Lecture 4

Conformal Mapping and Green's Theorem

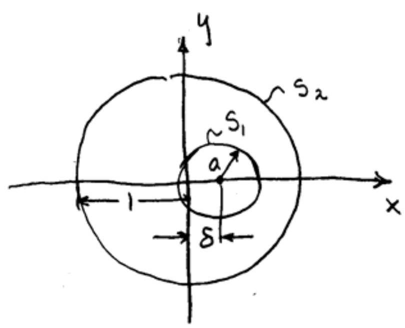
Today's topics

1. Solving electrostatic problems – continued
2. Why separation of variables doesn't always work
3. Conformal mapping
4. Green's theorem

The failure of separation of variables

1. Let's try to solve the following problem by separation of variables

$$\nabla^2\phi = 0 \qquad \phi(S_2) = 0 \qquad \phi(S_1) = 1$$



2. Use separation of variables in cylindrical coordinates

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial r^2} = 0$$

3. We assume that

$$\phi(r,\theta) = \sum_0^\infty \left(a_n r^n + b_n r^{-n}\right)\cos n\theta$$

4. To satisfy the boundary condition on S_2 we require $\phi(1,\theta)=0$. Therefore $a_n=-b_n$ and ϕ reduces to

$$\phi(r,\theta)=\sum_0^\infty b_n\left(r^n-r^{-n}\right)\cos n\theta$$

5. To satisfy the boundary condition on S_1 we require $\phi\left[r_1(\theta),\theta\right]=1$
6. We need to calculate $r_1(\theta)$. This is done by noting that the equation describing S_1 is given by $(x-\delta)^2+y^2=a^2$
7. Transform to cylindrical coordinates: $x=r\cos\theta \quad y=r\sin\theta$
8. The equation for the surface $r=r_1$ becomes $r_1^2-2\delta r_1\cos\theta+\delta^2=a^2$.
9. Solve for r_1

$$r_1(\theta)=\delta\cos\theta+\left(a^2-\delta^2\sin^2\theta\right)^{1/2}$$

10. Return to the boundary condition. We determine the coefficients b_n by Fourier analysis. Multiply $\phi\left[r_1(\theta),\theta\right]=1$ by $(1/2\pi)\int\cos m\theta\, d\theta$. The boundary condition becomes

$$\sum_0^\infty \frac{b_n}{2\pi}\int_0^{2\pi}\left(r_1^n-r_1^{-n}\right)\cos n\theta\cos m\theta\, d\theta=\frac{1}{2\pi}\int_0^{2\pi}\cos m\theta\, d\theta$$

11. The right hand side is easily evaluated

$$RHS=\delta_m$$

12. The left hand side becomes

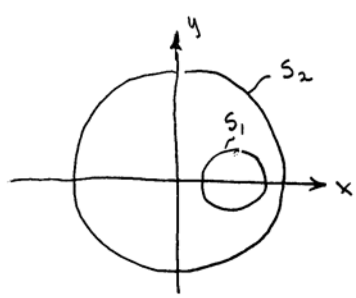
$$\begin{aligned}LHS&=\sum_{n=0}^\infty M_{mn}b_n\\ M_{mn}&=\frac{1}{2\pi}\int_0^{2\pi}\left[r_1^n(\theta)-r_1^{-n}(\theta)\right]\cos n\theta\cos m\theta\, d\theta\end{aligned}$$

13. The problem is thus reduced to a simple linear algebra problem for the coefficients b_n

$$\overleftrightarrow{\mathbf{M}} \cdot \mathbf{b} = \mathbf{v} = [1,\ 0,\ 0,\ \ldots\ 0]$$

$$\mathbf{b} = \overleftrightarrow{\mathbf{M}}^{-1} \cdot \mathbf{v}$$

14. We simply have to invert the matrix $\overleftrightarrow{\mathbf{M}}$ to find the coefficients $\mathbf{b}$. This is in general a very simple numerical procedure.
15. One might therefore think that separation of variables should work in a wide variety of cases, the only added difficulty being the need to numerically invert a matrix. Not a very big deal.
16. This is an incorrect conclusion. What happens if you try this procedure? For small values of δ the procedure works well and a correct set of $\mathbf{b}$ are calculated.
17. However, above a certain critical value of δ your PC will tell you that the inverse matrix does not exist. You cannot determine the values of $\mathbf{b}$. No amount of fancy numerical tricks will help. The inverse truly does not exist.
18. What has gone wrong? We have tried to write the solution using the wrong class of expansion functions. Even though the expansion functions satisfy $\nabla^2\phi = 0$ their radius of convergence does not include the entire domain of interest.
19. Below is a simple intuitive example that shows the problem. For the problem under consideration assume a large value of δ so that S_1 does not enclose the origin.



