

Lecture 2

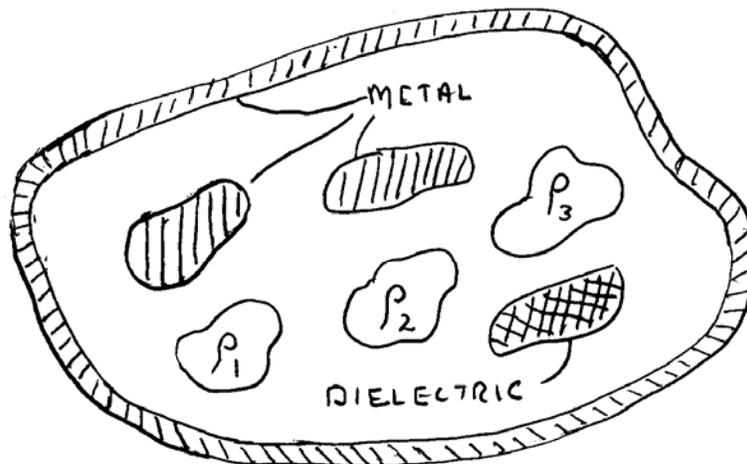
Solving Electrostatic Problems

Today's topics

1. Learn how to solve electrostatic problems
2. Overview of solution methods
3. Simple 1-D problems
4. Reduce Poisson's equation to Laplace's equation
5. Capacitance
6. The method of images

Overview

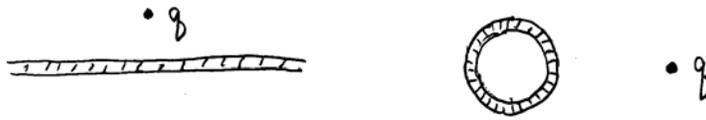
1. Illustrated below is a fairly general problem in electrostatics. Many practical problems are special cases of this general problem.



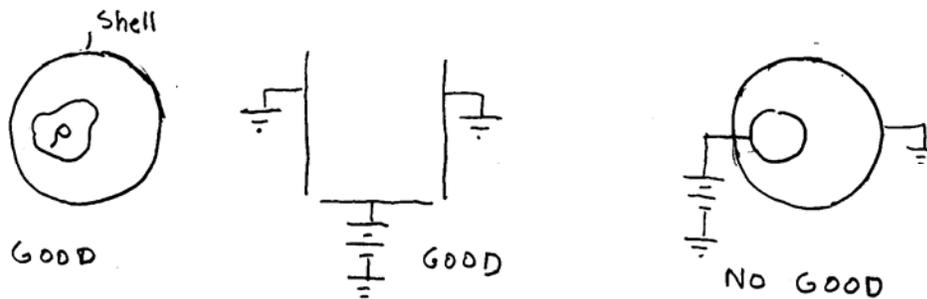
2. Where to start? This is a non-trivial question because there are integral and differential formulations of the electrostatic problem as well as a variety of mathematical procedures.
3. Here is a summary of the various methods and some related comments
 - a. Integral method Simple but needs special symmetry
 - b. Method of images Simple but needs special symmetry
 - c. Separation of variables Slightly complicated, more general geometry, ρ
 - d. Conformal mapping Elegant, but only useful in special geometries
 - e. Green's function More complicated but very general geometry
 - f. Purely numerical General, but hidden difficulties
4. The integral method:
 - a. The integral method works well for a charge distribution that is specified and there are no conductors or dielectrics materials present.
 - b. In this case one simply needs to evaluate an integral either analytically or numerically.
 - c. The shape and distribution of the charge density can be multidimensional



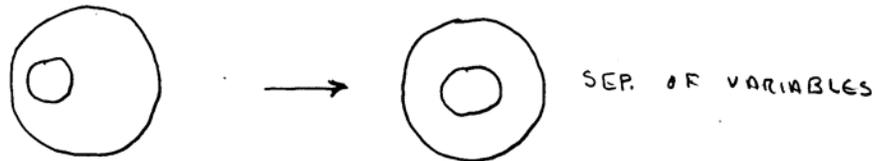
- d. The integral procedure works well when the charge distributions are isolated in free space but is not too useful when conductors are present, particularly when they do not have a very simple geometric shape.
5. The method of images:
 - a. The method of images is a simple technique that is very useful, but only in very specialized geometries with a lot of symmetry.



