Lecture 7

Applications of Magnetostatics

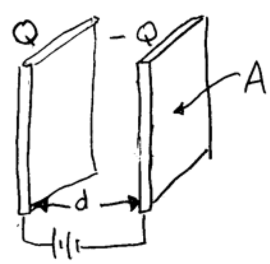
Today's topics

1. Inductance
2. Magnetic materials
3. Boundary conditions
4. Shielding

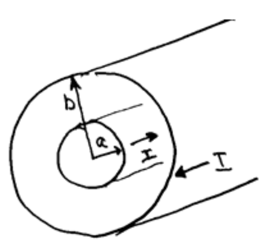
Inductance

1. Recall how to calculate capacitance from electrostatics

$$\frac{1}{2}CV^2 = \int \frac{\varepsilon_0 E^2}{2} d\mathbf{r}' \qquad \text{(more fundamental)}$$
$$CV = Q$$



2. The definitions are the same if the plates are located in a vacuum region
3. For two parallel plates $C = \varepsilon_0 A / d$
4. As a simple magnetostatic analog let's calculate the inductance of a finite diameter wire carrying a current I surrounded by a perfectly conducting shell carrying a return current $-I$.



5. First we calculate the magnetic field inside and outside the wire.

6. Inside the wire $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ while outside the wire $\nabla \times \mathbf{B} = 0$.
7. By symmetry we see that the non-trivial field components are $\mathbf{B} = B_\theta \mathbf{e}_\theta$ and $\mathbf{J} = J_z \mathbf{e}_z$.
8. If the current in the wire is uniform then $J_z = I / \pi a^2$
9. Apply Ampere's law in the wire

$$\frac{1}{r}\frac{d}{dr}(rB_\theta) = \mu_0 J_z$$
$$B_\theta = \frac{\mu_0 I}{2\pi}\frac{r}{a^2} + \frac{c}{r} = \frac{\mu_0 I}{2\pi}\frac{r}{a^2}$$

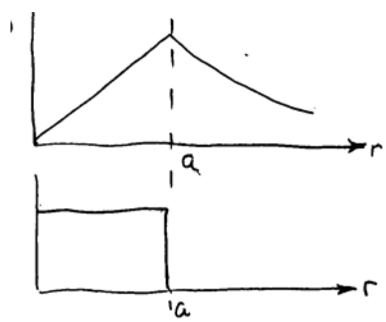
10. Outside the wire

$$\frac{1}{r}\frac{d}{dr}(rB_\theta) = 0$$
$$B_\theta = \frac{c}{r}$$

11. Match across $r = a$: $[\![B_\theta]\!]_a = 0 \quad \rightarrow \quad c = \mu_0 I / 2\pi$
12. Therefore

$$B_\theta = \frac{\mu_0 I}{2\pi r}$$

13. Below is a plot of B_θ due to the wire

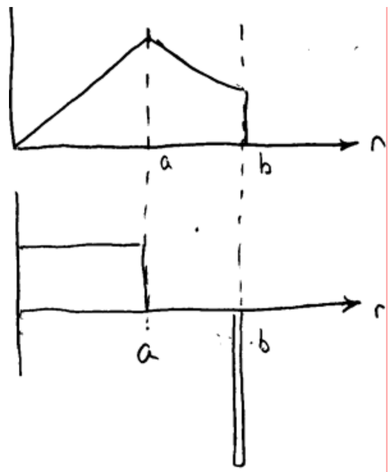


14. Now let's calculate the field due to the return current

Inside the shell: $\nabla \times \mathbf{B} = 0 \quad \rightarrow \quad B_\theta = c / r = 0$

Outside the shell: $\nabla \times \mathbf{B} = 0 \quad \rightarrow \quad B_\theta = c / r = -\mu_0 I / 2\pi r$

15. The total field is found by superposition and is shown below.



Calculating the inductance

1. In general there is both internal and external inductance.
2. This makes it a little more complicated than calculating capacitance between thin conducting plates.
3. Here is a simple widely used definition of external inductance

$$L_e = \frac{\psi_e}{I} \qquad \psi_e = \text{ external flux}$$

4. Evaluate the external flux

$$\psi_e = \int \mathbf{B}\cdot\mathbf{n}\,dS = \int_0^L dz \int_a^b B_\theta dr = \frac{\mu_0 IL}{2\pi}\int_a^b \frac{dr}{r} = \frac{\mu_0 IL}{2\pi}\ln\frac{b}{a}$$