20. To satisfy the boundary condition on S_2 we require both the r^n and r^{-n} solutions. However, the r^{-n} solutions diverge at the origin which is contained in the region where we need a solution.
21. Although the potential at $r = 0$ is finite and smooth we are attempting to describe this behavior by summing over an infinite set of divergent functions. This will not work and is the source of the problem.

22. OK – can we find a better set of expansion functions? In general the answer is NO!!! Only for special simple geometries with a high degree of symmetry do expansion techniques based on separation of variables work.
23. What do we do? An elegant procedure that works in certain cases makes use of conformal mapping techniques. A very general procedure makes use of Green's theorem. The key point of the Green's function procedure is that the method guarantees the existence of "good" surface expansion functions, thus always leading to the desired solution.

Conformal mapping

1. The conformal mapping procedure can be used to convert a complicated geometry into a much simpler geometry.
2. The mapping function is defined as $w = f(z)$ where $w = u + iv$ and $z = x + iy$ are complex variables.
3. The procedure works extremely well in 2-D geometries. The reason, as shown shortly, is that the 2-D Laplace's equation $\phi_{xx} + \phi_{yy} = 0$ transforms in the new coordinates to $\phi_{uu} + \phi_{vv} = 0$.
4. In the new coordinates ϕ also satisfies Laplace's equation. The beauty is that in the coordinate system the geometry is much simpler and it is thus much easier to find a solution.
5. What are the main limitations on the conformal mapping technique?
6. First, it only works well in a 2-D geometry. Furthermore the geometry must be such that Laplace's equation can be written as $\phi_{xx} + \phi_{yy} = 0$ (or the equivalent r,θ cylindrical system $\phi_{rr} + \phi_r / r + \phi_{\theta\theta} / r^2 = 0$).
7. It does not, for instance, work for the 2-D r,z system where Laplace's equation has the form $\phi_{rr} + \phi_r / r + \phi_{zz} = 0$. This equation cannot be transformed into the form $\phi_{xx} + \phi_{yy} = 0$ and the procedure is not very useful.
8. The second problem is that even if the geometry of interest satisfies $\phi_{xx} + \phi_{yy} = 0$ we must still find the mapping function. For certain relatively simple cases the mapping function can be found analytically.
9. In more complicated cases the mapping function can be determined numerically. In fact there are standard numerical packages which carry out this task.

10. Consequently finding the mapping function can be slightly inconvenient but is possible in a wide variety of cases as long as Laplace's equation in the relevant geometry can be written as $\phi_{xx} + \phi_{yy} = 0$.

Review of complex variables

1. The key point that we need to prove is that the Laplacian in x, y coordinates transforms into the Laplacian in u, v coordinates for a conformal transformation.
2. To do this we need the Cauchy-Riemann equations which are obtained as follows.
3. Consider the conformal mapping $w = f(z)$ where $w = u + iv$ and $z = x + iy$.
4. The derivation is as follow

$$\begin{aligned}
&w = u + iv = f(z) \\
&\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{df}{dz}\frac{\partial z}{\partial x} = \frac{df}{dz} \\
&\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} = \frac{df}{dz}\frac{\partial z}{\partial y} = i\frac{df}{dz}
\end{aligned}$$

5. Now equate the real and imaginary parts of the two expressions for df/dz yielding the Cauchy-Riemann equations

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\
\frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}
\end{aligned}$$

6. Straightforward differentiation of these relations shows that both u and v satisfy Laplace's equation.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

7. The next step is to transform Laplace's equation in x, y coordinates into u, v coordinates.

8. For a general transformation (not necessarily a conformal transformation) it follows that if $u = u(x, y)$ and $v = v(x, y)$, application of the chain rule for derivatives leads to

$$\phi_{xx} = \phi_{uu} u_x^2 + 2\phi_{uv} u_x v_x + \phi_{vv} v_x^2 + \phi_u u_{xx} + \phi_v v_{xx}$$
$$\phi_{yy} = \phi_{uu} u_y^2 + 2\phi_{uv} u_y v_y + \phi_{vv} v_y^2 + \phi_u u_{yy} + \phi_v v_{yy}$$