- b. A nice feature is that when the method works the solution is usually in the form of a simple closed analytic form.
- 6. Separation of variables:
 - a. Separation of variables is a good way to solve a reasonably large class of problems.
 - b. In general the solution is obtained as a summation of individual separated solutions; that is the solution can be expressed as a sum of expansion functions with appropriately determined coefficients (e.g. Fourier series, Bessel series)
 - c. For separation of variables to work the geometry must have some symmetry although the charge density can have an arbitrary shape and distribution.



- 7. Conformal mapping:
 - a. Conformal mapping is an elegant procedure when it works
 - b. It is a good technique in 2-D rectangular (x, y) or cylindrical (r, θ) geometries with sufficient symmetry that a mapping function can be found that transforms the boundaries into a simple rectangle or circle. A cylindrical (r, z) geometry is not useful.



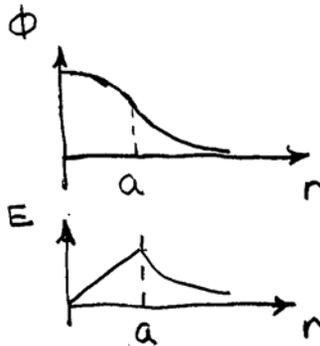
- c. There are relatively few geometries where the mappings are simple.
 - d. There are numerical techniques for calculating the mapping function in more general cases but then the simplicity and elegance is lost.
8. Green's function procedure
- a. The Green's function procedure is a very powerful technique that works in a wide variety of cases. It is a great procedure, worth understanding.
 - b. The technique is relatively complicated mathematically.
 - c. We have to understand Green's theorem.
 - d. We have to learn how to solve an integral equation.
 - e. Both of these problems will be discussed in the next lecture.
9. Pure numerical solution
- a. The pure numerical solution works very well if you know what you are doing.
 - b. Most of my experience has led me to the conclusion that relying on the numerical procedure too early, without really thinking about the problem, will result in unsatisfactory results.
 - c. Once you have a sufficient understanding of the problem and a reasonable expectation of the qualitative behavior of the solution, then the numerical approach can be very powerful and useful.

Simple 1-D problems

1. Find the potential from a sphere of uniform charge. We did this in the first lecture. The solutions, repeated here for convenience are given by

$$\phi = \frac{q}{4\pi\epsilon_0 r} \quad r \geq a$$

$$\phi = \frac{q}{4\pi\epsilon_0 a} \left(\frac{3}{2} - \frac{1}{2} \frac{r^2}{a^2} \right) \quad r \leq a$$



2. Find the potential from a cylindrical rod of uniform charge.
- Start with Poisson's equation in a cylindrical geometry

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{d\phi}{dr} \right) = -\frac{\rho}{\epsilon_0}$$



- Choose the free potential constant so that $\phi(a) = 0$. The solution inside and outside the rod is given by

$$\begin{aligned}
 r \geq a \quad \phi &= c_1 + c_2 \ln r = c_2 \ln \frac{r}{a} \\
 r \leq a \quad \phi' &= -\frac{\rho}{\epsilon_0} \frac{r}{2} \\
 \phi &= -\frac{\rho}{\epsilon_0} \frac{r^2}{4} + c_3 = \frac{\rho}{4\epsilon_0} (a^2 - r^2)
 \end{aligned}$$

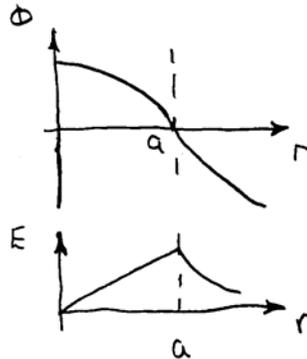
- c. Note that the constants have been chosen so that $\phi(a) = 0$ in both regions.
- d. The remaining constant is determined by equating ϕ' across $r = a$ corresponding to no surface charge.

