DEPARMENT OF NUCLEAR ENGINEERING

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

NOTE L.4

"INTRODUCTION TO STRUCTURAL MECHANICS"

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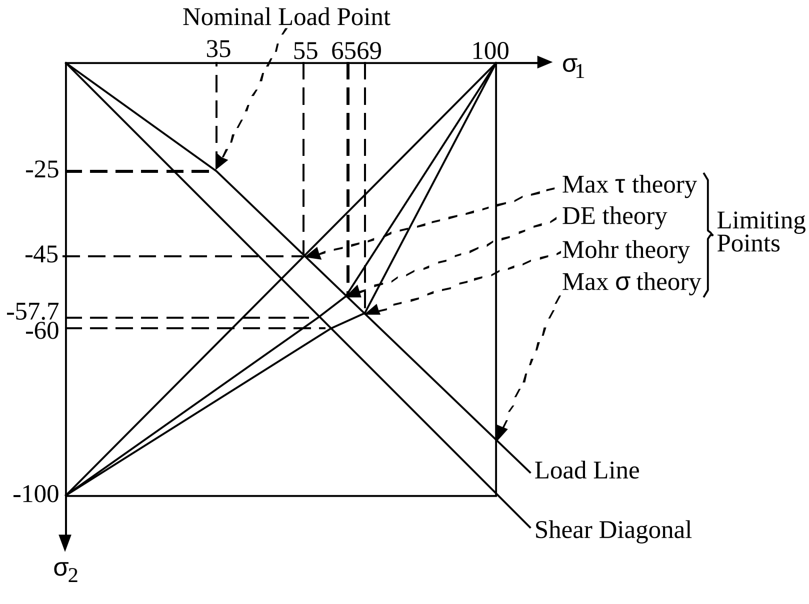

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Rev 12/23/03

22.312 ENGINEERING OF NUCLEAR REACTORS

Fall 2003

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"INTRODUCTION TO STRUCTURAL MECHANICS"

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1979, 1993, 1994, 1995, 1996, 1998, 2000, 2003

12/23/03

TABLE OF CONTENTS

TABLE OF CONTENTS (continued)

“INTRODUCTION TO STRUCTURAL MECHANICS”

M. S. Kazimi, N.E. Todreas and L. Wolf

1. DEFINITION OF CONCEPTS

Structural mechanics is the body of knowledge describing the relations between external forces, internal forces and deformation of structural materials. It is therefore necessary to clarify the various terms that are commonly used to describe these quantities. In large part, structural mechanics refers to solid mechanics because a solid is the only form of matter that can sustain loads parallel to the surface. However, some considerations of fluid-like behavior (creep) are also part of structural mechanics.

Forces are vector quantities, thus having direction and magnitude. They have special names (Fig. 1) depending upon their relationship to a reference plane:

a) Compressive forces act normal and into the plane;

b) Tensile forces act normal and out of the plane; and

c) Shear forces act parallel to the plane.

Pairs of oppositely directed forces produce twisting effects called moments.

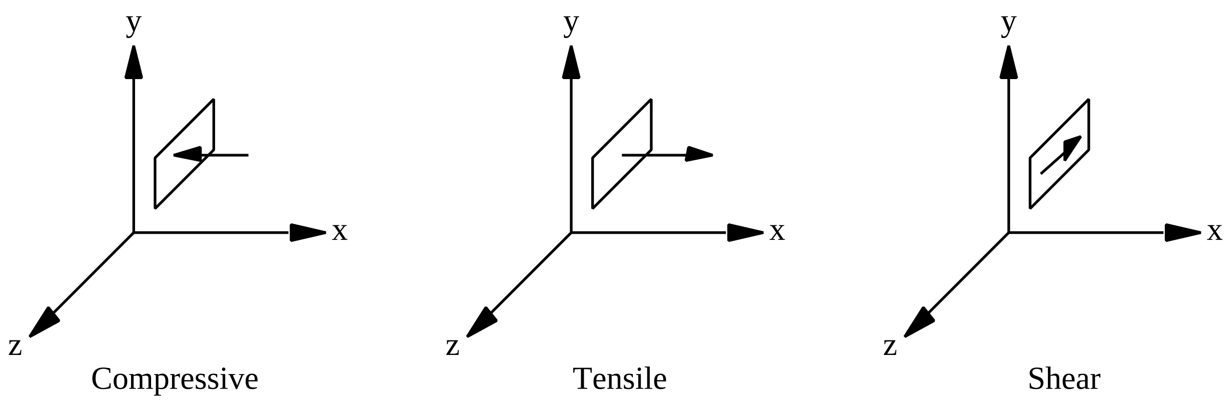


Figure 1. Definition of Forces.

The mathematics of stress analysis requires the definition of coordinate systems. Fig. 2 illustrates a right-handed system of rectangular coordinates.

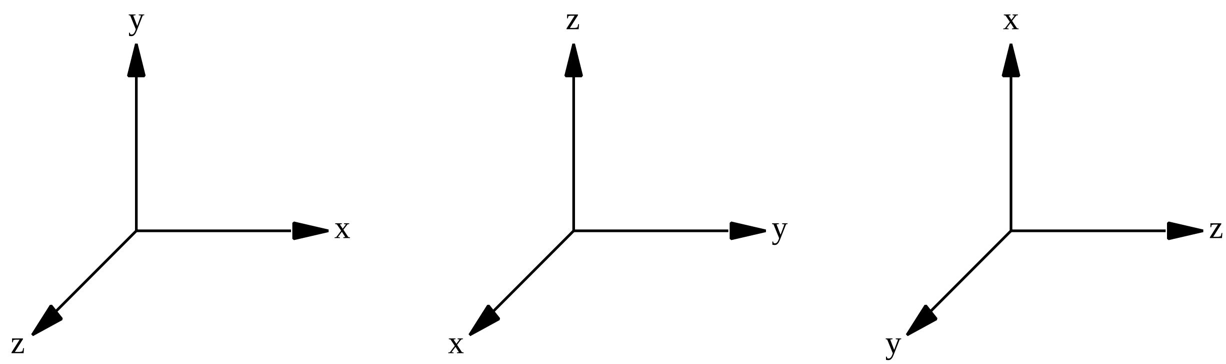


Figure 2. Right-handed System of Rectangular Coordinates.

In the general case, a body as shown in Fig. 3 consisting of an isolated group of particles will be acted upon by both external or surface forces, and internal or body forces (gravity, centrifugal, magnetic attractions, etc.)

If the surface and body forces are in balance, the body is in static equilibrium. If not, accelerations will be present, giving rise to inertia forces. By D'Alembert's principle, the resultant of these inertial forces is such that when added to the original system, the equation of equilibrium is satisfied.

The system of particles of Fig. 3 is said to be in equilibrium if every one of its constitutive particles is in equilibrium. Consequently, the resulting force on each particle is zero, and hence the vector sum of all the forces shown in Fig. 3 is zero. Finally, since we observe that the internal forces occur in self-canceling pairs, the first necessary condition for equilibrium becomes that the vector sum of the external forces must be zero.