5. The inductance is thus given by

$$L = \frac{\psi_e}{I} = \frac{\mu_0 L}{2\pi}\ln\frac{b}{a}$$

6. We get the same result from the more basic definition

$$\frac{1}{2}LI^2 = \int \frac{B^2}{2\mu_0}\,d\mathbf{r}'$$

7. The details are as follows

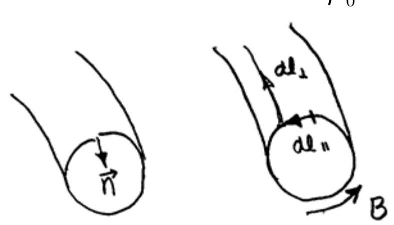
$$\int \frac{B^2}{2\mu_0} d\mathbf{r}' = \frac{1}{2\mu_0}\left(\frac{\mu_0 I}{2\pi}\right)^2 \int_a^b \frac{rdr}{r^2}\int_0^{2\pi} d\theta \int_0^L dz = \frac{\mu_0 I^2}{8\pi^2}\left(\ln\frac{b}{a}\right)(2\pi L)$$

$$L_e = \frac{\mu_0 L}{2\pi}\ln\frac{b}{a}$$

Equivalence of the two definitions

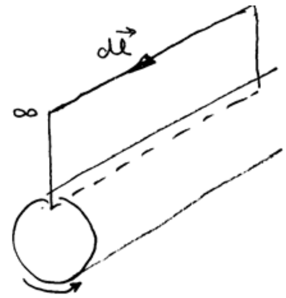
1. The equivalence is shown as follows

$$\begin{aligned}
\frac{1}{2}L_e I^2 &= \int \frac{B^2}{2\mu_0} d\mathbf{r}' = \frac{1}{2\mu_0}\int (\nabla\times\mathbf{A})^2\, d\mathbf{r}' \\
&= \frac{1}{2\mu_0}\int \left[\nabla\cdot(\mathbf{A}\times\nabla\times\mathbf{A}) - \mathbf{A}\cdot\nabla\times\nabla\times\mathbf{A}\right] d\mathbf{r}' \\
&= \frac{1}{2\mu_0}\int \nabla\cdot(\mathbf{A}\times\nabla\times\mathbf{A})\, d\mathbf{r}' = \frac{1}{2\mu_0}\int \mathbf{A}\times\mathbf{B}\cdot\mathbf{n}\, dS \\
&= \frac{1}{2\mu_0}\int \mathbf{A}\times\mathbf{B}\cdot d\mathbf{l}_{\parallel}\times d\mathbf{l}_{\perp} = \frac{1}{2\mu_0}\int \mathbf{A}\cdot\mathbf{B}\times d\mathbf{l}_{\parallel}\times d\mathbf{l}_{\perp} \\
&= \frac{1}{2\mu_0}\int \left[-\left(\mathbf{B}\cdot d\mathbf{l}_{\perp}\right)\left(\mathbf{A}\cdot d\mathbf{l}_{\parallel}\right) + \left(\mathbf{B}\cdot d\mathbf{l}_{\parallel}\right)\left(\mathbf{A}\cdot d\mathbf{l}_{\perp}\right)\right] \\
&= \frac{1}{2\mu_0}\int \left[\left(\mathbf{B}\cdot d\mathbf{l}_{\parallel}\right)\left(\mathbf{A}\cdot d\mathbf{l}_{\perp}\right)\right] = \frac{\mu_0 I}{2\mu_0}\int \mathbf{A}\cdot d\mathbf{l}_{\perp}
\end{aligned}$$



2. The last term is further simplified by noting that

$$\psi_e = \int \mathbf{B}\cdot d\mathbf{S} = \int \nabla\times\mathbf{A}\cdot d\mathbf{S} = \int \mathbf{A}\cdot d\mathbf{l}_{\perp}$$



3. Therefore,

$$\frac{1}{2}L_e I^2 = \frac{1}{2}I\psi_e \qquad \rightarrow \qquad \psi_e = L_e I$$