$$\frac{c_1}{a} = -\frac{\rho a}{2\epsilon_0}$$

- e. The potential outside the rod can now be written as

$$\begin{aligned}
 \phi &= -\frac{\lambda}{2\pi\epsilon_0} \ln \frac{r}{a} \\
 \lambda &= \pi a^2 \rho = q/L = \text{charge/length}
 \end{aligned}$$

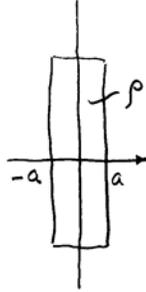
- f. Below is a plot of the solutions which look similar to the spherical case



3. Find the potential due to a uniform charge sheet

a. Start with Poisson's equation in slab geometry

$$\frac{d^2\phi}{dx^2} = -\frac{\rho}{\epsilon_0}$$



b. Note that by symmetry ϕ has even symmetry about $x = 0$. Also choose the free potential constant so that $\phi(0) = 0$. Then, the solutions inside and outside the slab are given by

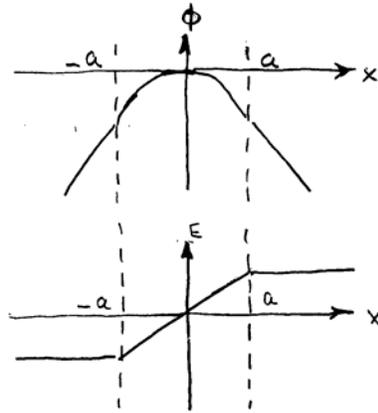
$$\begin{aligned} \text{Outside } |x| \geq a & \quad \phi_{\pm} = c_1 \pm c_2 x \\ & \quad \phi'_{\pm} = \pm c_2 \\ \text{Inside } |x| \leq a & \quad \phi = -\frac{\rho}{\epsilon_0} \frac{x^2}{2} \\ & \quad \phi' = -\frac{\rho}{\epsilon_0} x \end{aligned}$$

c. The constants c_1, c_2 are obtained by matching ϕ and ϕ' across $x = a$

$$\begin{aligned} \llbracket \phi' \rrbracket_a = 0 & \quad c_2 = -\frac{\rho a}{\epsilon_0} \\ \llbracket \phi \rrbracket_a = 0 & \quad c_1 = \frac{\rho a^2}{2\epsilon_0} \end{aligned}$$

d. The solution is thus given by

$$\begin{aligned} \phi &= \frac{\rho a^2}{\epsilon_0} \left(\frac{1}{2} - \frac{|x|}{a} \right) & |x| \geq a \\ \phi &= -\frac{\rho a^2}{2\epsilon_0} \left(\frac{x^2}{a^2} \right) & |x| \leq a \end{aligned}$$



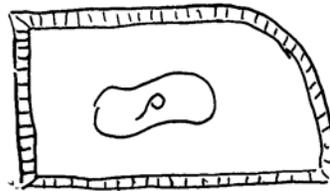
e. Note that the electric field outside the sheet is a constant

$$E = \frac{\sigma}{2\epsilon_0}$$

$$\sigma = 2\rho a = q / A = \text{charge/area}$$

Convert Poisson's equation into Laplace's equation

1. Consider next the more general problem of a closed surface surrounding an arbitrary distribution of charge.



2. The electrostatic problem is formulated as follows

$$\nabla^2\phi = -\frac{\rho}{\epsilon_0}$$

$$\phi(S) = \text{specified function on the surface}$$

3. Note that $\phi(S) = \text{const.}$ for a perfect conductor. Why?

4. Now let $\phi(\mathbf{r}) = \phi_p(\mathbf{r}) + \phi_h(\mathbf{r})$ where $\phi_p(\mathbf{r})$ satisfies

$$\phi_p(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}' \quad \rightarrow \quad \nabla^2 \phi_p = -\frac{\rho}{\epsilon_0}$$

