Lecture 5

Solving Problems using Green's Theorem

Today's topics

1. Show how Green's theorem can be used to solve general electrostatic problems
2. Dielectrics

A well known application of Green's theorem

1. Last time we derived Green's theorem.
2. We also derived the free space Green's function in a sphere and cylinder.
3. These functions were then used to derive the integral form of electrostatics from which the potential is derived by an integral involving the charge density.
4. This was reassuring but we already knew these results from prior work.
5. Today we focus on the more interesting and general problem of solving multi-dimensional electrostatic problems in complex geometries, including the presence of conductors (and dielectrics)
6. Let's set up a typical problem. We want to solve

$$\nabla^2\phi = -\rho/\varepsilon_0 \qquad \phi(S) = \phi_S = \text{ known}$$

7. Although we could solve this problem numerically it becomes inconvenient and computationally time consuming to do so for a large variety of boundary conditions $\phi_S$. Often this is what we need to do.
8. It is time consuming because each new boundary condition requires a whole new numerical calculation.
9. Green's theorem helps if we now change the boundary condition on $G$ from the free space condition at infinity to a different one specified on $S$.

$$\text{Old} \qquad \nabla^2 G = \delta(\mathrm{r}' - \mathrm{r}) \qquad G(\infty) = 0$$

$$\text{New} \qquad \nabla^2 G = \delta(\mathrm{r}' - \mathrm{r}) \qquad G(S) = 0$$

10. Green's theorem for an arbitrary interior point becomes

$$\phi(\mathrm{r}) = -\frac{1}{\varepsilon_0}\int_{V'} G\rho(\mathrm{r}')d\mathrm{r}' + \int_{S'}\left[\phi(S')\frac{\partial G}{\partial n'} - G\frac{\partial \phi(S')}{\partial n'}\right]dS'$$
$$= -\frac{1}{\varepsilon_0}\int_{V'} G\rho(\mathrm{r}')d\mathrm{r}' + \int_{S'}\phi(S')\frac{\partial G}{\partial n'}dS'$$

11. For any $\phi_S$ we need to evaluate a volume integral and a surface integral to determine $\phi$, a simple numerical task.
12. If we want to redo the problem for a different $\phi_S$ we only need to re-evaluate the lower dimensional surface integral.
13. This seems too good to be true! What is the catch?
14. In general it is of comparable difficulty to determine the Green's function satisfying $G(S) = 0$ as it is to solve the original problem. This is a major stumbling block.

A less well known but more important application

1. We show below how to use Green's theorem to solve the general problem without having to deal with the complicated problem of determining $G$ such that $G(S) = 0$.
2. Let's return to the original problem $\nabla^2\phi = -\rho/\varepsilon_0$.
3. For generality assume that either $\phi(S')$ or $\partial\phi(S')/\partial n'$ is specified.
4. The first step is to convert from Poisson's equation to Laplace's equation. We define $\phi(\mathrm{r}) = \phi_p(\mathrm{r}) + \phi_h(\mathrm{r})$ where

$$\phi_p(\mathrm{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\mathrm{r}')}{|\mathrm{r}-\mathrm{r}'|}d\mathrm{r}' \qquad \text{for 3-D}$$
$$= \frac{1}{2\pi\varepsilon_0}\int_S \rho(\mathrm{r}')\ln|\mathrm{r}-\mathrm{r}'|\,dS' \qquad \text{for 2-D}$$

5. For simplicity call $\phi_h(\mathrm{r}) = \psi(\mathrm{r})$. The homogeneous solution satisfies

$$\nabla^2\psi = 0 \qquad \psi(S) = \phi(S) - \phi_p(S) \qquad \text{or} \qquad \frac{\partial\psi(S)}{\partial n} = \frac{\partial\phi(S)}{\partial n} - \frac{\partial\phi_p(S)}{\partial n}$$

6. Green's theorem becomes

$$\alpha\psi(\mathrm{r}) = \int_{S'}\left(\psi'\frac{\partial G}{\partial n'} - G\frac{\partial\psi'}{\partial n'}\right)dS'$$

$$\alpha = \begin{cases} 1 & \text{Interior point} \\ 0 & \text{Exterior point} \\ 1/2 & \text{Surface point} \end{cases}$$