4. Both definitions are equivalent for vacuum fields (i.e. when $\nabla\times\nabla\times\mathbf{A}=0$)

Internal inductance

1. For the internal inductance L_i we always must use the general definition

$$\frac{1}{2}L_i I^2 = \int \frac{B^2}{2\mu_0}d\mathbf{r}' = \frac{2\pi L}{2\mu_0}\int_0^a rdr\left(\frac{\mu_0 I}{2\pi}\frac{r}{a^2}\right)^2 = \frac{\mu_0 I^2 L}{16\pi}$$
$$L_i = \frac{\mu_0 L}{8\pi}$$

2. What does the definition $\psi_i = L_i I$ give?

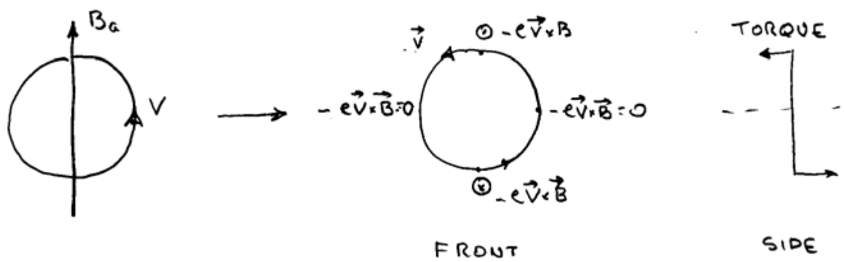
$$\psi_i = \int B_\theta dr dz = \frac{\mu_0 IL}{2\pi}\int_0^a \frac{r}{a^2}dr = \frac{\mu_0 IL}{4\pi}$$
$$L_i = \frac{\mu_0 L}{4\pi}$$

3. This is an incorrect answer.
4. For distributed currents we must use the general energy definition to calculate inductance.

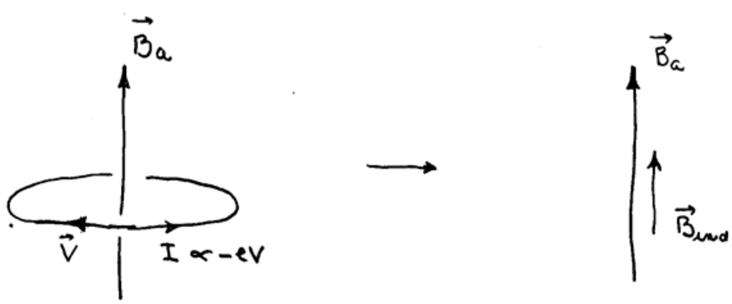
Magnetic materials

1. Recall that in electrostatics an applied electric field induced a small electric dipole whose electric field direction was opposite to the applied field.
2. There is a magnetic analog when a material is placed in a DC magnetic field.

3. What happens when such atoms are placed in an applied magnetic field $\mathbf{B}_a$? A torque develops that tends to flip the electron current so that the current flows in a plane perpendicular to $\mathbf{B}_a$.



4. The flips to a new position where the torque is zero.



5. The direction of the flipping is such as to enhance the field (paramagnetic).
6. As in electrostatics we can introduce the property of permeability. This is analogous to polarizability which allowed us to introduce a relative dielectric constant.

7. The comparisons are as follows:

Electrostatics	Magnetostatics
$\mathbf{E}_{ind} = -\chi_p \mathbf{E}$	$\mathbf{B}_{ind} = +\chi_m \mathbf{B}$
$\nabla \cdot \mathbf{E} = \frac{\rho_{free}}{\varepsilon_0} + \frac{\rho_{ind}}{\varepsilon_0}$	$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}_{free} + \mu_0 \mathbf{J}_{ind}$
$\nabla \cdot \mathbf{E}_{ind} = \frac{\rho_{ind}}{\varepsilon_0}$	$\nabla \times \mathbf{B}_{ind} = \mu_0 \mathbf{J}_{ind}$
$\nabla \cdot (\mathbf{E} - \mathbf{E}_{ind}) = \frac{\rho_{free}}{\varepsilon_0}$	$\nabla \times (\mathbf{B} - \mathbf{B}_{ind}) = \mu_0 \mathbf{J}_{free}$
$\nabla \cdot \left[(1+\chi_p)\mathbf{E}\right] = \frac{\rho_{free}}{\varepsilon_0}$	$\nabla \times \left[(1-\chi_m)\mathbf{B}\right] = \mu_0 \mathbf{J}_{free}$
$\mathbf{D} = (1+\chi_p)\varepsilon_0 \mathbf{E} = \varepsilon \mathbf{E} = \varepsilon_r \varepsilon_0 \mathbf{E}$	$\mathbf{H} = (1-\chi_m)\mathbf{B}/\mu_0 = \mathbf{B}/\mu = \mathbf{B}/\mu_r \mu_0$
$\nabla \cdot \mathbf{D} = \rho$	$\nabla \times \mathbf{H} = \mathbf{J}$

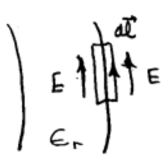
8. Note that for typical materials $\mathbf{E}$ is reduced by the dielectric effect while $\mathbf{B}$ is increased by the paramagnetic effect.