9. We now add the equations together, assume the transformation corresponds to a conformal mapping, and make use of the Cauchy-Riemann equations. We find

$$\phi_{xx} + \phi_{yy} = (\phi_{uu} + \phi_{vv}) \left|\frac{df}{dz}\right|^2$$
$$\left|\frac{df}{dz}\right|^2 = |u_x + iv_x|^2 = u_x^2 + v_x^2$$

10. Therefore if ϕ satisfies Laplace's equation in x, y coordinates it also does so in u, v coordinates.

$$\frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2} = 0$$

11. This is the beauty of the conformal mapping.

The mapping function

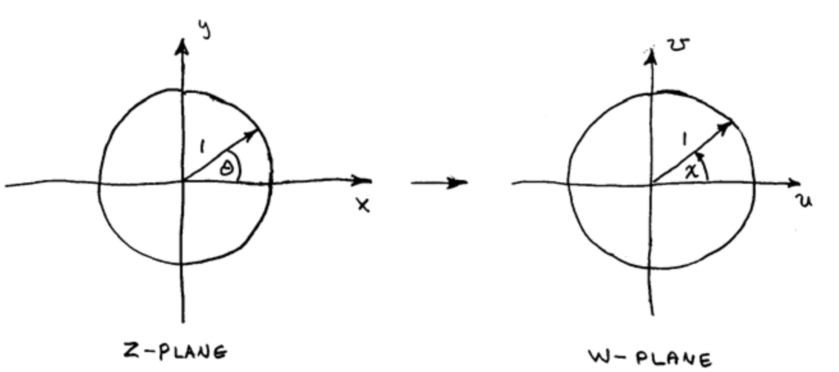
1. Consider next the shifted circle problem previously discussed.
2. What we want to find is a mapping function that has two properties: (1) it maps the outer unit circle in the x, y plane into the unit circle in the u, v plane, and (2) it maps the inner shifted circle in the x, y plane into a centered circle in the u, v plane.

3. If we can find such a mapping then our task will be to solve Laplace's equation in a coordinate system where the boundaries are two concentric circles, usually a very simple task.
4. The correct mapping function is given by

$$w = \frac{z - \alpha}{1 - \alpha z}$$

where α is a free parameter to be determined.

5. Let's first check the mapping of the unit circle. In the x, y plane the unit circle is given by $x = \cos\theta, y = \sin\theta$ which is equivalent to $z = e^{i\theta}$. In the u, v plane we let $u = R\cos\chi, v = R\sin\chi$ which is equivalent to $w = R\,\mathrm{e}^{i\chi}$.



6. We need to verify that the mapping leads to $R = 1$, a unit circle in the u, v plane. Straightforward substitution shows that

$$\mathrm{Re}^{i\chi} = \frac{\cos\theta - \alpha + i\sin\alpha}{1 - \alpha\cos\alpha - i\alpha\sin\alpha}$$

7. Multiply each side of the equation by its complex conjugate.

$$R^2 = \frac{1 - 2\alpha\cos\theta + \alpha^2}{1 - 2\alpha\cos\theta + \alpha^2} = 1$$

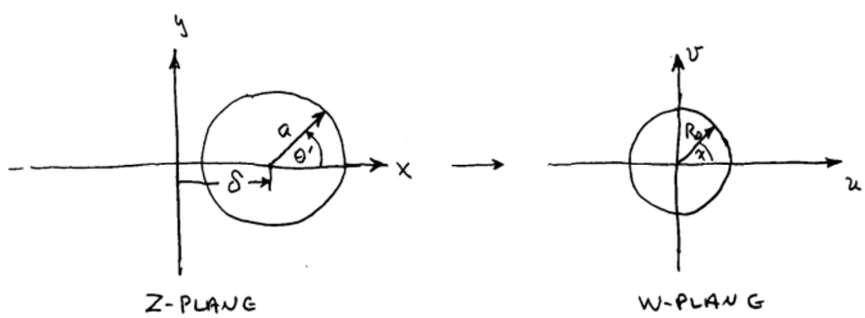
8. The unit circle maps into the unit circle for any value of α.
9. Consider now the inner shifted circle $x = \delta + a\cos\theta', y = a\sin\theta'$. This maps into