7. Now, choose the observation point to lie on the surface so that $\alpha = 1/2$. Then

$$\frac{1}{2}\psi(S) = \int_{S'}\left[\psi(S')\frac{\partial G}{\partial n'} - G\frac{\partial\psi(S')}{\partial n'}\right]dS'$$

8. Choose the Green's function to correspond to the free space Green's function. This is easy to do. We already know this function. No complicated boundary conditions on $G$ are required.
9. Since we know $G$ it is an easy task to calculate $\partial G/\partial n'$.
10. If we know $\partial\psi(S')/\partial n'$ then Green's theorem yields an integral equation for $\psi(S')$.
11. Similarly if we know $\psi(S')$ we have an integral equation for $\partial\psi(S')/\partial n'$.
12. This is the desired formulation. If we assume that we can solve the integral equation then we will know both $\psi(S')$ and $\partial\psi(S')/\partial n'$. Thus $\psi$ and hence $\phi$ can be easily found by using Green's theorem for an internal point and simply evaluating known integrals.
13. The next step is to show how to solve the integral equation, guaranteeing that we will always be able to avoid the problem of choosing improper expansion functions as was the case using separation of variables.

Solving the integral equation

1. The integral equation is a linear equation. Therefore, expansion techniques are a good approach.
2. Note that in a 3-D problem we need to solve the integral equation on a closed 2-D surface bounding the volume of interest. For a 2-D problem we need to solve the integral equation on a closed 1-D curve bounding the surface of interest.
3. Here is the absolutely critical point!!! On a closed surface or curve the solutions must be periodic. Therefore, we are guaranteed that a Fourier series must exist that can represent any arbitrary boundary data.
4. For example for a 2-D problem where $l$ is the arc length along the boundary, the potential $\psi = \psi(l)$ can always be written as

$$\psi(l) = \sum_{-\infty}^{\infty} \psi_m e^{im\theta}$$

5. The existence of the Fourier series guarantees that the problem of improper expansion functions is eliminated.
6. Furthermore, one does not have to use the angle $\theta$ as the independent variable. We could choose any other angle $v = v(\theta)$ that might be more convenient (i.e. could put more resolution in certain sections along the curve)

Details of the procedure

1. There are a fair number of details to obtain the solution to the integral equation. We demonstrate the steps for a general 2-D problem using an elliptical surface as a special example.
2. Assume the boundary curve is parameterized in terms of an arbitrary angle-like variable $v$ (i.e. $0 \le v \le 2\pi$) as follows

$$x = x(v) = a\cos v$$
$$y = y(v) = b\sin v$$

3. To use Green's theorem note that $dS' = dl'dz' = L_z dl'$. Since the $L_z$ factor cancels everywhere, we hereafter suppress it for convenience.
4. The following geometric relation for vector arc length is needed for the solution (where over dot denotes $d/dv$)

$$
\begin{aligned}
d\mathrm{l} &= dx\,\mathrm{e}_x + dy\,\mathrm{e}_y = \left(\dot{x}\,\mathrm{e}_x + \dot{y}\,\mathrm{e}_y\right)dv \\
&= \left(-a\sin v\,\mathrm{e}_x + b\cos v\,\mathrm{e}_y\right)dv
\end{aligned}
$$

5. From this we find that

$$
\begin{aligned}
\mathrm{t} &= \frac{\dot{x}\,\mathrm{e}_x + \dot{y}\,\mathrm{e}_y}{\left(\dot{x}^2 + \dot{y}^2\right)^{1/2}} = \frac{-a\sin v\,\mathrm{e}_x + b\cos v\,\mathrm{e}_y}{\left(a^2\sin^2 v + b^2\cos^2 v\right)^{1/2}} && \text{unit tangent} \\
\mathrm{n} = \mathrm{t}\times\mathrm{e}_z &= \frac{\dot{y}\,\mathrm{e}_x - \dot{x}\,\mathrm{e}_y}{\left(\dot{x}^2 + \dot{y}^2\right)^{1/2}} = \frac{b\cos v\,\mathrm{e}_x + a\sin v\,\mathrm{e}_y}{\left(a^2\sin^2 v + b^2\cos^2 v\right)^{1/2}} && \text{unit normal} \\
dl &= \left(\dot{x}^2 + \dot{y}^2\right)^{1/2} dv = \left(a^2\sin^2 v + b^2\cos^2 v\right)^{1/2} dv && \text{arc length}
\end{aligned}
$$