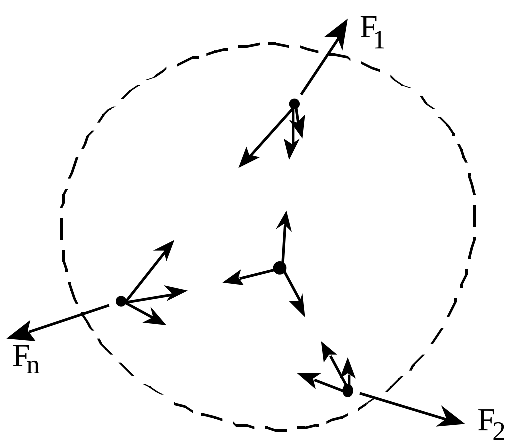


Figure 3. An Isolated System of Particles Showing External and Internal Forces (Ref. 1, Fig 1.12).

$$\vec{F}_1 + \vec{F}_2 + \ldots + \vec{F}_n = \sum_n \vec{F}_n = 0 \quad (1.1a)$$

The total moment of all the forces in Fig. 3 about an arbitrary point 0 must be zero since the vector sum of forces acting on each particle is zero. Again, observe that internal forces occur in self-canceling pairs along the same line of action. This leads to the second condition for equilibrium: the total moment of all the external forces about an arbitrary point 0 must be zero.

$$\vec{r}_1\vec{F}_1 + \vec{r}_2\vec{F}_2 + \ldots + \vec{r}_n\vec{F}_n = \sum_n \vec{r}_n\vec{F}_n = 0 \quad (1.1b)$$

where $\vec{r}_n$ extends from point 0 to an arbitrary point on the line of action of force $\vec{F}_n$.

In Fig. 4A, an arbitrary plane, aa, divides a body in equilibrium into regions I and II. Since the force acting upon the entire body is in equilibrium, the forces acting on part I alone must be in equilibrium.

In general, the equilibrium of part I will require the presence of forces acting on plane aa. These internal forces applied *to* part I *by* part II are distributed continuously over the cut surface, but, in general, will vary over the surface in both direction and intensity.

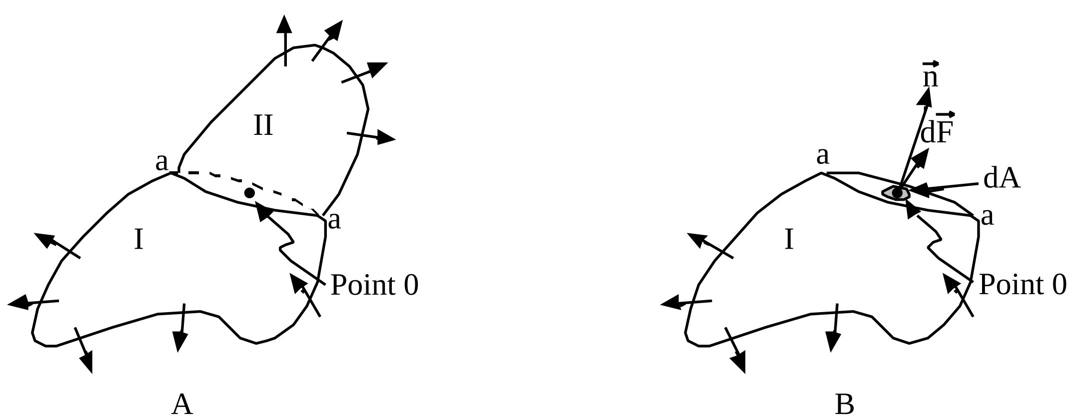


Figure 4. Examination of Internal Forces of a Body in Equilibrium.

Stress is the term used to define the intensity and direction of the internal forces acting at a particular point on a given plane.

Metals are composed of grains of material having directional and boundary characteristics. However, these grains are usually microscopic and when a larger portion of the material is considered, these random variations average out to produce a macroscopically uniform material.

Macroscopic uniformity = homogenous,

If there are no macroscopic direction properties the material is isotropic.

<u>Definition of Stress</u> (mathematically), (Fig. 4B) [see Ref. 1, p. 203]

$$\vec{T}^{(\vec{n})} = \text{stress at point 0 on plane aa whose normal is } \vec{n} \text{ passing through point 0}$$

$$= \lim_{dA \to 0} \frac{d\vec{F}}{dA} \text{ where } d\vec{F} \text{ is a force acting on area dA.}$$

[Reference 1 uses the notation $\vec{T}^{(\vec{n})}$ to introduce the concept that $\vec{T}^{(\vec{n})}$ is a stress vector]

NOTE: Stress is a point value.

The stress acting at a point on a specific plane is a vector. Its direction is the limiting direction of force $d\vec{F}$ as area dA approaches zero. It is customary to resolve the stress vector into two components whose scalar magnitudes are:

normal stress component σ: acting perpendicular to the plane
shear stress component τ: acting in the plane.

1.1 <u>Concept of State of Stress</u>

The selection of different cutting planes through point 0 would, in general, result in stresses differing in both direction and magnitude. Stress is thus a second-order tensor quantity, because not only are <u>magnitude</u> and <u>direction</u> involved but also the <u>orientation of the plane</u> on which the stress acts is involved.

NOTE: A complete description of the magnitudes and directions of stresses on all possible planes through point 0 constitutes the <u>state of stress</u> at point 0.

NOTE: A knowledge of maximum stresses alone is not always sufficient to provide the best evaluation of the strength of a member. The orientation of these stresses is also important.

In general, the overall stress state must be determined first and the maximum stress values derived from this information. The state of stress at a point can normally be determined by computing the stresses acting on certain conveniently oriented planes passing through the point of interest. Stresses acting on any other planes can then be determined by means of simple, standardized analytical or graphical methods. Therefore, it is convenient to consider the three mutually perpendicular planes as faces of a cube of infinitesimal size which surround the point at which the stress state is to be determined.

Figure 5 illustrates the general state of 3D stress at an arbitrary point by illustrating the stress components on the faces of an infinitesimal cubic element around the point.

Infinitesimal Cube about Point of Interest

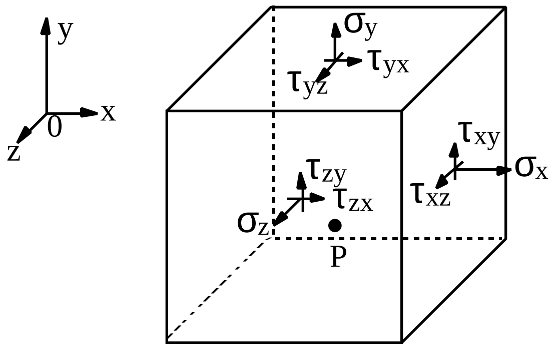


Figure 5. Stress Element Showing General State of 3D Stress at a Point Located Away from the Origin.

Notation Convention for Fig. 5:

Normal stresses are designated by a single subscript corresponding to the outward drawn normal to the plane that it acts upon.

The rationale behind the double-subscript notation for shear stresses is that the first designates the plane of the face and the second the direction of the stress. The plane of the face is represented by the axis which is normal to it, instead of the two perpendicular axes lying in the plane.

Stress components are positive when a positively-directed force component acts on a positive face or a negatively-directed force component acts on a negative face. When a positively-directed force component acts on a negative face or a negatively-directed force component acts on a positive face, the resulting stress component will be negative. A face is positive when its outwardly-directed normal vector points in the direction of the positive coordinate axis (Ref. 1, pp. 206-207). All stresses shown in Fig. 5 are positive. Normal stresses are positive for tensile stress and negative for compressive stress. Figure 6 illustrates positive and negative shear stresses.

NOTE: τ_{yx} equals τ_{xy}.

