Lecture 1

Basics of Electrostatics

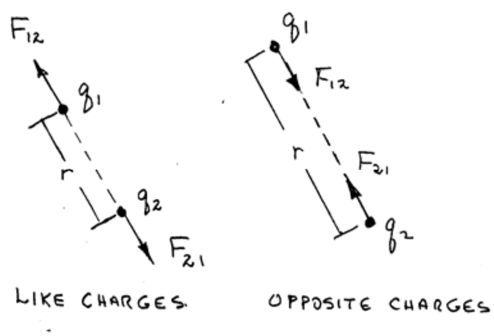
Introduction

1. The first topic of interest is "electrostatics"
2. The topic covered today include
    a. The Coulomb force
    b. Definition of the electric field
    c. Gauss's law
    d. Poisson's equation
    e. Electrostatics in integral and differential form
    f. A simple problem
3. What is electrostatics?
    a. DC behavior – no time variation or waves
    b. No magnetic field or currents
    c. Study of the behavior of stationary electric charges and the resulting electric fields
    d. Electrostatics can be formulated in an integral form. This is elegant but usually not very useful except in special geometries with simple boundary conditions
    e. Electrostatics can be formulated in differential form. This is much more useful for actually solving problems. The solution techniques also apply to many other areas of physics (e.g. mechanics, thermodynamics, aeronautics, chemical engineering, etc.)

The basis of electrostatics

1. The basis of electrostatics is the Coulomb force between two charged particles.

2. Four our purposes we should view the Coulomb force as an experimentally determined relationship based on a large number of observations. It is a basic postulate.
3. Coulombs law is shown below



4. The force is directed along the line of centers – that is, along r
5. The force on charge 1 due to charge 2 is proportional to $q_1 q_2$. Similarly, the force on charge 2 due to charge 1 is proportional to $q_2 q_1$. Therefore it follows that $|F_{12}| = |F_{21}|$.
6. The force is inversely proportional to the distance between the charges.
7. Coulomb's law in SI units is given by

$$\mathbf{F}_{12} = -\mathbf{F}_{21} = (\pm\, ?)\frac{q_1 q_2}{4\pi\varepsilon_0 r^2}\mathbf{e}_r$$

$$\varepsilon_0 = 8.85 \times 10^{-12} \ Farads / m$$

$$e = 1.6 \times 10^{-19} \ Coulombs$$

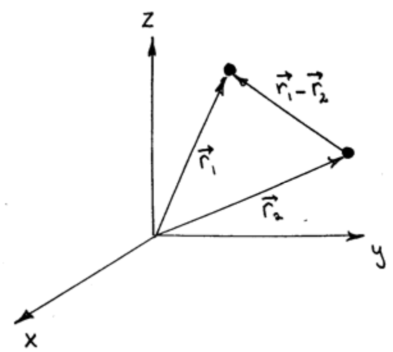
$$\mathrm{F} = Newtons$$

8. How do we determine the right sign? The correct sign is based on the observation that like charges repel and opposite charges attract.
9. Let's set up a coordinate system where the various geometric vectors have the directions illustrated below

$$\mathbf{r}_1 = x_1\mathbf{e}_x + y_1\mathbf{e}_y + z_1\mathbf{e}_z$$

$$\mathbf{r}_2 = x_2\mathbf{e}_x + y_2\mathbf{e}_y + z_2\mathbf{e}_z$$

$$\mathbf{r}_{12} = \mathbf{r}_1 - \mathbf{r}_2$$



10. From this diagram it follows that

$$\mathbf{F}_{12} = -\mathbf{F}_{21} = +\frac{q_1 q_2}{4\pi\varepsilon_0}\frac{\mathbf{r}_{12}}{r_{12}^3}$$