6. Similarly

$$
\begin{aligned}
\left|\mathrm{r} - \mathrm{r}'\right| &= \left\{\left[x(v') - x(v)\right]^2 + \left[y(v') - y(v)\right]^2\right\}^{1/2} \\
&= \left[a^2\left(\cos v' - \cos v\right)^2 + b^2\left(\sin v' - \sin v\right)^2\right]^{1/2}
\end{aligned}
$$

7. From this we can evaluate the Green's function

$$
\begin{aligned}
G &= \frac{1}{4\pi}\ln\left\{\left[x(v') - x(v)\right]^2 + \left[y(v') - y(v)\right]^2\right\} \\
&= \frac{1}{4\pi}\ln\left[a^2\left(\cos v' - \cos v\right)^2 + b^2\left(\sin v' - \sin v\right)^2\right]
\end{aligned}
$$

8. We also need the normal derivative of the Green's function.

$$
\begin{aligned}
\left(\dot{x}'^2 + \dot{y}'^2\right)^{1/2} \mathrm{n}' \cdot \nabla' G &= \frac{1}{4\pi}\left(\dot{y}' \frac{\partial}{\partial x'} - \dot{x}' \frac{\partial}{\partial y'}\right) \ln\left[\left(x'-x\right)^2 + \left(y'-y\right)^2\right] \\
&= \frac{1}{2\pi} \frac{\dot{y}'\left(x'-x\right) - \dot{x}'\left(y'-y\right)}{\left(x'-x\right)^2 + \left(y'-y\right)^2} \\
&= \frac{ab}{2\pi} \frac{1-\cos\left(v'-v\right)}{a^2\left(\cos v' - \cos v\right)^2 + b^2\left(\sin v' - \sin v\right)^2}
\end{aligned}
$$

7. Observe that $G$ has a logarithmic singularity when $v' \to v$. However, this is an integrable singularity

$$
\int \ln x \, dx = x \ln x - x = \text{ finite}
$$

8. Because of this one might think that $\partial G / \partial n'$ would be singular as $1/\left|\mathrm{r} - \mathrm{r}'\right|$ when $v' \to v$. It is actually finite. Using L'Hospital's rule twice we find that for the case of the ellipse

$$
\underset{v' \to v}{L} \left(\dot{x}'^2 + \dot{y}'^2\right)^{1/2} \mathrm{n}' \cdot \nabla' G = \frac{ab}{4\pi} \frac{1}{a^2 \sin^2 v + b^2 \cos^2 v}
$$

9. In fact it can be shown that in general $\partial G / \partial n'$ is finite on any surface as $\left|\mathrm{r} - \mathrm{r}'\right| \to 0$.

$$
\underset{v' \to v}{L} \left(\dot{x}'^2 + \dot{y}'^2\right)^{1/2} \mathrm{n}' \cdot \nabla' G = \frac{1}{4\pi} \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^2 + \dot{y}^2}
$$

10. Let's return now to the integral equation of interest which can be written as

$$
\begin{aligned}
\frac{1}{2}\psi(v) &= \int \left[\psi\left(v'\right)\frac{\partial G}{\partial n'} - G \frac{\partial \psi\left(v'\right)}{\partial n'}\right] dl' \\
&= \int_0^{2\pi} \left[\psi'\left(\dot{x}'^2 + \dot{y}'^2\right)^{1/2} \mathrm{n}' \cdot \nabla G - G\left(\dot{x}'^2 + \dot{y}'^2\right)^{1/2} \mathrm{n}' \cdot \nabla \psi'\right] dv'
\end{aligned}
$$