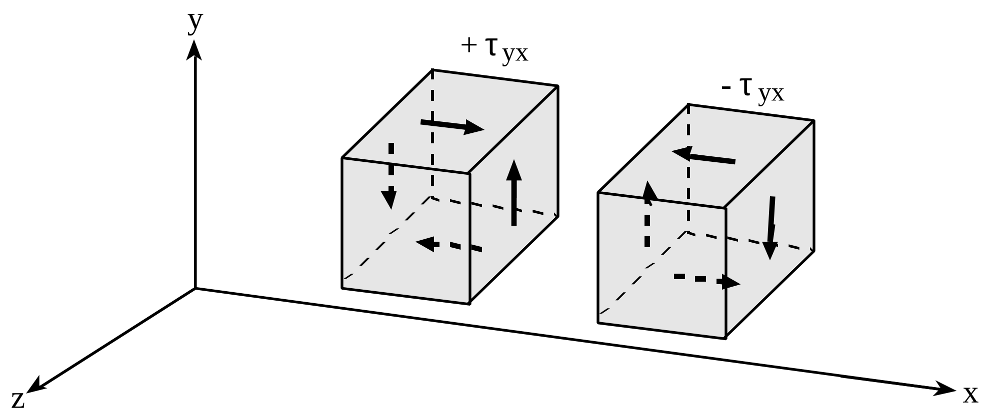


Figure 6. Definition of Positive and Negative τ_{yx}.

Writing the state of stress as tensor S:

$$S = \begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{pmatrix} \quad \text{9-components} \tag{1.2}$$

However, we have three equal pairs of shear stress:

$$\tau_{xy} = \tau_{yx}, \tau_{xz} = \tau_{zx}, \tau_{yz} = \tau_{zy} \tag{1.3}$$

Therefore, six quantities are sufficient to describe the stresses acting on the coordinate planes through a point, i.e., the triaxial state of stress at a point. If these six stresses are known at a point, it is possible to compute from simple equilibrium concepts the stresses on any plane passing through the point [Ref. 2, p. 79].

1.2 Principal Stresses, Planes and Directions

The tensor S becomes a symmetric tensor if Eq. 1.3 is introduced into Eq. 1.2. A fundamental property of a symmetrical tensor (symmetrical about its principal diagonal) is that there exists an orthogonal set of axes 1, 2, 3 (called principal axes) with respect to which the tensor elements are all zero except for those on the principal diagonal:

$$S' = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix} \tag{1.4}$$

Hence, when the tensor represents the state of stress at a point, there always exists a set of mutually perpendicular planes on which only normal stress acts. These planes of zero shear stress are called principal planes, the directions of their outer normals are called principal directions, and the stresses acting on these planes are called principal stresses. An element whose faces are principal planes is called a principal element.

For the general case, the principal axes are usually numbered so that:

$$\sigma_1 \geq \sigma_2 \geq \sigma_3$$

1.3 Basic Considerations of Strain

The concept of strain is of fundamental importance to the engineer with respect to the consideration of deflections and deformation.

A component may prove unsatisfactory in service as a result of excessive deformations, although the associated stresses are well within the allowable limits from the standpoint of fracture or yielding.

Strain is a directly measurable quantity, stress is not.

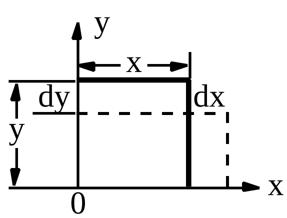
Concept of Strain and State of Strain

Any physical body subjected to forces, i.e., stresses, deforms under the action of these forces.

Strain is the direction and intensity of the deformation at any given point with respect to a specific plane passing through that point. Strain is therefore a quantity analogous to stress.

State of strain is a complete definition of the magnitude and direction of the deformation at a given point with respect to all planes passing through the point. Thus, state of strain is a tensor and is analogous to state of stress.

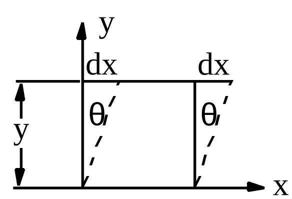
For convenience, strains are always resolved into normal components, ε, and shear components, γ (Figs. 7 & 8). In these figures the original shape of the body is denoted by solid lines and the deformed shape by the dashed lines. The change in length in the x-direction is dx, while the change in the y-direction is dy. Hence, ε_x, ε_y and γ are written as indicated in these figures.



$$\varepsilon_x = \lim_{x \to 0} \frac{dx}{x}$$

$$\varepsilon_y = \lim_{y \to 0} \frac{dy}{y}$$

Figure 7. Deformation of a Body where the x-Dimension is Extended and the y-Dimension is Contracted.



$$\gamma_{yx} = \lim_{y \to 0} \frac{dx}{y} = \tan\theta \approx \epsilon$$

Figure 8. Plane Shear Strain.

Subscript notation for strains corresponds to that used with stresses. Specifically,

- γ_{yx}: shear strain resulting from taking adjacent planes perpendicular to the y-axis and displacing them relative to each other in the x-direction (Fig. 9).
- $\varepsilon_x, \varepsilon_y$: normal strains in x- and y-directions, respectively.

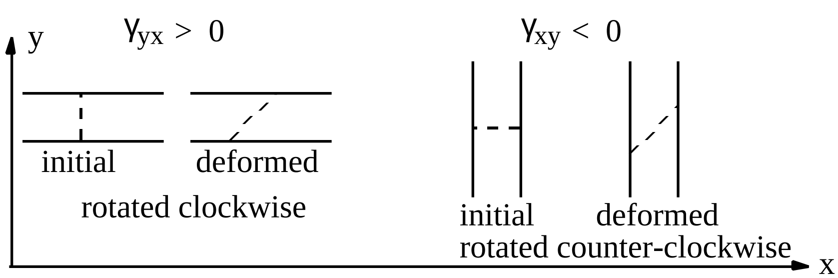


Figure 9. Strains resulting from Shear Stresses $\pm\gamma_{xy}$.

Sign conventions for strain also follow directly from those for stress: positive normal stress produces positive normal strain and vice versa. In the above example (Fig. 7), $\varepsilon_x > 0$, whereas $\varepsilon_y < 0$. Adopting the positive clockwise convention for shear components, $\gamma_{xy} < 0$, $\gamma_{yx} > 0$. In Fig. 8, the shear is γ_{yx} and the rotation is clockwise.

NOTE: Half of γ_{xy}, γ_{xz}, γ_{yz} is analogous to τ_{xy}, τ_{xz} and τ_{yz}, whereas ε_x is analogous to σ_x.

$$T = \begin{pmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{yx} & \varepsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{zx} & \frac{1}{2}\gamma_{zy} & \varepsilon_z \end{pmatrix}$$

It may be helpful in appreciating the physical significance of the fact that τ is analogous to $\gamma/2$ rather than with γ itself to consider Fig. 10. Here it is seen that each side of an element changes in slope by an angle $\gamma/2$.

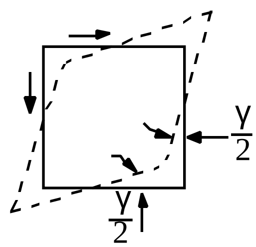


Figure 10. State of Pure Shear Strain

1.4 Plane Stress

In Fig. 11, all stresses on a stress element act on only two pairs of faces. Further, these stresses are practically constant along the z-axis. This two-dimensional case is called biaxial stress or plane stress.