11. When q_1 and q_2 have the same sign, the force on q_1 due to q_2 is along $+\mathbf{r}_{12}$

Definition of the electric field

1. A useful way to think about the force is to imagine that each charge produces a "force field" which acts on any charges present.
2. Therefore, by definition charge q_2 produces a force field $\mathbf{E}$ over all space defined by

$$\mathbf{E}(\mathbf{r}) = \frac{q_2}{4\pi\varepsilon_0}\frac{\mathbf{r}-\mathbf{r}_2}{|\mathbf{r}-\mathbf{r}_2|^3}$$

3. To check this assume a charge q_1 is placed at $\mathbf{r} = \mathbf{r}_1$. Then, as expected, the force due to charge 2 on charge 1 is given by

$$\mathbf{F}_{12} = q_1\mathbf{E} = \frac{q_1 q_2}{4\pi\varepsilon_0}\frac{\mathbf{r}_1-\mathbf{r}_2}{|\mathbf{r}_1-\mathbf{r}_2|^3}$$

4. The introduction of $\mathbf{E}$ would not be very useful if we only considered situations with two charges
5. The electric field becomes useful when we consider the interaction of many charges and make use of the powerful principle that the net force on any charge is obtained by linear superposition.
6. Superposition cannot be proved but is found to be an excellent experimental observation, valid on all but the sub-atomic scale.
7. What exactly does linear superposition mean? Consider a system with three charges. Then the force on charge 1 due to charges 2 and 3 is given by

$$\mathbf{F}_1 = \mathbf{F}_{12} + \mathbf{F}_{13}$$

8. The forces simply add together.
9. If the force were

$$\mathbf{F}_1 = \frac{F_{12}^2\mathbf{F}_{12} + F_{13}^2\mathbf{F}_{13}}{F_{12}^2 + F_{13}^2}$$

   then superposition would not be valid.

10. The conclusion from this discussion is that if a large number of charges is present then the force on a given charge q due to all the other charges can be written as

$$\mathbf{F} = q\mathbf{E}$$
$$\mathbf{E} = \sum_i \frac{q_i}{4\pi\varepsilon_0}\frac{\mathbf{r} - \mathbf{r}_i}{\left|\mathbf{r} - \mathbf{r}_i\right|^3}$$

Continuous charge distributions

1. Clearly it is impractical as well as useless to determine the electric field by summing over 10^{20} individual charges.

2. For the common situation involving a large number of charges it is convenient and highly accurate to replace the 10^{20} charges with a smooth, smeared out, continuous distribution of charge, called the charge density, having the units $Coulomb / m^3$
3. Physically, the electric field outside the charge distribution cannot depend on the precise location of any individual charge. The same is true for the electric field within the charge distribution if there are enough total charges present so that the net field due to the bulk of charges dominates the field from a few nearest neighbors.
4. How do we make the transition from discrete charges to a continuous distribution? The procedure is as follows.

$$\sum_i \quad \rightarrow \quad \int$$
$$q_i \quad \rightarrow \quad \rho(x',y',z')dx'dy'dz' = \rho d\mathbf{r}'$$

5. Here ρ is the charge density measured in $Coulombs / m^3$
6. Therefore, the definition of the electric field $\mathbf{E}$ for a continuous distribution of charges is given by

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int \rho(\mathbf{r}')\frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}d\mathbf{r}'$$

The scalar potential

1. So far we have been talking about forces and the fact that the electric field $\mathbf{E}$ is a useful way to describe these forces.
2. The specific inverse square law dependence of the Coulomb force allows us to introduce a scalar potential. This function is very convenient in terms of mathematical simplification when we want to solve actual problems.
3. The potential $\phi(\mathbf{r})$ is introduced by noting the following relationship

a. $$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{1/2}}$$

b. $$\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \left(\mathbf{e}_x\frac{\partial}{\partial x} + \mathbf{e}_y\frac{\partial}{\partial y} + \mathbf{e}_z\frac{\partial}{\partial z}\right)\frac{1}{\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{1/2}}$$

$$= -\frac{(x-x')\mathbf{e}_x + (y-y')\mathbf{e}_y + (z-z')\mathbf{e}_z}{\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{3/2}}$$

$$= -\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}$$

4. Therefore it follows that

$$\mathbf{E}(\mathbf{r}) = -\frac{1}{4\pi\varepsilon_0}\int \rho(\mathbf{r}')\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}'$$