$$R^2 = \frac{a^2 + (\delta - \alpha)^2 + 2a(\delta - \alpha)\cos\theta'}{\alpha^2 a^2 + (\alpha\delta - 1)^2 - 2\alpha a(\alpha\delta - 1)\cos\theta'}$$
$$= \frac{a^2 + (\delta - \alpha)^2}{\alpha^2 a^2 + (\alpha\delta - 1)^2}\left[\frac{1 + A\cos\theta'}{1 + C\cos\theta'}\right]$$

where

$$A = \frac{2a(\delta - \alpha)}{a^2 + (\delta - \alpha)^2}$$
$$C = \frac{2\alpha a(\alpha\delta - 1)}{\alpha^2 a^2 + (\alpha\delta - 1)^2}$$

10. In order for the surface to be a concentric circle in the u, v plane we require that the mapping lead to $R = R_0 = const.$ There cannot be any θ' dependence in R. To satisfy this constraint we must choose α such that $A = C$. Then the θ' dependence will cancel out.



11. After a slightly tedious calculation we obtain a quadratic equation for α.

$$\alpha^2 - \left(\frac{\delta^2 + 1 - a^2}{\delta}\right)\alpha + 1 = 0$$

12. The root that corresponds to $R_0 < 1$ is given by

$$\alpha = \frac{1}{2}\left\{\frac{\delta^2 + 1 - a^2}{\delta} - \left[\left(\frac{\delta^2 + 1 - a^2}{\delta}\right)^2 - 4\right]^{1/2}\right\}$$

13. Also, keep in mind that for topologically consistent solutions to exist δ, a must lie in the ranges $0 < \delta < 1$ and $0 < a < 1 - \delta$.

14. For a given δ, a we have now determined α and the mapping is completely defined.

The solution

1. The solution to the problem in the u, v plane is easily found. We let $u = R\cos\chi, v = R\sin\chi$. Laplace's equation and the boundary conditions become

$$\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\phi}{\partial R}\right)+\frac{1}{R^2}\frac{\partial^2\phi}{\partial\chi^2}=0 \qquad \phi(1)=0 \qquad \phi(R_0)=1$$

2. Because of the χ symmetry in the boundary conditions it follows that the solution is of the form $\phi(R,\chi)=\phi(R)$.
3. The solution is given by

$$\phi(R)=\frac{\ln R}{\ln R_0}$$

4. We can transform back into the x, y plane by noting that

$$R^2 = \frac{(x-\alpha)^2+y^2}{(\alpha x-1)^2+\alpha^2 y^2}$$
$$R_0^2 = \frac{(\delta-\alpha)^2+a^2}{(\alpha\delta-1)^2+\alpha^2 a^2}$$

5. To visualize the potential surfaces note that a $\phi = const.$ surface corresponds to an $R = const.$ surface. The relation giving $R = R(x,y)$ can be easily inverted yielding

$$\left[x-\frac{\alpha(1-R^2)}{1-\alpha^2R^2}\right]^2+y^2=1-\frac{(1-R^2)(1-\alpha^4R^2)}{(1-\alpha^2R^2)^2}$$

6. The potential surfaces in the x, y plane are a sequence of shifted circles
7. This completes the solution.

Green's theorem

1. What is the purpose of Green's theorem?
2. In an infinite space problem Green's theorem converts a PDE $\left(\nabla^2\phi = -\rho/\varepsilon_0\right)$ into a simple integral evaluation. This is good but we already know how to do this.
3. For boundary value problems Green's theorem converts a PDE $\left(\nabla^2\phi = -\rho/\varepsilon_0\right)$ into a simple integral evaluation. This is good but in general it is difficult to find the Green's function with finite boundaries.
4. However, if one knows the Green's function for a fixed set of boundaries, it is easy to calculate the solution for a wide range of boundary conditions.
5. For boundary value problems Green's theorem using the simple free space Green's function converts the PDE $\left(\nabla^2\phi = -\rho/\varepsilon_0\right)$ into an integral equation. This is good because we know the Green's function, but unpleasant because we have to solve an integral equation.
6. For simple geometries the integral equation is easy to solve. This is good.
7. In general geometries, solving the integral equation is often the best approach. This may be hard to believe but it is true.