11. We shall solve this equation by Fourier analysis leading to a relation between $\psi(v)$ and $\partial\psi(v)/\partial n$
12. The expansion is as follows

$$\psi(v) = \sum_{-\infty}^{\infty} a_m e^{imv}$$
$$\left(\dot{x}^2 + \dot{y}^2\right)^{1/2} \mathrm{n}\cdot\nabla\psi(v) = \sum_{-\infty}^{\infty} b_m e^{imv}$$

13. The goal now is to find a relation between the $a_m$ and $b_m$. One quantity is given by the boundary condition. The other is obtained by solving the integral equation.
14. Let us assume that $\psi_S = \psi(v)$ is specified on the surface. This means that we know the $a_m$ coefficients.

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} \psi(v) e^{-inv} dv$$

15. Now Fourier analyze the Green's function and its normal derivative on the boundary curve

$$G\left(v,v'\right) = \frac{1}{2\pi}\sum_{m,n} B_{mm'} e^{imv-im'v'}$$
$$\left(\dot{x}'^2 + \dot{y}'^2\right)^{1/2} \mathrm{n}'\cdot\nabla' G\left(v,v'\right) = \frac{1}{2\pi}\sum_{m,n} A_{mm'} e^{imv-im'v'}$$

16. The matrix elements $A_{mm'}, B_{mm'}$ are known quantities that can be evaluated numerically in a straightforward manner.

$$B_{nn'} = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} G\left(v,v'\right)e^{-inv+inv'} dv\, dv'$$
$$A_{nn'} = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} \left(\dot{x}'^2 + \dot{y}'^2\right)^{1/2} \mathrm{n}'\cdot\nabla' G\left(v,v'\right)e^{-inv+inv'} dv\, dv'$$

17. These expansions are substituted into Green's theorem

$$\frac{1}{2}\sum_m a_m e^{imv} = \frac{1}{2\pi}\sum_{n',p,p'} a_{n'}A_{pp'}\int_0^{2\pi} e^{in'v'}e^{ipv-ip'v'}dv' - \frac{1}{2\pi}\sum_{n',p,p'} b_{n'}B_{pp'}\int_0^{2\pi} e^{in'v'}e^{ipv-ip'v'}dv'$$

18. Carry out Fourier analysis by multiplying by

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-inv}dv$$

19. The various terms are evaluated as follows

$$\frac{1}{2}\sum_m a_m \frac{1}{2\pi}\int_0^{2\pi} e^{i(m-n)v}dv = \frac{1}{2}a_n$$

$$\frac{1}{2\pi}\sum_{n',p,p'} a_{n'}A_{pp'}\frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} e^{i(n'-p')v'}e^{i(p-n)v}dv'dv = \sum_{n'} A_{nn'}a_{n'}$$

$$\frac{1}{2\pi}\sum_{n',p,p'} b_{n'}B_{pp'}\frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} e^{i(n'-p')v'}e^{i(p-n)v}dv'dv = \sum_{n'} B_{nn'}b_{n'}$$

20. Combine terms

$$\frac{1}{2}a_n = \sum_{n'}(A_{nn'}a_{n'} - B_{nn'}b_{n'})$$

21. In compact form this can be written as

$$\left(\frac{1}{2}\overleftrightarrow{\mathrm{I}} - \overleftrightarrow{\mathrm{A}}\right)\cdot \mathrm{a} = \overleftrightarrow{\mathrm{B}}\cdot \mathrm{b}$$

22. Consequently, if $\psi(v)$ is specified, then the Fourier coefficients for the normal derivative are given by

$$\mathrm{b} = \left[\overleftrightarrow{\mathrm{B}}^{-1}\cdot\left(\frac{1}{2}\overleftrightarrow{\mathrm{I}} - \overleftrightarrow{\mathrm{A}}\right)\right]\cdot \mathrm{a}$$

23. Conversely, if the normal derivative $\partial\psi(v)/\partial n$ is given then the Fourier coefficients of the potential are given by

$$\mathrm{a} = \left[\left(\frac{1}{2}\overleftrightarrow{\mathrm{I}} - \overleftrightarrow{\mathrm{A}}\right)^{-1} \cdot \overleftrightarrow{\mathrm{B}}\right] \cdot \mathrm{b}$$

24. Once both a and b are known, then $\psi(\mathrm{r})$ can be found at any interior point by evaluating the now known Green's function integrals.