5. Note that the ∇ comes outside the integral since it is a function of the unprimed coordinates, implying that

$$\mathbf{E}(\mathbf{r}) = -\frac{1}{4\pi\varepsilon_0}\nabla\int \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}'$$

6. From this form we see that we can introduce a scalar potential $\phi(\mathbf{r})$ defined by

$$\mathbf{E}(\mathbf{r}) = -\nabla\phi(\mathbf{r})$$

where

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}'$$

7. This relation determines the potential function in terms of the charge density.

Gauss's law

1. The potential relation given above is known as Gauss' law. It is an integral form of electrostatics.
2. This form is useful if we know, by one way or another, the charge distribution $\rho(\mathrm{r}')$. A straightforward integration then yields $\phi(r)$.
3. Gauss's law is also valid when conductors are present. However, it is usually not very useful in these situations because we do not know the charge distributions on the conductors. Instead, we most often know the potentials on the surfaces of the conductors.
4. When the potentials are specified we shall see that the differential form of electrostatics is a more useful formulation.
5. The last point to note is the vector identity $\nabla \times \nabla(scalar) = 0$. It then follows that in electrostatics $\nabla \times \mathrm{E} = \nabla \times (\nabla\phi) = 0$. The conclusion is that any electrostatic electric field is curl free.

The differential form of electrostatics

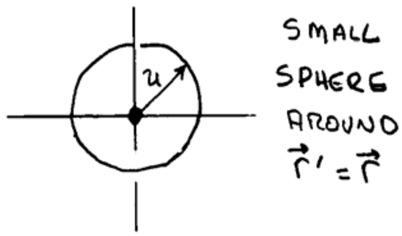
1. The differential form of electrostatics can be derived from Gauss's law by noting the following identity which is valid for $\mathrm{r} \neq \mathrm{r}'$

$$\begin{aligned}
\nabla^2 \frac{1}{|\mathrm{r}-\mathrm{r}'|} &= \nabla \cdot \nabla \frac{1}{|\mathrm{r}-\mathrm{r}'|} = -\nabla \cdot \frac{(x-x')\mathrm{e}_x + (y-y')\mathrm{e}_y + (z-z')\mathrm{e}_z}{\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{3/2}} \\
&= -\frac{\partial}{\partial x}\frac{x-x'}{|\mathrm{r}-\mathrm{r}'|^3} - \frac{\partial}{\partial y}\frac{y-y'}{|\mathrm{r}-\mathrm{r}'|^3} - \frac{\partial}{\partial z}\frac{z-z'}{|\mathrm{r}-\mathrm{r}'|^3} \\
&= -\frac{3}{|\mathrm{r}-\mathrm{r}'|^3} + \frac{3}{2}\left[\frac{2(x-x')^2 + 2(y-y')^2 + 2(z-z')^2}{|\mathrm{r}-\mathrm{r}'|^5}\right] \\
&= 0!!
\end{aligned}$$

2. We have to be careful near $\mathbf{r} = \mathbf{r}'$ since there is a delta function contribution. This can be seen as follows using the divergence theorem.

$$\int \nabla^2 \frac{1}{|\mathbf{r}-\mathbf{r}'|} d\mathbf{r}' = \int \nabla'^2 \frac{1}{|\mathbf{r}-\mathbf{r}'|} d\mathbf{r}' = \int \nabla' \cdot \nabla' \frac{1}{|\mathbf{r}-\mathbf{r}'|} d\mathbf{r}' = \int \mathbf{n}' \cdot \nabla' \frac{1}{|\mathbf{r}-\mathbf{r}'|} dS'$$