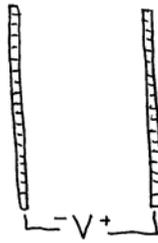
5. Note that $\phi_p(\mathbf{r})$ is a particular solution. It satisfies Poisson's equation and takes into account the charge distribution ρ . However, it does not satisfy the boundary condition. Even so, it is easy to calculate since all that is needed is the evaluation of an integral.
6. The full solution for ϕ is found by solving for ϕ_h which is the homogeneous solution, satisfying

$$\begin{aligned} \nabla^2 \phi_h &= 0 \\ \phi_h(S) &= \phi(S) - \phi_p(S) \end{aligned}$$

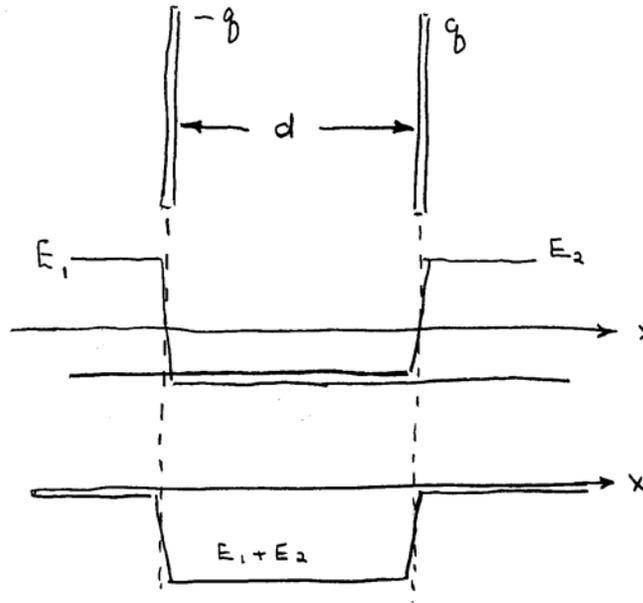
7. The problem is thus reduced to solving Laplace's equation with a modified boundary condition on the surface.

Capacitance

1. A capacitor is a circuit element that stores electrostatic energy. This energy can be provided by a charging circuit (e.g. a battery) and can be discharged through other circuit elements (e.g. a resistor).
2. Below we calculate the capacitance between two parallel plates. We then generalize the definition to arbitrary geometry. We also show how capacitance is related to the stored electrostatic energy.
3. Consider two parallel plates with a voltage across them as shown below.



4. The voltage places a charge $+q = +\sigma A$ on the right hand plate and an equal but opposite charge $-q = -\sigma A$ on the left hand plate. They must be equal and opposite because of the conservation of charge.
5. The total electric field can be found by superposition as illustrated below.



$$E_{left} = -\frac{\sigma}{2\epsilon_0} + \frac{\sigma}{2\epsilon_0} = 0$$

$$E_{right} = +\frac{\sigma}{2\epsilon_0} - \frac{\sigma}{2\epsilon_0} = 0$$

$$E_{center} = -\frac{\sigma}{2\epsilon_0} - \frac{\sigma}{2\epsilon_0} = -\frac{\sigma}{\epsilon_0}$$

6. Now, the voltage between the two plates is defined as

$$V = -\int \mathbf{E} \cdot d\mathbf{l} = -\int_0^d E dx = -Ed = \frac{\sigma d}{\epsilon_0} = \frac{qd}{\epsilon_0 A}$$

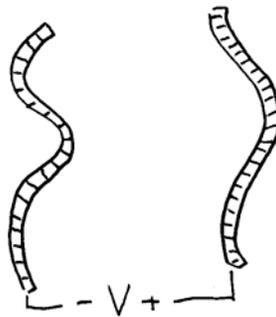
7. The simplest definition of capacitance C is $q = CV$ from which it follows that

$$C = \frac{\epsilon_0 A}{d}$$

8. Some of you may have seen an alternate definition of capacitance related to the stored electrostatic energy.

$$\frac{1}{2} CV^2 = \int \frac{\epsilon_0 E^2}{2} d\mathbf{r}'$$

9. Are these definitions the same? Are there conditions for them to be the same?
10. The energy definition is more basic.
11. We show the relation between the definitions by considering the following general problem involving two arbitrary shaped electrodes with a voltage across them.