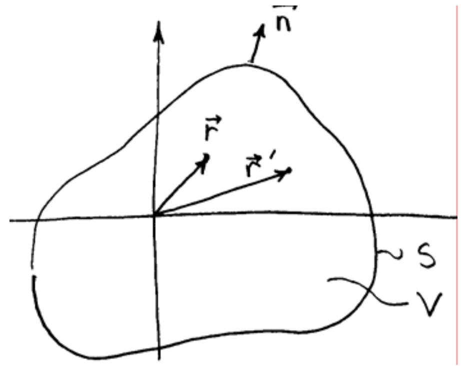
Derivation of Green's theorem

1. Consider two functions ϕ and G.
2. Note the following identity

$$\begin{aligned}\nabla\cdot(\phi\nabla G - G\nabla\phi) &= \phi\nabla^2 G + \nabla\phi\cdot\nabla G - G\nabla^2\phi - \nabla G\cdot\nabla\phi \\ &= \phi\nabla^2 G - G\nabla^2\phi\end{aligned}$$

3. Now integrate over a closed volume using the divergence theorem (We could easily include multiple boundaries but restrict the analysis to one boundary for simplicity)

$$\int_V \nabla\cdot\mathbf{A}'\,d\mathbf{r}' = \int_S \mathbf{n}'\cdot\mathbf{A}'\,dS'$$



4. Apply this to our identity

$$\int_V \left(\phi' \nabla'^2 G - G \nabla'^2 \phi'\right) d\mathbf{r}' = \int_S \left(\phi' \mathbf{n}' \cdot \nabla' G - G \mathbf{n}' \cdot \nabla' \phi'\right) dS'$$

5. Now assume that G is the Green's function satisfying

$$\nabla^2 G = \delta\left(\mathbf{r}' - \mathbf{r}\right) = \delta\left(x' - x\right)\delta\left(y' - y\right)\delta\left(z' - z\right)$$

6. For the moment let's not worry about the boundary conditions on G. Also note that $\mathbf{r}$ is the observation point and $\mathbf{r}'$ is the integration point.
7. Note that the delta function is defined such that

$$\int \delta\left(\mathbf{r}' - \mathbf{r}\right) d\mathbf{r}' = \begin{cases} 1 & \text{For an interior point} \\ 0 & \text{For an exterior point} \\ 1/2 & \text{For a surface point} \end{cases}$$

8. Now assume that ϕ satisfies Poisson's equation: $\nabla^2 \phi = -\rho / \varepsilon_0$
9. Evaluate the terms in Greens theorem

$$\int_V \phi(\mathbf{r}')\nabla'^2 G(\mathbf{r}'-\mathbf{r})\,d\mathbf{r}' = \int_V \phi(\mathbf{r}')\delta(\mathbf{r}'-\mathbf{r})\,d\mathbf{r}' = \alpha\phi(\mathbf{r})$$

$$\alpha = \begin{cases} 1 & \text{Interior point} \\ 0 & \text{Exterior point} \\ 1/2 & \text{Surface point} \end{cases}$$

$$\int_V G(\mathbf{r}'-\mathbf{r})\nabla'^2\phi(\mathbf{r}')\,d\mathbf{r}' = -\frac{1}{\varepsilon_0}\int_V G(\mathbf{r}'-\mathbf{r})\rho(\mathbf{r}')\,d\mathbf{r}'$$

$$\int_S \left(\phi'\mathbf{n}'\cdot\nabla' G - G\mathbf{n}'\cdot\nabla'\phi'\right)dS' = \int_S \left[\phi(S')\frac{\partial G}{\partial n'} - G\frac{\partial\phi}{\partial n'}\right]dS' \qquad \mathbf{n}'\cdot\nabla' \equiv \frac{\partial}{\partial n'}$$

10. Combine terms

$$\alpha\phi(\mathbf{r}) = -\frac{1}{\varepsilon_0}\int_V G(\mathbf{r}'-\mathbf{r})\rho(\mathbf{r}')\,d\mathbf{r}' + \int_S \left[\phi(S')\frac{\partial G}{\partial n'} - G\frac{\partial\phi}{\partial n'}\right]dS'$$

11. This is Green's theorem. Admittedly it doesn't look very helpful in its present form.
12. To show how it can be helpful let's calculate the infinite space Green's function for a sphere and a cylinder.

