attenuated (filtered out).

2. Using your knowledge of Fourier convolution, calculate the Fourier transforms of the following functions and draw both the real and imaginary spectra. k_0 is a real number.

   a. $\cos^2(k_0 z)$

$$\underbrace{\cos(k_o z)}_{\Updownarrow} \bullet \cos(k_o z)$$

$$\pi\left[\delta(k-k_o)+\delta(k+k_o)\right] \otimes \pi\left[\delta(k-k_o)+\delta(k+k_o)\right]$$

$$= \pi^2\left[\delta(k-2k_o)+2\delta(k)+\delta(k+2k_o)\right]$$

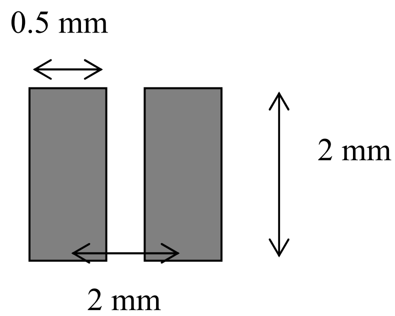
   b. $\sin^3(k_0 z)$

$$\underbrace{\sin(k_o z)}_{\Updownarrow} \bullet \sin(k_o z) \bullet \sin(k_o z)$$

$$\pi i\left[-\delta(k-k_o)+\delta(k+k_o)\right] \otimes \underbrace{\pi i\left[-\delta(k-k_o)+\delta(k+k_o)\right] \otimes \pi i\left[-\delta(k-k_o)+\delta(k+k_o)\right]}_{-\pi^2\left[\delta(k)-\delta(k-2k_o)-\delta(k-2k_o)\right]}$$

$$= -\pi^3 i\left[\delta(k+3k_o)-3\delta(k+k_o)+3\delta(k-k_o)-\delta(k-3k_o)\right]$$

3. The spot size of an X-ray source typically looks like 2 rectangles. Below is a schematic representation of a X-ray source. We expect that the image resolution will depend on this source distribution. Recall that we typically assume that the source is a infinitesimal point source radiating in all directions, here the source is a distributed source with each infinitesimal element radiating in all directions.



a. Describe how you would use a pin-hole camera to measure the spot size. In your analysis forget about the off axis effects (no oblique angle correction).

Place pin-hole parallel to surface of the source a distance, a, away and line up the pinhole with the center of the source. Place a photographic plate parallel to the pin-hole a distance, b, further from the source.

b. Given a detector with 1 mm x 1 mm spatial resolution and that you desire to characterize the spot size to a resolution of 100 µm, how will you set up the measurement (distances from source to pin-hole to detector, size of pin-hole).

Detector is 1mm x 1mm and source resolution is 100 µm, therefore need a gain of at least x 10. Thus, $b/a \geq 10$.

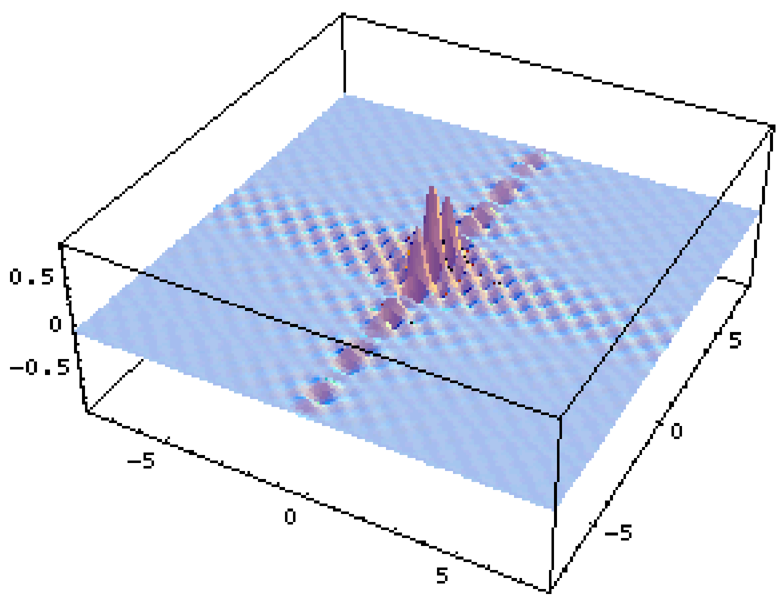
A small pinhole is about 50µm. The blurring at the detector is $\left(\frac{a+b}{a}\right)50\mu m$ which should be less than 1mm. So we see that b/a of 10 is fine.

c. What is the 2-dimensional Fourier Transform of the 2-dimensional function describing the above X-ray source distribution? Draw this and label the axes.

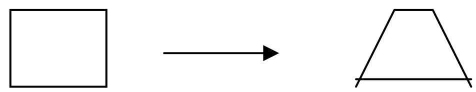
QuickTime™ and a
None decompressor
are needed to see this picture.

$$f(x,y) = \underbrace{TopHat(y)}_{\Updownarrow} \quad \bullet \quad \underbrace{\left[TopHat(4x) \otimes \left[\delta(x-1) + \delta(x+1)\right]\right]}_{\Updownarrow}$$

$$2\,\mathrm{sinc}(k_y) \quad \bullet \quad \frac{1}{2}\,\mathrm{sinc}\left(\frac{k_x}{4}\right)\cos(k_x)$$



d. The Fourier Transform of the X-ray source has pronounced oscillations. To remove these oscillations one can file down the edges which replaces a sharp edge by a triangular edge:



Show that in k-space the trapezoid function falls off faster than the TopHat function with increasing k (wave-number). You do not have to calculate the actual Fourier Transform to answer this.

To go from the original function to the desired function, convolve the original function with a narrower TopHat.

In the Fourier domain,

$$F\left\{TopHat\left(x/3\right)\otimes TopHat(x)\right\}$$
$$=6sinc(3k)\bullet 2sinc(k)$$
$$=\frac{12\sin(3k)\sin(k)}{k^2}$$

so this falls off as k^2 *rather than* k.