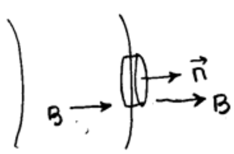
Boundary conditions

1. Let's determine the boundary conditions across the interface between a magnetic material and a vacuum region.
2. There are two conditions in analogy with electrostatics. The first is given by

$$\text{Electrostatics} \quad \nabla \times \mathbf{E} = 0 \quad \rightarrow \quad \oint \mathbf{E} \cdot d\mathbf{l} = 0 \quad \rightarrow \quad [\![\mathbf{n} \times \mathbf{E}]\!] = 0$$

$$\text{Magnetostatics} \quad \nabla \cdot \mathbf{B} = 0 \quad \rightarrow \quad \int \mathbf{B} \cdot \mathbf{n}\, dS = 0 \quad \rightarrow \quad [\![\mathbf{n} \cdot \mathbf{B}]\!] = 0$$

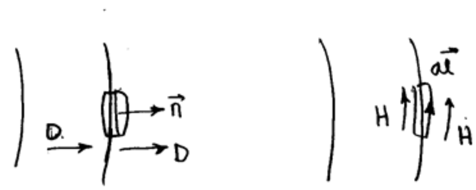




3. The second condition, assuming no surface charge or surface current is given by

$$\text{Electrostatics} \quad \nabla \cdot \mathbf{D} = \rho \quad \rightarrow \quad \int \mathbf{D} \cdot \mathbf{n}\, dS = 0 \quad \rightarrow \quad [\![\varepsilon \mathbf{n} \cdot \mathbf{E}]\!] = 0$$

$$\text{Magnetostatics} \quad \nabla \times \mathbf{H} = \mathbf{J} \quad \rightarrow \quad \oint \mathbf{H} \cdot d\mathbf{l} = 0 \quad \rightarrow \quad [\![\mathbf{n} \times \mathbf{B}/\mu]\!] = 0$$



Statics in a resistive medium

1. Consider a piece of metal with finite thickness.
2. It has a resistivity η because of the friction felt by electrons as they try to flow through the fixed lattice of ions.
3. Therefore a piece of metal carrying a current density $\mathbf{J}$ generates a resistive electric field given by the familiar ohm's law $\mathbf{E} = \eta \mathbf{J}$.
4. Static problems of this type are solved in the following order
5. First, in a static problem $\nabla \times \mathbf{E} = 0$ so that as before $\mathbf{E} = -\nabla \phi$. When no free charges flow in the conductor ϕ satisfies

$$\nabla^2 \phi = 0$$

6. We solve this equation first.
7. Assuming that ϕ is known we next turn to Ampere's law. For a magnetostatic problem we can again write $\mathbf{B} = \nabla \times \mathbf{A}$ with $\nabla \cdot \mathbf{A} = 0$. The vector potential then satisfies

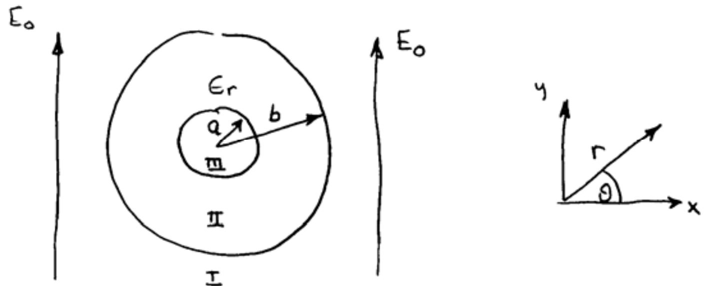
$$\nabla^2 \mathbf{A} = \frac{\mu_0}{\eta} \nabla \phi$$

8. Since we know ϕ we can now solve for $\mathbf{A}$

DC Shielding

1. We next examine the important practical question of DC shielding.
2. The goals are to see whether (1) dielectrics can shield electric fields and (2) magnetic materials can shield magnetic fields.