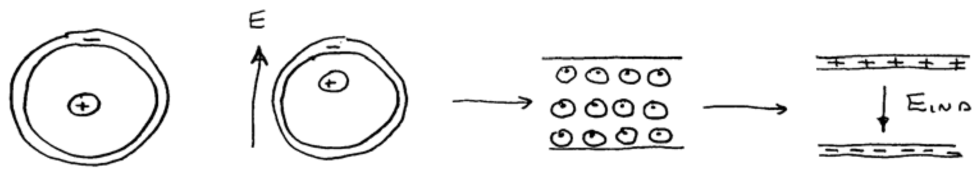
$$\psi(\mathrm{r}) = \int_{S'} \left(\psi' \frac{\partial G}{\partial n'} - G \frac{\partial \psi'}{\partial n'}\right) dS'$$

25. This complicated procedure has been used extensively used in the NSE fusion program.
26. One problem involved an accurate determination of the magnetic field in the presence of large amounts of iron in the PHENIX detector on the RHIC facility at Brookhaven National Laboratory.
27. Another application involves determining the best set of coil currents in the Alcator C-Mod poloidal field system to achieve a given desired plasma shape.

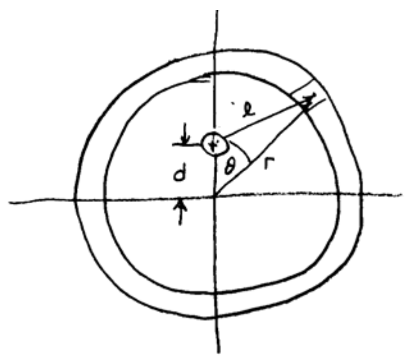
## Dielectrics

1. A new topic now – dielectrics
2. What is a dielectric?
3. A dielectric is an insulating material – one with no free charges and no conduction electrons (as in a metal)
4. Dielectrics consist of neutral atoms which become polarized when placed in an electric field.
5. We shall see that the direction of polarization is such as to cancel part of the applied field.

6. A simple physical picture is shown below




7. In a real material not every atom stays polarized. Other forces, such as thermal forces, are also present which tend to randomize the polarization. Thus the amount of polarization depends upon the detailed atomic structure of the material under consideration.
8. Let's see if we can create a model to determine the induced electric field in a simple atom. Keep in mind that this is a qualitative, not quantitative model.
9. In the diagram below assume the nucleus of the atom is infinitely massive (compared to the electron). An electron cloud encircles the nucleus with a radius $r_0$ determined by quantum mechanics.



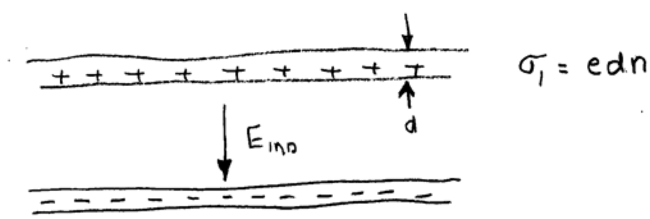

10. An electric field is applied causing a slight shift $d \ll r_0$ in the location of the electron cloud. There is now more cloud below the nucleus than above it. This generates a net Coulomb force on the cloud.
11. Note that direction of the electric field induced by this charge separation is opposite to that of the applied field.
12. We can approximate the relationship between $d$ and $E$ by a simple force balance as follows.
13. Assume the sphere of electron charge has a uniform charge density $\rho_0$ and a thickness $\Delta r \ll r_0$.
14. The net upward force on the cloud $F_z = \mathbf{F} \cdot \mathbf{e}_z = F \cos\theta$ is found by integrating Coulomb's force law over the volume of the cloud

$$F_z = qE = q\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \int_{r_0}^{r_0+\Delta r} r^2 dr \left(\frac{\rho_0}{4\pi\varepsilon_0 l^2}\cos\theta\right)$$
$$l^2 = r^2 + d^2 - 2rd\cos\theta \approx r_0^2 - 2r_0 d\cos\theta$$