12. Assume we have solved the electrostatic problem in the vacuum region. Specifically we have found the solution to

$$\nabla^2 \phi = 0 \quad \phi(S_1) = 0 \quad \phi(S_2) = V$$

13. As before assume there is no net charge in the system so that

$$q_1 = -q_2 \equiv -q.$$

14. Now let's use the energy definition for capacitance

$$\begin{aligned} \frac{1}{2}CV^2 &= \frac{\epsilon_0}{2} \int E^2 d\mathbf{r}' = \frac{\epsilon_0}{2} \int (\nabla\phi)^2 d\mathbf{r}' \\ &= \frac{\epsilon_0}{2} \int [\nabla \cdot (\phi\nabla\phi) - \phi\nabla^2\phi] d\mathbf{r}' \\ &= \frac{\epsilon_0}{2} \int [\nabla \cdot (\phi\nabla\phi)] d\mathbf{r}' \end{aligned}$$

15. Note that the $\nabla^2\phi$ term vanishes for a vacuum field

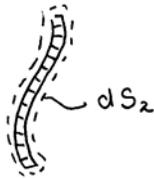
16. The next step is to simplify the last form using the divergence theorem

$$\begin{aligned} \frac{1}{2}CV^2 &= \frac{\epsilon_0}{2} \int [\nabla \cdot (\phi\nabla\phi)] d\mathbf{r}' \\ &= \frac{\epsilon_0}{2} \int \phi\mathbf{n} \cdot \nabla\phi dS_1 + \frac{\epsilon_0}{2} \int \phi\mathbf{n} \cdot \nabla\phi dS_2 \end{aligned}$$

17. From the BC we know that $\phi(S_1) = 0$ and $\phi(S_2) = V$. Therefore,

$$\frac{1}{2}CV^2 = \frac{\epsilon_0}{2} \int \phi\mathbf{n} \cdot \nabla\phi dS_2 = \frac{\epsilon_0 V}{2} \int \mathbf{n} \cdot \nabla\phi dS_2 = -\frac{\epsilon_0 V}{2} \int \mathbf{n} \cdot \mathbf{E} dS_2$$

18. The last term is further simplified by using Gauss's law around the second conductor; that is, from $\nabla \cdot \mathbf{E} = \rho / \epsilon_0$ it follows that



$$\begin{aligned} \int \nabla \cdot \mathbf{E} d\mathbf{r}' &= \int \frac{\rho}{\epsilon_0} d\mathbf{r}' = \frac{q}{\epsilon_0} \\ \int \mathbf{n}' \cdot \mathbf{E} dS_2 &= - \int \mathbf{n} \cdot \mathbf{E} dS_2 = \frac{q}{\epsilon_0} \end{aligned}$$

19. Here $\mathbf{n}' = -\mathbf{n}$. We must be careful to use the correct outward normal.

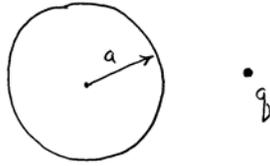
20. After substituting this result into the energy definition we obtain

$$\frac{1}{2}CV^2 = -\frac{\varepsilon_0 V}{2} \int \mathbf{n} \cdot \mathbf{E} dS_2 = -\left(\frac{\varepsilon_0 V}{2}\right)\left(-\frac{q}{\varepsilon_0}\right) \rightarrow q = CV$$

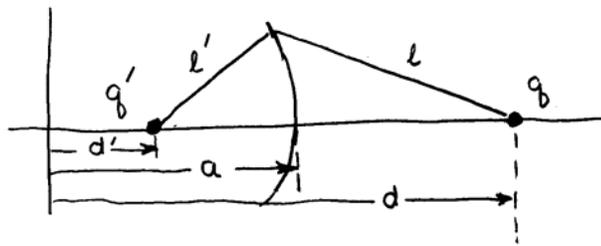
21. We see that for a vacuum field between the two plates both definitions are equivalent.