3. We can evaluate the integral by choosing the volume of interest to be a small sphere centered around $\mathbf{r} = \mathbf{r}'$ and taking the limit as the radius of the sphere approaches zero



4. For this choice the differential surface element becomes $dS' = u'^2 \sin\theta' d\theta' d\phi'$ where $u' = |\mathbf{r}-\mathbf{r}'|$. The integral becomes

$$\begin{aligned}\int \nabla^2 \frac{1}{|\mathbf{r}-\mathbf{r}'|} d\mathbf{r}' &= \int \mathbf{n}' \cdot \nabla' \frac{1}{|\mathbf{r}-\mathbf{r}'|} dS' \\ &= \int \mathbf{e}_{u'} \cdot \nabla' \frac{1}{|\mathbf{r}-\mathbf{r}'|} u'^2 \sin\theta' d\theta' d\phi' \\ &= \int \frac{\partial}{\partial u'}\left(\frac{1}{u'}\right) u'^2 \sin\theta' d\theta' d\phi' \\ &= -\int \sin\theta' d\theta' d\phi' \\ &= -4\pi\end{aligned}$$

5. Therefore we see that

$$\nabla^2 \frac{1}{|\mathrm{r} - \mathrm{r}'|} = 0 \qquad \text{for } \mathrm{r}' \neq \mathrm{r}$$

$$\int \nabla^2 \frac{1}{|\mathrm{r} - \mathrm{r}'|} d\mathrm{r}' = -4\pi \quad \text{for } \mathrm{r}' \to \mathrm{r}$$

6. This implies that

$$\nabla^2 \frac{1}{|\mathrm{r} - \mathrm{r}'|} = -4\pi\delta(x - x')\delta(y - y')\delta(z - z') = -4\pi\delta(\mathrm{r} - \mathrm{r}')$$

7. We can now obtain the differential form of Gauss's law as follows

   a. $\phi = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\mathrm{r}')}{|\mathrm{r} - \mathrm{r}'|} d\mathrm{r}'$
   b. $\mathrm{E} = -\nabla\phi = -\frac{1}{4\pi\varepsilon_0} \int \rho(\mathrm{r}') \nabla \frac{1}{|\mathrm{r} - \mathrm{r}'|} d\mathrm{r}'$
   c. $\nabla \cdot \mathrm{E} = -\frac{1}{4\pi\varepsilon_0} \int \rho(\mathrm{r}') \nabla^2 \frac{1}{|\mathrm{r} - \mathrm{r}'|} d\mathrm{r}'$
   d. $\nabla \cdot \mathrm{E} = -\frac{1}{4\pi\varepsilon_0} \int \rho(\mathrm{r}') [-4\pi\delta(\mathrm{r} - \mathrm{r}')] d\mathrm{r}'$
   e. $\nabla \cdot \mathrm{E} = \frac{\rho(\mathrm{r})}{\varepsilon_0}$

8. This is the differential form of electrostatics
9. If we now set $\mathrm{E} = -\nabla\phi$ we obtain Poisson's equation

$$\nabla^2\phi = -\frac{\rho(\mathrm{r})}{\varepsilon_0}$$

10. For the special case where $\rho(\mathrm{r}) = 0$ in some region of space then Poisson's equation reduces to

$$\nabla^2\phi = 0$$

and is known as Laplace's equation.

Summary of electrostatics

1. The goal in electrostatics problems is to determine the potential $\phi(r)$.
2. In the integral formulation

$$\phi(r) = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\mathrm{r}')}{|\mathrm{r}-\mathrm{r}'|}d\mathrm{r}'$$

3. In the differential formulation

$$\nabla^2\phi = -\frac{\rho(\mathrm{r})}{\varepsilon_0}$$

4. In either case the electric field is calculated by evaluating

$$\mathrm{E} = -\nabla\phi$$

5. For any electrostatic problem the electric field satisfies

$$\nabla\times\mathrm{E} = 0$$

6. Since E is curl free, Stokes theorem implies that around any closed loop

$$\oint\mathrm{E}\cdot d\mathrm{l} = 0$$

7. Similarly, using the divergence theorem leads to the result that in any closed volume

$$\int \mathbf{n} \cdot \mathbf{E} dS = \frac{Q}{\varepsilon_0}$$
$$Q = \int \rho(\mathbf{r}) d\mathbf{r}$$

where Q is the total charge enclosed in the volume.