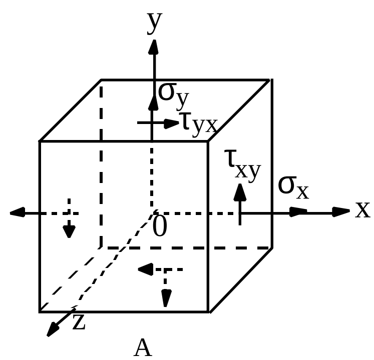


Figure 11. State of Plane Stress

The stress tensor for stresses in horizontal (x) and vertical (y) directions in the plane xy is:

$$S = \begin{pmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{pmatrix}$$

When the angle of the cutting plane through point 0 is varied from horizontal to vertical, the shear stresses acting on the cutting plane vary from positive to negative. Since shear stress is a continuous function of the cutting plane angle, there must be some intermediate plane in which the shear stress is zero. The normal stresses in the intermediate plane are the principle stresses. The stress tensor is:

$$S' = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$$

where the zero principal stress is always σ_3. These principal stresses occur along particular axes in a particular plane. By convention, σ_1 is taken as the larger of the two principal stresses. Hence

$$\sigma_1 \geq \sigma_2 \qquad \sigma_3 = 0$$

In general we wish to know the stress components in a set of orthogonal axes in the xy plane, but rotated at an arbitrary angle, α, with respect to the x, y axes. We wish to express these stress components in terms of the stress terms referred to the x, y axes, i.e., in terms of σ_x, σ_y, τ_{xy} and the angle of rotation α. Fig. 12 indicates this new set of axes, x_1 and y_1, rotated by α from the original set of axes x and y. The determination of the new stress components σ_{x_1}, σ_{y_1}, $\tau_{x_1y_1}$ is accomplished by considering the equilibrium of the small wedge centered on the point of interest whose faces are along the x-axis, y-axis and perpendicular to the x_1 axis. For the general case of expressing stresses σ_{x_1} and $\tau_{x_1y_1}$ at point 0 in plane x_1, y_1 in terms of stresses σ_x, σ_y, τ_{xy} and the angle α in plane x, y by force balances in the x_1 and y_1 directions, we obtain (see Ref. 1, p. 217):

$$\sigma_{x_1} = \sigma_x \cos^2 \alpha + \sigma_y \sin^2 \alpha + 2\tau_{xy} \sin \alpha \cos \alpha \tag{1.5a}$$

$$\tau_{x_1y_1} = (\sigma_y - \sigma_x) \sin \alpha \cos \alpha + \tau_{xy} (\cos^2 \alpha - \sin^2 \alpha) \tag{1.5b}$$

$$\sigma_{y_1} = \sigma_x \sin^2 \alpha + \sigma_y \cos^2 \alpha - 2\tau_{xy} \sin \alpha \cos \alpha \tag{1.5c}$$

where the angle α is defined in Fig. 12.

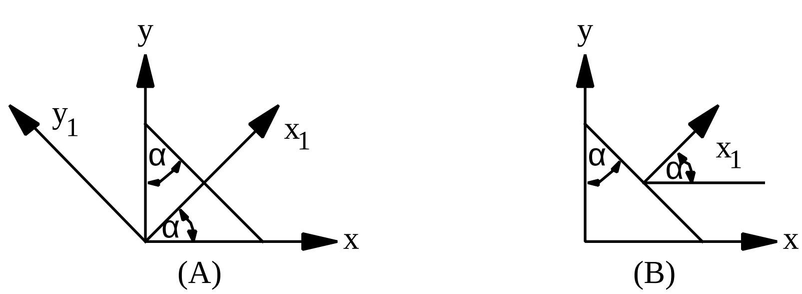


Figure 12. (A) Orientation of Plane x_1y_1 at Angle α to Plane xy; (B) Wedge for Equilibrium Analysis.

These are the transformation equations of stress for plane stress. Their significance is that stress components $\sigma_{x_1}, \sigma_{y_1}$ and $\tau_{x_1y_1}$ at point 0 in a plane at an arbitrary angle α to the plane xy are uniquely determined by the stress components σ_x, σ_y and τ_{xy} at point 0.

Eq. 1.5 is commonly written in terms of 2α. Using the identities:

$$\sin^2\alpha = \frac{(1 - \cos 2\alpha)}{2}\ ; \quad \cos^2\alpha = \frac{(1 + \cos 2\alpha)}{2}\ ; \quad 2 \sin\alpha \cos\alpha = \sin 2\alpha$$

we get

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\alpha + \tau_{xy} \sin 2\alpha \qquad (1.6a)$$

$$\tau_{x_1y_1} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\alpha + \tau_{xy} \cos 2\alpha \qquad (1.6b)$$

$$\sigma_{y_1} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos 2\alpha - \tau_{xy} \sin 2\alpha \qquad (1.6c)^*$$

The orientation of the principal planes, in this two dimensional system is found by equating $\tau_{x_1y_1}$ to zero and solving for the angle α.

1.5 Mohr's Circle

Equations 1.6a,b,c, taken together, can be represented by a circle, called Mohr's circle of stress. To illustrate the representation of these relations, eliminate the function of the angle 2α from Eq. 1.6a and 1.6b by squaring both sides of both equations and adding them. Before Eq. 1.6a is squared, the term $(\sigma_x + \sigma_y)/2$ is transposed to the left side. The overall result is (where plane y_1x_1 is an arbitrarily located plane, so that σ_{x_1} is now written as σ and $\tau_{x_1y_1}$ as τ):

$$[\sigma - 1/2\,(\sigma_x + \sigma_y)]^2 + \tau^2 = 1/4\,(\sigma_x - \sigma_y)^2 + \tau_{xy}^2 \qquad (1.7)$$

Now construct Mohr's circle on the σ and τ plane. The principal stresses are those on the σ-axis when $\tau = 0$. Hence, Eq. 1.7 yields the principal normal stresses σ_1 and σ_2 as:

$$\sigma_{1,2} = \underbrace{\frac{\sigma_x + \sigma_y}{2}}_{\text{center of circle}} \pm \underbrace{\sqrt{\tau_{xy}^2 + \left(\frac{\sigma_x - \sigma_y}{2}\right)^2}}_{\text{radius}} \qquad (1.8)$$

The maximum shear stress is the radius of the circle and occurs in the plane represented by a vertical orientation of the radius.

* σ_{y1} can also be obtained from σ_{x1} by substituting $\alpha + 90^\circ$ for α.

$$\tau_{max} = \pm\sqrt{\tau_{xy}^2 + \left(\frac{\sigma_x - \sigma_y}{2}\right)^2} = \frac{1}{2}(\sigma_1 - \sigma_2) \quad (1.9)$$

The angle between the principal axes x_1y_1 and the arbitrary axis, xy can be determined from Eq. 1.6b by taking x_1y_1 as principal axes. Hence, from Eq. 1.6b with $\tau_{x_1y_1} = 0$ we obtain:

$$2\alpha = \tan^{-1}\frac{2\tau_{xy}}{\sigma_x - \sigma_y} \quad (1.10)$$

From our wedge we know an arbitrary plane is located an angle α from the general x-, y-axis set and α ranges from zero to 180˚. Hence, for the circle with its 360˚ range of rotation, planes separated by α in the wedge are separated by 2α on Mohr's circle. Taking the separation between the principal axis and the x-axis where the shear stress is τ_{xy}, we see that Fig. 13A also illustrates the relation of Eq. 1.10. Hence, Mohr's circle is constructed as illustrated in Fig. 13A.