3. Start with the dielectric problem as illustrated below.



4. With no dielectric the interior electric field is obviously $\mathbf{E} = E_0 \mathbf{e}_y$.
5. With a shell of dielectric we have to solve the Laplace's equation in three regions and match across the interfaces.
6. In each region $\mathbf{E} = -\nabla\phi$ and $\nabla^2\phi = 0$.
7. The solutions are given by

$$
\begin{aligned}
&\text{I.} && \phi = -E_0 y + \frac{c_1}{r}\sin\theta = \left(-E_0 r + \frac{c_1}{r}\right)\sin\theta \\
& && \mathbf{E} = \left(E_0 + \frac{c_1}{r^2}\right)\sin\theta\,\mathbf{e}_r + \left(E_0 - \frac{c_1}{r^2}\right)\cos\theta\,\mathbf{e}_\theta \\
&\text{II.} && \phi = \left(-c_2 r + \frac{c_3}{r}\right)\sin\theta \\
& && \mathbf{E} = \left(c_2 + \frac{c_3}{r^2}\right)\sin\theta\,\mathbf{e}_r + \left(c_2 - \frac{c_3}{r^2}\right)\cos\theta\,\mathbf{e}_\theta \\
&\text{III.} && \phi = -c_4 r\sin\theta \\
& && \mathbf{E} = c_4\sin\theta\,\mathbf{e}_r + c_4\cos\theta\,\mathbf{e}_\theta
\end{aligned}
$$

8. The goal is to calculate $c_4 = E_{inside}$ and see how it compares in magnitude to the applied field E_0.
9. The matching conditions across $r = b$ yield two relations

$$
\begin{aligned}
&[\![E_\theta]\!] = 0 && E_0 - \frac{c_1}{b^2} = c_2 - \frac{c_3}{b^2} \\
&[\![\varepsilon_r E_r]\!] = 0 && E_0 + \frac{c_1}{b^2} = \varepsilon_r\left(c_2 + \frac{c_3}{b^2}\right)
\end{aligned}
$$

10. There are two similar conditions across $r = a$

$$[\![E_\theta]\!] = 0 \qquad c_4 = c_2 - \frac{c_3}{a^2}$$

$$[\![\varepsilon_r E_r]\!] = 0 \qquad c_4 = \varepsilon_r\left(c_2 + \frac{c_3}{a^2}\right)$$

11. These are 4 equations with four unknowns. We can easily solve them and evaluate c_4.

$$\frac{E_{inside}}{E_0} = \frac{4\varepsilon_r}{(\varepsilon_r + 1)^2 - (\varepsilon_r - 1)^2 (a/b)^2}$$

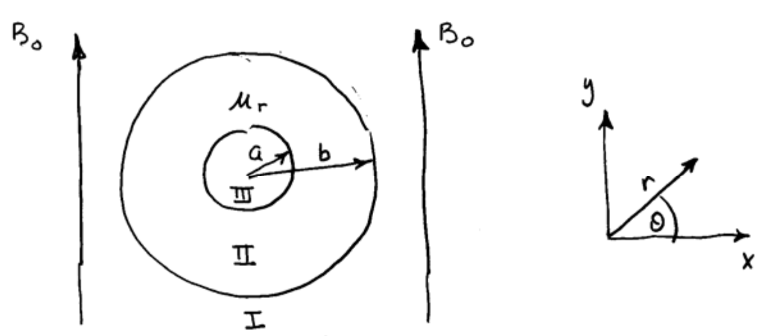
12. In the limit of a large dielectric constant $\varepsilon_r \gg 1$ we have

$$\frac{E_{inside}}{E_0} = \frac{4}{\varepsilon_r\left(1 - a^2/b^2\right)} \ll 1$$

13. The conclusion is that a material with a high dielectric constant does a good job shielding DC electric fields.

Magnetic shielding

1. We can now ask a similar question with respect to magnetostatics. Does a highly permeable material shield out magnetic fields?
2. The calculation is very similar to the electrostatic case. Consider the problem illustrated below.