A simple electrostatics problem

1. Find the electric field due to a uniform spherical charge using Poisson's equation.

$$\rho = \rho_0 \qquad 0 \le r \le R$$
$$\rho = 0 \qquad r \ge R$$
$$\rho_0 = \frac{Q}{(4/3)\pi R^3} = const.$$

2. Use Poisson's equation for a spherically symmetric system

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) = -\frac{\rho_0}{\varepsilon_0} \qquad \text{region I} \qquad 0 \le r \le R$$
$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) = 0 \qquad \text{region II} \qquad r \ge R$$

3. The boundary conditions are

$$\phi'(0) = 0 \qquad \text{regularity at the origin}$$
$$\phi(\infty) = 0 \qquad \text{choice of potential at } \infty \text{ is arbitrary}$$

4. Across the interface we assume that there are no surface charges. This plus the curl free property of $\mathbf{E}$ imply the following jump conditions across the interface.

$$\phi_I(R_-) = \phi_{II}(R_+)$$
$$\phi_I'(R_-) = \phi_{II}'(R_+)$$

5. The solution in terms of arbitrary constants is given by

$$\phi_I = c_1 + \frac{c_2}{r} - \frac{\rho_0}{\varepsilon_0}\frac{r^2}{6}$$
$$\phi_{II} = c_3 + \frac{c_4}{r}$$

6. The boundary conditions require that $c_2 = 0$ and $c_3 = 0$.
7. The jump conditions require that

$$[\![\phi]\!]_R = 0 \qquad c_1 - \frac{\rho_0}{\varepsilon_0}\frac{R^2}{6} = \frac{c_4}{R}$$
$$[\![\phi']\!]_R = 0 \qquad -\frac{\rho_0}{\varepsilon_0}\frac{R^2}{3} = -\frac{c_4}{R}$$

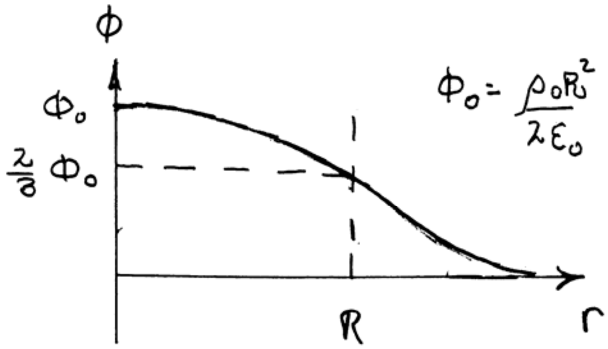
8. Solving for c_1, c_2 we find

$$c_1 = \frac{\rho_0}{\varepsilon_0}\frac{R^2}{2}$$
$$c_4 = \frac{\rho_0}{\varepsilon_0}\frac{R^3}{3}$$

9. The potential is thus given by

$$\phi_I = \frac{\rho_0 R^2}{\varepsilon_0}\left(\frac{1}{2} - \frac{1}{6}\frac{r^2}{R^2}\right)$$

$$\phi_{II} = \frac{\rho_0 R^2}{3\varepsilon_0}\left(\frac{R}{r}\right)$$



10. Note that the potential far from the sphere is the same as the potential generated by an equivalent charge q at the origin whose value is

$$\phi = \frac{q}{4\pi\varepsilon_0 r} \quad \rightarrow \quad q = \frac{4}{3}\pi R^3 \rho_0$$