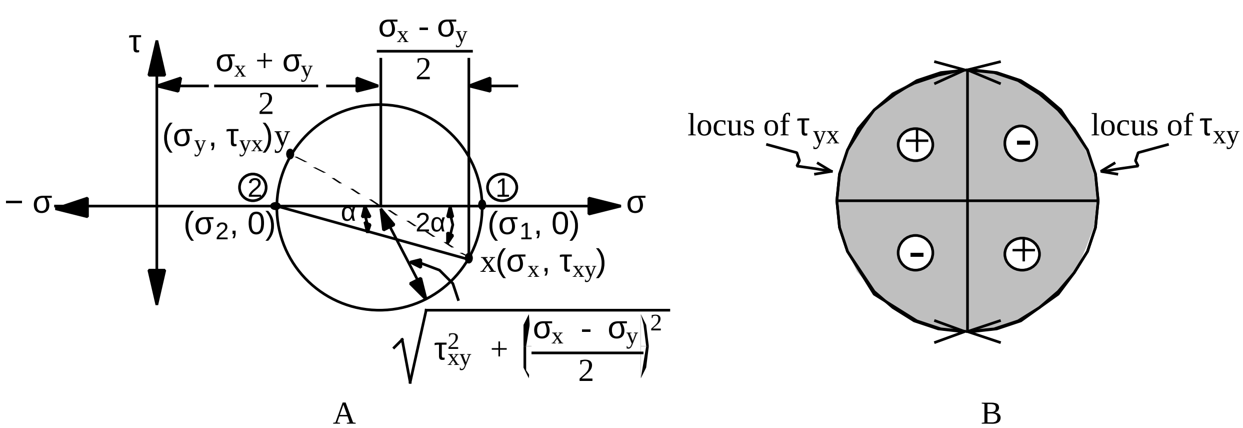


Figure 13. (A) Mohr's Circle of Stress; (B) Shear Stress Sign Convection for Mohr's Circle.*

When the principal stresses are known and it is desired to find the stresses acting on a plane oriented at angle φ from the principal plane numbered 1 in Fig. 14, Eq. 1.6 becomes:

$$\sigma_\varphi = \frac{\sigma_1 + \sigma_2}{2} \pm \frac{\sigma_1 - \sigma_2}{2}\cos 2\varphi \quad (1.11)$$

and

$$\tau_\varphi = -\frac{\sigma_1 - \sigma_2}{2}\sin 2\varphi \quad (1.12)$$

* For shear stresses, the sign convection is complicated since τ_{yx} and τ_{xy} are equal and of the same sign (see Figs 5 and 6, and Eq. 1.3), yet one must be plotted in the convention positive y-direction and the other in the negative y-direction (note the τ-axis replaces the y-axis in Fig. 13A). The convention adopted is to use the four quadrants of the circle as follows: positive τ_{xy} is plotted in the fourth quadrant and positive τ_{yx} in the second quadrant. Analogously, negative τ_{xy} lies in the first quadrant. Further, the shear stress at 90˚ on the circle is positive while that at 270˚ is negative. [Ref. 1, p. 220]
Note that the negative shear stress of Eq. 1.12 is plotted in the first and third quadrants of Fig. 14, consistent with this sign convention.

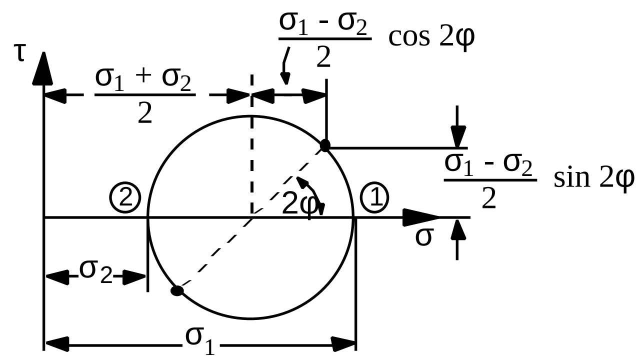


Figure 14. Mohr's Circle Representation of Eqs. 1.11 & 1.12.

Example 1:

Consider a cylindrical element of large radius with axial (x direction) and torque loadings which produce shear and normal stress components as illustrated in Fig. 15. Construct Mohr's circle for this element and determine the magnitude and orientation of the principal stresses.

From this problem statement, we know the coordinates σ_x, τ_{xy} are 60, 40 ksi and the coordinates σ_y, τ_{yx} are 0, 40 ksi. Further, from Eq. 1.10, $2\alpha = \tan^{-1} 2(40)/(60) = 53^\circ$. From these values and the layout of Fig. 13, we can construct Mohr's circle for this example as shown in Fig. 16. Points 1 and 2 are stresses on principal planes, i.e., principal stresses are 80 and -20 ksi and points S and S' are maximum and minimum shear stress, i.e., 50 and -50 ksi. These results follow from Eqs. 1.8 and 1.9 as:

From Eq. 1.8 $\quad \sigma_1, \sigma_2 = \frac{60 + 0}{2} \pm \sqrt{40^2 + \left(\frac{60 - 0}{2}\right)^2} = 80, -20 \text{ ksi}$

From Eq. 1.9 $\quad \tau_{max} = \pm \sqrt{40^2 + \left(\frac{60 - 0}{2}\right)^2} = \pm 50 \text{ ksi}$

From Fig. 13A $\quad \sigma$ at plane of $\tau_{max} = (60 + 0)/2 = 30$ ksi, i.e., at S and S'

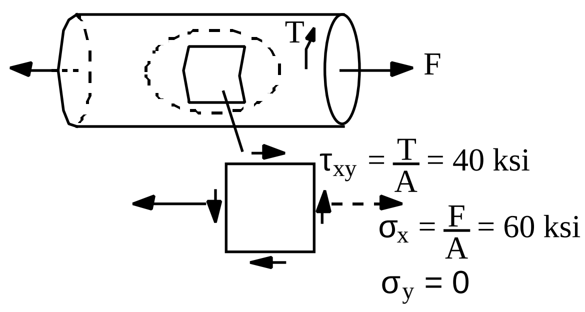


Figure 15. Example 1 Loadings.

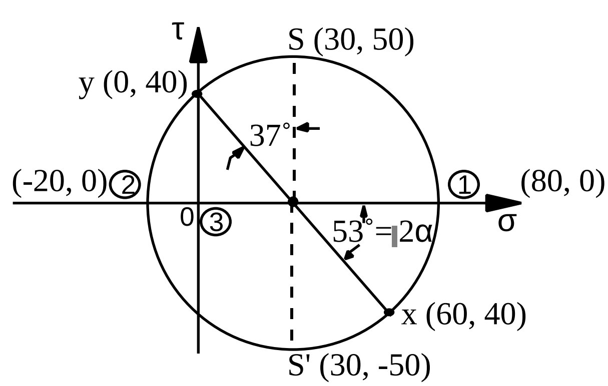


Figure 16. Mohr's Circle for Example 1.

The principal element, i.e., its faces are principal planes, and the maximum shear element are shown in Fig. 17.

NOTE: The angle between points y and S is the complement of 53˚ or 37˚, and plane y is rotated 37˚/2 counter-clockwise from plane S.

NOTE: The procedure for determining the orientation of these elements: point 1 is 53˚ counter-clockwise from point x on the circle; hence the plane represented by 1 is half of 53˚ counter-clockwise from plane x. Similarly, the S plane is 143˚/2 counter-clockwise of plane x, or 37˚/2 clockwise of plane y.

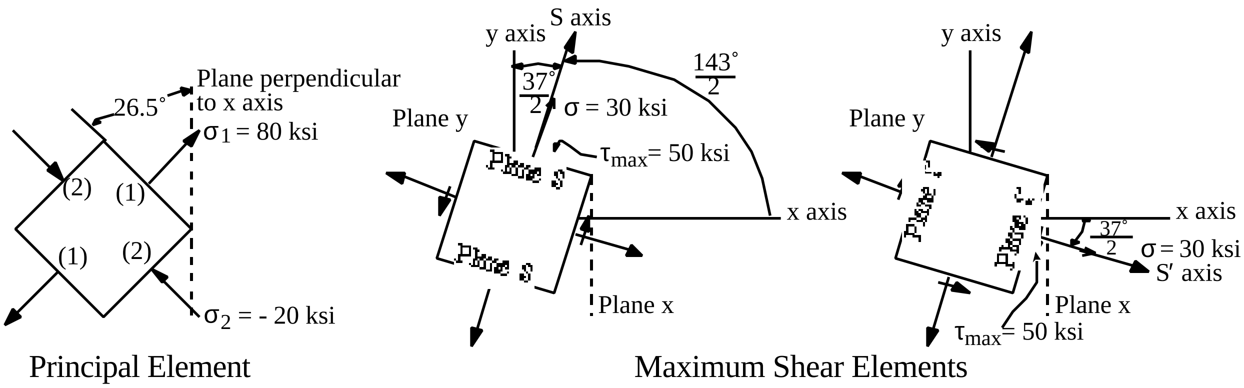


Figure 17. Key Elements for Example 1.