3. Clearly with no permeable material the field on the inside is just B_0.
4. When a permeable material is present we again have to solve a three region problem. The solutions in each region satisfy $\mathbf{B} = \nabla \times (A\mathbf{e}_z)$ with $\nabla^2 A = 0$.
5. The solutions are given by

$$\text{I.} \qquad A = -B_0 x + \frac{c_1}{r}\cos\theta = \left(-B_0 r + \frac{c_1}{r}\right)\cos\theta$$

$$\mathbf{B} = \left(B_0 - \frac{c_1}{r^2}\right)\sin\theta\,\mathbf{e}_r + \left(B_0 + \frac{c_1}{r^2}\right)\cos\theta\,\mathbf{e}_\theta$$

$$\text{II.} \qquad A = \left(-c_2 r + \frac{c_3}{r}\right)\cos\theta$$

$$\mathbf{B} = \left(c_2 - \frac{c_3}{r^2}\right)\sin\theta\,\mathbf{e}_r + \left(c_2 + \frac{c_3}{r^2}\right)\cos\theta\,\mathbf{e}_\theta$$

$$\text{III.} \qquad A = -c_4 r\cos\theta$$

$$\mathbf{B} = c_4\sin\theta\,\mathbf{e}_r + c_4\cos\theta\,\mathbf{e}_\theta$$

6. We again have matching conditions across the two interfaces.

$$[\![B_r]\!]_b = 0 \qquad B_0 + \frac{c_1}{b^2} = c_2 + \frac{c_3}{b^2}$$

$$[\![B_\theta / \mu]\!]_b = 0 \qquad B_0 - \frac{c_1}{b^2} = \frac{1}{\mu_r}\left(c_2 - \frac{c_3}{b^2}\right)$$

$$[\![B_r]\!]_a = 0 \qquad c_4 = c_2 + \frac{c_3}{a^2}$$

$$[\![B_\theta / \mu]\!]_a \qquad c_4 = \frac{1}{\mu_r}\left(c_2 - \frac{c_3}{a^2}\right)$$

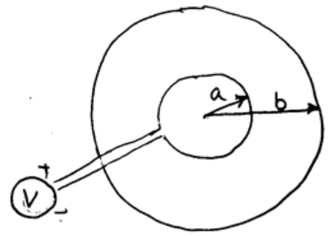
7. Solve for c_4 to find the interior field.

$$\frac{B_{inside}}{B_0} = \frac{4\mu_r}{(\mu_r+1)^2 - (\mu_r-1)^2(a/b)^2} \approx \frac{4}{\mu_r\left(1 - a^2/b^2\right)} \ll 1$$

8. A material with a high permeability $(\mu_r \gg 1)$ does a good job shielding out DC magnetic fields.
9. What happens if I try to shield electric or magnetic fields with a conductor which has $\varepsilon_r = \mu_r = 1$?

Conducting materials

1. To learn a little about conductors consider the single turn solenoidal magnetic illustrated below.



2. Our goal is to calculate the electric field, magnetic field, and current density. We start with the electric field in the magnet which by symmetry is in the θ direction.
3. The electric field equation reduces to

$$\nabla \times \mathbf{E} = 0 \quad \rightarrow \quad \frac{1}{r}\frac{drE_\theta}{dr} = 0 \quad \rightarrow \quad E_\theta = \frac{c_1}{r}$$

4. The constant c_1 is found by noting that around the circumference of the coil

$$V = -\int_0^{2\pi} E_\theta r dr \quad \rightarrow \quad c_1 = -\frac{V}{2\pi} \quad \rightarrow \quad E_\theta = -\frac{V}{2\pi r}$$

5. The current density in the magnet is then given by

$$J_\theta = \frac{E_\theta}{\eta} = -\frac{V}{2\pi\eta r}$$

6. The magnetic field in the magnet satisfies Ampere's law

$$\frac{dB_z}{dr} = -\mu_0 J_\theta = \frac{\mu_0 V}{2\pi\eta r}$$

7. The solution is

$$B_z = \frac{\mu_0 V}{2\pi\eta}\ln r + c_2$$

8. The integration constant c_2 is found by noting that in a long solenoid the field is zero outside the coil. Therefore,

$$B_z = \frac{\mu_0 V}{2\pi\eta} \ln\frac{r}{b}$$

9. The magnetic field in the coil is uniform. Its value is given by

$$B_z(r)_{inside} = \frac{\mu_0 V}{2\pi\eta} \ln\frac{a}{b}$$

10. The fields are sketched below.