15. This integral can easily be evaluated in the limits $d \ll r_0$ and $\Delta r \ll r_0$. We obtain (with $q = -e$)

$$qE = -\frac{q^2 d}{6\pi\varepsilon_0 r_0^3} \qquad \rightarrow \qquad d = \left(\frac{6\pi\varepsilon_0 r_0^3}{e}\right)E$$

16. Here we have balanced the shift in the orbit $d$ caused by the electric field $E$ (equal to the applied field plus the induced field due to all other atoms) against the attractive Coulomb force.
17. Next, note that in a dielectric the negative charge due to a downward shift of one electron is balanced by a deficit of negative charge from the atom located one layer lower. There is only a net effect at the surfaces of the dielectric where no further compensating charges are available.




18. As shown, this produces a surface charge whose value is estimated by assuming that on average the number density of atoms in the material is $n$ particles per cubic meter.
19. Each electron carries a charge equal to $q = -e$. The total number of unbalanced electrons is related to the average shift $d$ due to the polarization. Thus $nV = nAd$. The total charge in the unbalanced layer is $qnV$ which is equivalent to a surface charge density $\sigma_2 = qnV / A = qnd$
20. Clearly there is also a net deficit of electrons on the upper edge of dielectric producing a surface charge $\sigma_1 = -\sigma_2$.
21. The net macroscopic effect of the polarization is to induce a macroscopic opposing electric field within the dielectric which is calculated as the field between two equal and opposite surface charges.

$$E_{ind} = \frac{\sigma_1}{\varepsilon_0} = -\frac{edn}{\varepsilon_0} = -\left(6\pi r_0^3 n\right)E$$

22. We now introduce the concept of the “relative dielectric constant” as follows. Write down the 1-D form of Poisson’s equation.

$$\frac{\partial \varepsilon_0 E}{\partial z} = \frac{\rho_{free}}{\varepsilon_0} + \frac{\rho_{ind}}{\varepsilon_0}$$

23. Here, $\rho_{ind}$ represents the induced surface charge due to the polarization while $\rho_{free}$ represents any other free charge that may be present in the material (e.g. such as due to a beam of charged particles propagating through the material)
24. If we integrate Poisson’s equation across the dielectric we obtain

$$\varepsilon_0 E = \int \rho_{free} dz + \varepsilon_0 E_{ind}$$

25. This can be rewritten as

$$\varepsilon_0 \left(1 + \chi_p\right) E = \int \rho_{free} dz$$
$$\chi_p = 6\pi n r_0^3$$

26. We can now define the relative dielectric constant as $\varepsilon_r = 1 + \chi_p$ and Poisson’s equation becomes

$$\nabla \cdot (\varepsilon \mathrm{E}) = \rho$$

27. For a simple dielectric we show that after all this work we simply replace $\varepsilon_0$ with $\varepsilon = \varepsilon_0 \varepsilon_r$ .

Boundary conditions for a dielectric

1. It is customary in E&M theory to introduce the displacement vector $\mathbf{D} = \varepsilon \mathbf{E}$ so that Poisson's equation becomes

$$\nabla \cdot \mathbf{D} = \rho$$
$$\mathbf{D} = \varepsilon \mathbf{E}$$

2. The boundary conditions across a dielectric-vacuum interface are conveniently expressed in terms of $\mathbf{E}$ and $\mathbf{D}$. They are found as follows.
3. Consider the area integral shown below and use the fact that in electrostatics $\nabla \times \mathbf{E} = 0$. Stoke's vector theorem then implies that

$$\int \nabla \times \mathbf{E} \cdot d\mathbf{S} = \oint \mathbf{E} \cdot d\mathbf{l} = 0 \quad \rightarrow \quad [\![\mathbf{n} \times \mathbf{E}]\!] = 0$$

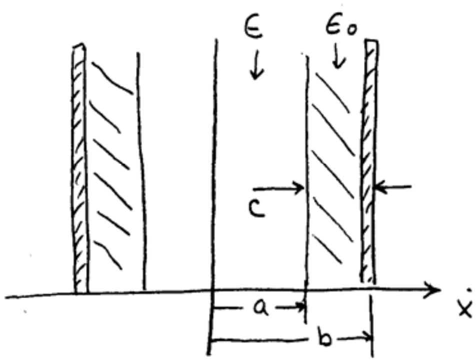

4. Next, we integrate Poisson's equation over the volume shown below, assuming that no infinitesimally thin free surface charges exist.