Mohr's circle clearly illustrates these additional points:

1. the principal stresses represent the extreme values of normal stress at the point in question;
2. $\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_2)$;
3. planes of maximum shear are always 45˚ removed from the principal planes; and
4. normal stresses acting on maximum shear planes S and S' are equal to the algebraic average of principal stresses σ_1 and σ_2, i.e., $1/2\ (\sigma_1 + \sigma_2)$.

1.6 Octahedral Planes and Stresses

Figure 18 illustrates the orientation of one of the eight octahedral planes which are associated with a given stress state. Each of the octahedral planes cut across one of the corners of a principal element, so that the eight planes together form an octahedron.

The stresses acting on these planes have interesting and significant characteristics. First of all, identical normal stresses act on all eight planes. By themselves the normal stresses are therefore said to be hydrostatic and tend to compress or enlarge the octahedron but not distort it.

$$\sigma_{oct} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} \quad (1.13a)$$

Shear stresses are also identical. These distort the octahedron without changing its volume. Although the octahedral shear stress is smaller than the highest principal shear stress, it constitutes a single value that is influenced by all three principal shear stresses. <u>Thus, it is important as a criterion for predicting yielding of a stressed material</u>.

$$\tau_{oct} = \frac{1}{3}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{1/2} \quad (1.13b)$$

In cases in which $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{xz}$ and τ_{yz} are known:

$$\sigma_{oct} = \frac{\sigma_x + \sigma_y + \sigma_z}{3} \quad (1.14a)$$

$$\tau_{oct} = \frac{1}{3}\left[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6\left(\tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2\right)\right]^{1/2} \quad (1.14b)$$

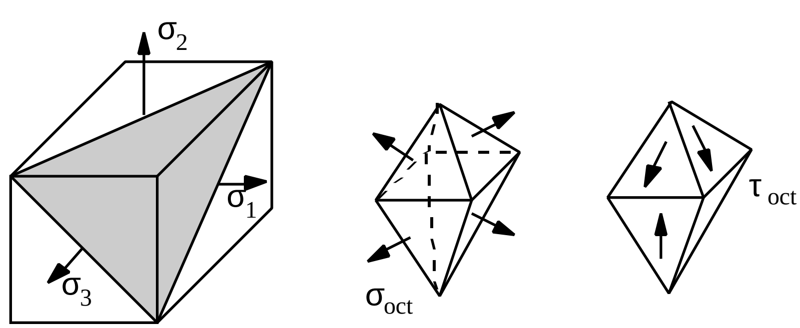


Figure 18. Octahedral Planes Associated with a Given Stress State.

1.7 <u>Principal Strains and Planes</u>

Having observed the correspondence between strain and stress, it is evident that with suitable axis transformation one obtains an expression for the strain tensor T ' which is identical to that of stress tensor S' except that the terms in the principal diagonal are $\varepsilon_1, \varepsilon_2, \varepsilon_3$. Hence, recalling from Section 1.3 that $\gamma/2$ corresponds to τ, strain relations analogous to Eqs. 1.8, 1.9 and 1.10 can be written as follows:

$$\varepsilon_1, \varepsilon_2 = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{1}{2}\gamma_{xy}\right)^2 + \left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2} \quad (1.15)$$

$$\gamma_{max} = \pm 2\sqrt{\left(\frac{1}{2}\gamma_{xy}\right)^2 + \left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2} \quad (1.16)$$

$$2\alpha = \tan^{-1}\frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y} \quad (1.17)$$

Conclusion

The preceding sections *dealt separately* with the concepts of stress and strain at a point. These considerations involving stresses and strains separately are general in character and applicable to bodies composed of any continuous distribution of matter.

NOTE: No material properties were involved in the relationships, hence they are applicable to water, oil, as well as materials like steel and aluminum.

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2. ELASTIC STRESS-STRAIN RELATIONS

The relationships between these quantities are of direct importance to the engineer concerned with design and stress analysis. Generally two principal types of problems exist:

1. Determination of the stress state at a point from a known strain state—the problem encountered when stresses are to be computed from experimentally determined strains.

2. Determination of the state of strain at a point from a known stress state—the problem commonly encountered in design, where a part is assured to carry certain loads, and strains must be computed with regard to critical clearances and stiffnesses.

We limit ourselves to solids loaded in the elastic range. Furthermore, we shall consider only materials which are isotropic, i.e., materials having the same elastic properties in all directions. Most engineering materials can be considered as isotropic. Notable exceptions are wood and reinforced concrete.

2.1 Generalized Hooke's Law

Let us consider the various components of stress one at a time and add all their strain effects. For a uni-axial normal stress in the x direction, σ_x, the resulting normal strain is

$$\varepsilon_x = \frac{\sigma_x}{E} \tag{2.1}$$

where E is Young's modulus or the modulus of elasticity.

Additionally this stress produces lateral contraction, i.e., ε_y and ε_z, which is a fixed fraction of the longitudinal strain, i.e.,

$$\varepsilon_y = \varepsilon_z = -\nu\varepsilon_x = -\nu\frac{\sigma_x}{E}\ . \tag{2.2}$$

This fixed fraction is called Poisson's ratio, ν. Analogous results are obtained from strains due to σ_y and σ_z.

The shear-stress components produce only their corresponding shear-strain components that are expressed as:

$$\gamma_{zx} = \frac{\tau_{zx}}{G}\ , \qquad \gamma_{xy} = \frac{\tau_{xy}}{G}\ , \qquad \gamma_{yz} = \frac{\tau_{yz}}{G} \tag{2.3a,b,c}$$

where the constant of proportionality, G, is called the shear modulus.

For a linear-elastic isotropic material with all components of stress present:

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right] \tag{2.4a}$$

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu(\sigma_z + \sigma_x)\right] \tag{2.4b}$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_x + \sigma_y)\right] \tag{2.4c}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$ (2.5a) same as (2.3)

$$\gamma_{yz} = \frac{\tau_{yz}}{G} \tag{2.5b}$$

$$\gamma_{zx} = \frac{\tau_{zx}}{G} \tag{2.5c}$$

These equations are the generalized Hooke’s law.