The method of images

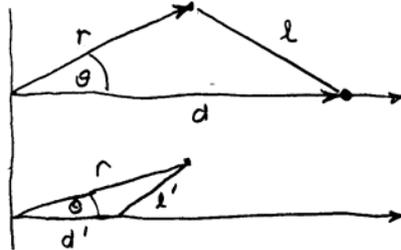
1. The method of images is a nice way to solve problems. The difficulty is that it really only works well in a very few cases with special symmetry.
2. The method is usually applied to situations where there is a known charge near a perfectly conducting surface.
3. Three examples are as follows: (1) a point charge above a conducting sheet, (2) a line charge parallel to a conducting cylinder, and (3) a point charge outside a conducting sphere.
4. The method of images involves some luck. The shape of the surface must have the right symmetry so that it can be replaced by a simple, finite collection of charges known as the image charges.
5. Luck is required because the potential due to the given charge plus the image charges must be such that the potential on the original conducting surface just turns out to be a constant.
6. One can often replace a general shaped conductor with an infinite set of charges but this is not so useful. Only when a small, finite number of charges is required is the replacement useful and corresponds to the method of images.
7. Let's demonstrate the procedure by calculating the potential outside a perfectly conducting sphere of radius a in the presence of a point charge q as shown below.



8. The solution for this problem involves placing an image charge q' at a cleverly chosen location d' inside the sphere.
9. There is no obvious way to derive this result. One simply postulates a q' and d' and then tests whether or not values can be found that make the potential constant on the surface of the sphere.
10. To carry out the test, consider the geometry below.



11. The potential at any point outside the sphere is given by



$$\phi = \frac{q}{4\pi\epsilon_0 l} + \frac{q'}{4\pi\epsilon_0 l'}$$

$$l^2 = r^2 + d^2 - 2dr \cos \theta$$

$$l'^2 = r^2 + d'^2 - 2d'r \cos \theta$$

12. The potential can be rewritten as

$$\phi = \frac{1}{4\pi\epsilon_0} \left[\frac{q}{(r^2 + d^2)^{1/2}} \frac{1}{\left(1 - \frac{2dr}{r^2 + d^2} \cos \theta\right)^{1/2}} + \frac{q'}{(r^2 + d'^2)^{1/2}} \frac{1}{\left(1 - \frac{2d'r}{r^2 + d'^2} \cos \theta\right)^{1/2}} \right]$$

13. Can we now find a q' and d' that makes $\phi(a, \theta, \phi) = 0$?

14. To do this we must set

$$\frac{da}{d^2 + a^2} = \frac{d'a}{d'^2 + a^2}$$

$$\frac{q}{(d^2 + a^2)^{1/2}} = -\frac{q'}{(d'^2 + a^2)^{1/2}}$$

15. The first equation is a quadratic equation for d' which can be factored as follows

$$d'^2 - \left(\frac{d^2 + a^2}{d}\right)d' + a^2 = (d' - d)\left(d' - \frac{a^2}{d}\right) = 0$$

16. Clearly the non-trivial solution is

$$d' = \frac{a^2}{d}$$

17. The charge q' is easily found from the second constraint equation

$$q' = -q \left(\frac{d'^2 + a^2}{d^2 + a^2}\right) = -q \left(\frac{a}{d}\right)$$

18. The final expression for the potential can now be written as

$$\phi = \frac{1}{4\pi\epsilon_0} \left\{ \frac{1}{(r^2 + d^2 - 2rd \cos \theta)^{1/2}} - \frac{a/d}{\left[r^2 + \frac{a^4}{d^2} - 2\left(\frac{a^2}{d}\right)r \cos \theta \right]^{1/2}} \right\}$$