Free space spherical Green's function

1. G satisfies: $\nabla^2 G = \delta(\mathbf{r}'-\mathbf{r}) = \delta(x'-x)\delta(y'-y)\delta(z'-z)$
2. Try a solution of the form $G = G(u)$ where $u^2 = (x'-x)^2 + (y'-y)^2 + (z'-z)'$
3. A short calculation shows that G satisfies

$$\frac{1}{u^2}\frac{\partial}{\partial u}\left(u^2\frac{\partial G}{\partial u}\right) = \delta(u)$$

4. The solution for G for $u \neq 0$ is given by

$$G(u) = \frac{c_1}{u} + c_2 = \frac{c_1}{u} \qquad \text{for } G(\infty) = 0$$

5. The value of c_1 is found by integrating around a small sphere located at $u = 0$. The terms are

$$\int \nabla^2 G d\mathbf{r} = \int \mathbf{n} \cdot \nabla G \, dS = \int \frac{\partial G}{\partial u} u^2 \sin\theta \, d\theta d\phi = -4\pi c_1$$
$$\int \delta(u) \, d\mathbf{r} = 1$$

6. Therefore $c_1 = -1/4\pi$ and

$$G\left(\mathbf{r}' - \mathbf{r}\right) = -\frac{1}{4\pi} \frac{1}{\left|\mathbf{r} - \mathbf{r}'\right|}$$

7. This is the spherical Green's function.

Free space cylindrical Green's function

1. The cylindrical Green's function is found in a similar way
2. G satisfies $\dfrac{\partial^2 G}{\partial x^2} + \dfrac{\partial^2 G}{\partial y^2} = \delta\left(x' - x\right)\delta\left(y' - y\right)$
3. Let $G = G(u)$ where $u^2 = \left(x' - x\right)^2 + \left(y' - y\right)^2$
4. G satisfies $\dfrac{1}{u}\dfrac{\partial}{\partial u}\left(u \dfrac{\partial G}{\partial u}\right) = \delta(u)$
5. The solution is $G(u) = c_1 \ln u + c_2 = c_1 \ln u \qquad \text{for } G(1) = 0$
6. Find c_1 by integrating over a small circle located at $u = 0$. The separate terms are

$$\int \nabla'^2 G \, d\mathbf{r}' = \int \mathbf{n}' \cdot \nabla' G \, dS' = \int \frac{\partial G}{\partial u'} u' \, d\theta' dz' = 2\pi L_z c_1$$
$$\int \delta\left(u'\right) d\mathbf{r}' = \int \delta\left(x' - x\right)\delta\left(y' - y\right) dx' dy' dz' = L_z$$

7. Therefore $c_1 = 1/2\pi$ and the Green's function is given by

$$G = \frac{1}{2\pi} \ln u = \frac{1}{4\pi} \ln\left[\left(x' - x\right)^2 + \left(y' - y\right)^2\right]$$

Using Green's theorem for infinite space problems (i.e. no conductors)

1. Consider a 3-D domain where the volume extends to infinity. This is the case of the spherical Green's function.
2. Recall that for localized charges (i.e. no sources at ∞) both ϕ and G vanish at infinity.
3. Consequently both terms in the surface integral vanish

$$\int_S \left[\phi\left(S'\right)\frac{\partial G}{\partial n'} - G\frac{\partial \phi}{\partial n'}\right] dS' = 0 \qquad \text{as } r \to \infty$$

4. For any point within the domain we must take $\alpha = 1$ and Green's theorem reduces to

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho\left(\mathbf{r}'\right)}{\left|\mathbf{r}-\mathbf{r}'\right|} d\mathbf{r}'$$

5. This is correct but we already knew the result from previous lectures.
6. A similar result follows from the cylinder

$$\phi(\mathbf{r}) = \frac{1}{2\pi\varepsilon_0}\int_V \rho\left(\mathbf{r}'\right)\ln\left|\mathbf{r}-\mathbf{r}'\right| dS'$$

7. Here $\mathbf{r} = x\,\mathbf{e}_x + y\,\mathbf{e}_y, \mathbf{r}' = x'\,\mathbf{e}_x + y'\,\mathbf{e}_y, dS' = dx'dy'$ are two dimensional functions. The total line charge density λ is given by $\lambda = \int \rho\left(\mathbf{r}'\right) dS' \; Coul/m$.
8. So far we have done a lot of work verifying what we already know. In the next lecture we will learn how to use Green's theorem to solve far more difficult problems.