It can also be shown (Ref 1, p. 285) that for an isotropic materials, the properties G, E and ν are related as:

$$G = \frac{E}{2(1 + \nu)} \; . \tag{2.6}$$

Hence,

$$\gamma_{xy} = \frac{2(1 + \nu)}{E}\tau_{xy} \tag{2.7a}$$

$$\gamma_{yz} = \frac{2(1 + \nu)}{E}\tau_{yz} \tag{2.7b}$$

$$\gamma_{zx} = \frac{2(1 + \nu)}{E}\tau_{zx} \; . \tag{2.7c}$$

Equations 2.4 and 2.5 may be solved to obtain stress components as a function of strains:

$$\sigma_x = \frac{E}{(1 + \nu)(1 - 2\nu)}\left[(1 - \nu)\varepsilon_x + \nu(\varepsilon_y + \varepsilon_z)\right] \tag{2.8a}$$

$$\sigma_y = \frac{E}{(1 + \nu)(1 - 2\nu)}\left[(1 - \nu)\varepsilon_y + \nu(\varepsilon_z + \varepsilon_x)\right] \tag{2.8b}$$

$$\sigma_z = \frac{E}{(1 + \nu)(1 - 2\nu)}\left[(1 - \nu)\varepsilon_z + \nu(\varepsilon_x + \varepsilon_y)\right] \tag{2.8c}$$

$$\tau_{xy} = \frac{E}{2(1 + \nu)}\gamma_{xy} = G\gamma_{xy} \tag{2.9a}$$

$$\tau_{yz} = \frac{E}{2(1 + \nu)}\gamma_{yz} = G\gamma_{yz} \tag{2.9b}$$

$$\tau_{zx} = \frac{E}{2(1 + \nu)}\gamma_{zx} = G\gamma_{zx} \; . \tag{2.9c}$$

For the first three relationships one may find:

$$\sigma_x = \frac{E}{(1+\nu)}\left[\varepsilon_x + \frac{\nu}{(1-2\nu)}(\varepsilon_x + \varepsilon_y + \varepsilon_z)\right] \tag{2.10a}$$

$$\sigma_y = \frac{E}{(1+\nu)}\left[\varepsilon_y + \frac{\nu}{(1-2\nu)}(\varepsilon_x + \varepsilon_y + \varepsilon_z)\right] \tag{2.10b}$$

$$\sigma_z = \frac{E}{(1+\nu)}\left[\varepsilon_z + \frac{\nu}{(1-2\nu)}(\varepsilon_x + \varepsilon_y + \varepsilon_z)\right] . \tag{2.10c}$$

For special case in which the x, y, z axes coincide with principal axes 1, 2, 3, we can simplify the strain set, Eqs. 2.4 and 2.5, and the stress set Eqs. 2.8 and 2.9, by virtue of all shear strains and shear stresses being equal to zero.

$$\varepsilon_1 = \frac{1}{E}[\sigma_1 - \nu(\sigma_2 + \sigma_3)] \tag{2.11a}$$

$$\varepsilon_2 = \frac{1}{E}[\sigma_2 - \nu(\sigma_3 + \sigma_1)] \tag{2.11b}$$

$$\varepsilon_3 = \frac{1}{E}[\sigma_3 - \nu(\sigma_1 + \sigma_2)] \tag{2.11c}$$

$$\sigma_1 = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_1 + \nu(\varepsilon_2 + \varepsilon_3)] \tag{2.12a}$$

$$\sigma_2 = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_2 + \nu(\varepsilon_3 + \varepsilon_1)] \tag{2.12b}$$

$$\sigma_3 = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_3 + \nu(\varepsilon_1 + \varepsilon_2)] . \tag{2.12c}$$

For biaxial-stress state, one of the principal stresses (say σ_3) = 0, Eqs. 2.11a,b,c become:

$$\varepsilon_1 = \frac{1}{E}(\sigma_1 - \nu\sigma_2) \tag{2.13a}$$

$$\varepsilon_2 = \frac{1}{E}(\sigma_2 - \nu\sigma_1) \tag{2.13b}$$

$$\varepsilon_3 = -\frac{\nu}{E}(\sigma_1 + \sigma_2) . \tag{2.13c}$$

In simplifying Eqs. 2.12a,b,c for the case of $\sigma_3 = 0$, we note from Eq. 2.12c that for σ_3 to be zero,

$$\varepsilon_3 = -\frac{\nu}{1-\nu}(\varepsilon_1 + \varepsilon_2) . \tag{2.14}$$

Substituting this expression into the first two of Eqs. 2.12a,b,c gives:

$$\sigma_1 = \frac{E}{1-\nu^2}(\varepsilon_1 + \nu\varepsilon_2) \tag{2.15a}$$

$$\sigma_2 = \frac{E}{1-\nu^2}(\varepsilon_2 + \nu\varepsilon_1) \tag{2.15b}$$

$$\sigma_3 = 0 . \tag{2.15c}$$

In case of uniaxial stress Eqs. 2.13 and 2.15 must, of course reduce to:

$$\varepsilon_1 = \frac{1}{E}\sigma_1 \tag{2.16a}$$

$$\varepsilon_2 = \varepsilon_3 = -\frac{\nu}{E}\sigma_1 \tag{2.16b}$$

$$\sigma_1 = E\varepsilon_1 \tag{2.17a}$$

$$\sigma_2 = \sigma_3 = 0 . \tag{2.17b}$$

2.2 Modulus of Volume Expansion (Bulk Modulus)

k may be defined as the ratio between hydrostatic stress (in which $\sigma_1 = \sigma_2 = \sigma_3$) and volumetric strain (change in volume divided by initial volume), i.e.,

$$k = \sigma/(\Delta V/V). \tag{2.18}$$

NOTE: Hydrostatic compressive stress exists within the fluid of a pressurized hydraulic cylinder, in a rock at the bottom of the ocean or far under the earth's surface, etc.

Hydrostatic tension can be created at the interior of a solid sphere by the sudden application of uniform heat to the surface, the expansion of the surface layer subjecting the interior material to triaxial tension. For $\sigma_1 = \sigma_2 = \sigma_3 = \sigma$, Eqs. 2.11a,b,c show that: $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon = \frac{\sigma}{E}(1 - 2\nu)$.

This state of uniform triaxial strain is characterized by the absence of shearing deformation; an elemental cube, for example, would change in size but remain a cube. The size of an elemental cube initially of unit dimension would change from 1^3 to $(1 + \varepsilon)^3$ or to $1 + 3\varepsilon + 3\varepsilon^2 + \varepsilon^3$. If we consider normal structural materials, ε is a quantity sufficiently small, so that ε^2 and ε^3 are completely negligible, and the volumetric change is from 1 to $1 + 3\varepsilon$. The volumetric strain, $\Delta V/V$, is thus equal to 3ε or to:

$$\frac{\Delta V}{V} = 3\varepsilon = \frac{3\sigma(1 - 2\nu)}{E} . \tag{2.19}$$

Hence,

$$k \equiv \frac{\sigma}{3\varepsilon} = \frac{E}{3(1 - 2\nu)} . \tag{2.20}$$

Now $\nu \leq 0.5$, so that k cannot become negative. A simple physical model of a representative atomic crystalline structure gives:

$$\nu = 1/3 , \tag{2.21}$$

so that

$$\boxed{k = E} . \tag{2.22}$$

3. THIN-WALLED CYLINDERS AND SPHERE

3.1 Stresses

- Stresses in a Thin-Walled Cylinder:

Consider a portion of the cylinder sufficiently remote from the ends to avoid end effects. The equation of equilibrium of radial forces (Fig. 19) acting on the element is:

$$p_i\left(r_i d\varphi dL\right) = 2\sigma_{t,av} t dL\left(\sin\frac{d\varphi}{2}\right) \quad (3.1)$$

yielding

$$\sigma_{t,av} = \frac{p_i r_i}{t}\ . \quad (3.2)$$

when sin ($d\varphi/2$) is approximated by $d\varphi/2$, which is valid for small $d\varphi$. This is a correct expression for the average tangential stress (hoop stress) regardless of cylinder thickness.

NOTE: It is only when this equation is used as an approximation for the maximum tangential stress, $\sigma_{t,max}$, that the thickness t must be very small in comparison with the cylinder diameter.