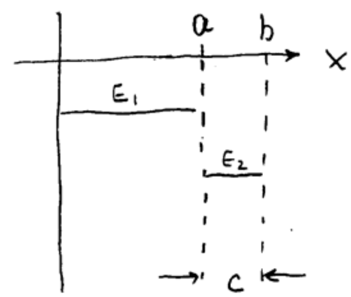
$$\int \nabla \cdot \mathbf{D} d\mathbf{r}' = \int \mathbf{D} \cdot \mathbf{n} dS = 0 \quad \rightarrow \quad [\![\mathbf{n} \cdot \mathbf{D}]\!] = 0$$


5. Across a dielectric-vacuum interface the tangential electric field and normal displacement vector are continuous.

The dielectric filled capacitor

1. As a simple application of dielectrics consider the dielectric filled capacitor as shown below. The goal is to calculate the capacitance of the system and the voltage profile.
2. In particular, the electric field in each region is a constant and we want determine their values from which the other information can then be easily obtained.




3. The solution is obtained as follows. First use the voltage relation

$$V = -\int_0^b E dz \qquad \rightarrow \qquad V = -2(E_1 a + E_2 c) \qquad c = b - a$$

4. Second, from symmetry we see that the condition $[\![\mathrm{n} \times \mathrm{E}]\!] = 0$ across the interface is automatically satisfied.
5. Third, across the interface the condition on the displacement vector reduces to

$$[\![\mathrm{n} \cdot \mathrm{D}]\!] \qquad \rightarrow \qquad \varepsilon_r E_1 = E_2$$

6. We can solve these two simultaneous equations for $E_1$ and $E_2$

$$
\begin{aligned}
E_1 &= -\frac{V}{2}\frac{1}{a + \varepsilon_r c} \\
E_2 &= -\frac{V}{2}\frac{\varepsilon_r}{a + \varepsilon_r c}
\end{aligned}
$$

7. The capacitance can easily be calculated from the energy definition

$$
\begin{aligned}
\frac{1}{2}CV^2 &= \int \frac{\varepsilon E^2}{2} d\mathrm{r} \\
&= A\varepsilon_0 \left(E_2^2 c + \varepsilon_r E_1^2 a\right) \\
&= \frac{A\varepsilon_0 V^2}{4}\left[\frac{\varepsilon_r^2 c}{(a + \varepsilon_r c)^2} + \frac{\varepsilon_r a}{(a + \varepsilon_r c)^2}\right] \\
&= \frac{A\varepsilon_0 V^2}{4}\frac{\varepsilon_r}{a + \varepsilon_r c}
\end{aligned}
$$

8. We see that

$$
C = \frac{\varepsilon_0 A}{2a}\left(\frac{\varepsilon_r}{1 + \varepsilon_r c / a}\right)
$$

9. The voltage drop across each region can now be easily calculated.

$$
\begin{aligned}
V_1 &= -2E_1 a = \frac{V}{1 + \varepsilon_r c / a} \\
V_2 &= -2E_2 c = \frac{V}{1 + \varepsilon_r c / a}\frac{\varepsilon_r c}{a}
\end{aligned}
$$

10. Lastly, consider the interesting limit of a strongly diamagnetic material $\varepsilon_r \gg 1$

11. Then

$$\frac{V_1}{V} \approx \frac{a}{c\varepsilon_r} \ll 1$$
$$\frac{V_2}{V} \approx 1$$

12. Note that most of the voltage drop occurs across the vacuum. It had better be a good vacuum to avoid breakdown.