Equation 3.1 can be modified to give a good quick estimate of the maximum tangential stress due to internal pressure for cylinders of moderate thickness by replacing r_i with r_{av}, which we label R:

$$R = r_i + \frac{t}{2}\ . \quad (3.3)$$

Thus,

$$\sigma_{t,max} \approx \frac{p_i R}{t} \quad (3.4)$$

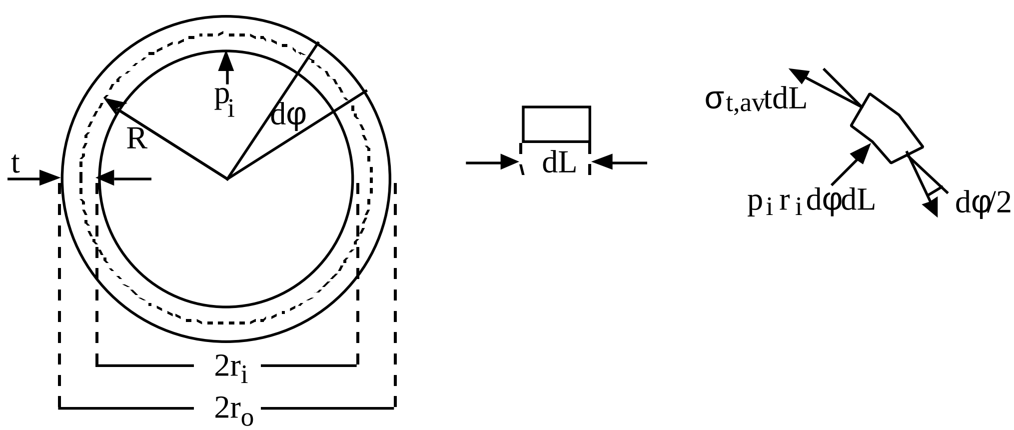


Figure 19. Radial Forces in an Internally Pressurized Thin-Walled Cylinder.

The following are errors in estimating $\sigma_{t,max}$ from Eqs. 3.2 or 3.4 compared to the thick-walled cylinder results:

t/r_i	% Error Using Eq. 3.2	% Error Using Eq. 3.4
0.1	5 (low)	0.5 (low)
0.2	10 (low)	1.0 (low)
0.5	23 (low)	3.8 (low)
1.0	40 (low)	10.2 (low)

If the ends are closed, the cylinder is also subjected to an axial force of magnitude $p_i \pi r_i^2$. This is distributed over an area of cross section that can be expressed as:

$$A = \pi\left(r_o^2 - r_i^2\right) = 2\pi R t . \tag{3.5}$$

Thus the average axial tensile stress can be expressed by:

$$\sigma_{a,av} = \frac{p_i r_i^2}{r_o^2 - r_i^2} = \frac{p_i r_i^2}{2 r_{av} t} . \tag{3.6}$$

For the thin-walled case, $r_i \approx r_{av} \equiv R$ since $r_i = R(1 - 0.5t/R)$. Hence, Eq. 3.6 reduces to:

$$\sigma_{a,av} = \frac{p_i R}{2t} . \tag{3.7}$$

Thus the axial stress is approximately half the tangential stress.

- Stresses in a Thin-Walled Sphere:

  From a force balance these stresses can be determined to be

$$\sigma_{a,av} = \sigma_{t,av} = \frac{pR}{2t} \tag{3.8}$$

3.2 Deformation and Strains

The deformation of the diameter of a thin-walled cylinder or sphere caused by the internal pressure is to be determined. Note that the state of stress at the inner surface is triaxial and is not plane stress. The principal stresses of the inner surface are $\sigma_1 = \sigma_\varphi$, $\sigma_2 = \sigma_z$ and $\sigma_3 = -p$. However, $\sigma_3 = -p$ is so small compared to σ_1 and σ_2 that it can be neglected when considering strains. *Thus, the state of stress in thin-walled pressure vessels is usually considered to be plane stress*.

The deformation of such pressure vessels is influenced by the tangential and axial (transverse for the sphere) stresses, and hence use must be made of the following relations obtained from Eqs. 2.4, 2.5 and 2.8, respectively, with $\sigma_y = 0$.

$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{\nu\sigma_z}{E} \tag{3.9a}$$

$$\varepsilon_y = -\frac{\nu}{E}(\sigma_x + \sigma_z) \tag{3.9b}$$

$$\varepsilon_z = \frac{\sigma_z}{E} - \frac{\nu\sigma_x}{E} \tag{3.9c}$$

$$\gamma = \frac{\tau_{xy}}{G} \tag{3.9d}$$

$$\sigma_x = \frac{E}{\left(1 - \nu^2\right)}(\varepsilon_x + \nu\varepsilon_z) \tag{3.10a}$$

$$\sigma_z = \frac{E}{\left(1 - \nu^2\right)}(\varepsilon_z + \nu\varepsilon_x) \tag{3.10b}$$

to express Hooke's law for plane stress. Equations 3.10a,b are obtained from Eqs. 2.8a and 2.8c upon inspection of ε_y evaluated from Eq. 2.8b with σ_y taken as zero.

Let σ_φ, σ_z, ε_φ and ε_z represent the tangential and axial stress and strain, respectively, in the wall. The substitution of these symbols in Eqs. 3.9a,b,c, i.e., $x \equiv \varphi$ and $z = z$, gives:

$$\varepsilon_\varphi = \frac{\sigma_\varphi}{E} - \nu\frac{\sigma_z}{E} \tag{3.11}$$

$$\varepsilon_z = \frac{\sigma_z}{E} - \nu\frac{\sigma_\varphi}{E} . \tag{3.12}$$

- Closed-End Cylinder:

For the strains in the closed-end cylinder, the values of σ_φ and σ_z as derived in Eqs. 3.4 and 3.8, respectively, are substituted into Eqs. 3.11 and 3.12 to give:

$$\varepsilon_\varphi = \frac{1}{E}\left(\frac{pR}{t} - \nu\frac{pR}{2t}\right) = \frac{pR}{2Et}(2 - \nu) \tag{3.13}$$

$$\varepsilon_z = \frac{1}{E}\left(\frac{pR}{2t} - \nu\frac{pR}{t}\right) = \frac{pR}{2Et}(1 - 2\nu) . \tag{3.14}$$

Let the change in length of radius R be Δr when the internal pressure is applied. Then the change in length of the circumference is $2\pi\Delta r$. But the circumferential strain, ε_φ, is, by definition, given by the following equation:

$$\varepsilon_\varphi = \frac{2\pi\Delta r}{2\pi R} = \frac{\Delta r}{R} . \tag{3.15}$$

By combining Eqs. 3.15 and 3.13 we get:

$$\Delta r = \frac{pR^2}{2Et}(2 - \nu) . \tag{3.16}$$

The change in length, $\Delta\ell$, for a closed-end cylinder is equal to:

$$\Delta\ell = \varepsilon_z \ell \tag{3.17}$$

or

$$\Delta\ell = \frac{pR\ell}{2Et}(1 - 2\nu) \tag{3.18}$$

- Sphere:

$$\sigma_z = \sigma_\varphi = \frac{pR}{2t} \tag{3.19}$$

Thus, from Eq. 3.11:

$$\varepsilon_\varphi = \frac{1}{E}\left(\frac{pR}{2t} - \nu\frac{pR}{2t}\right) = \frac{pR}{2Et}(1 - \nu) . \tag{3.20}$$

By combining Eq. 3.20 with Eq. 3.15, we obtain:

$$\Delta r = \frac{pR^2}{2Et}(1 - \nu) . \tag{3.21}$$

3.3 End Effects for the Closed-End Cylinder

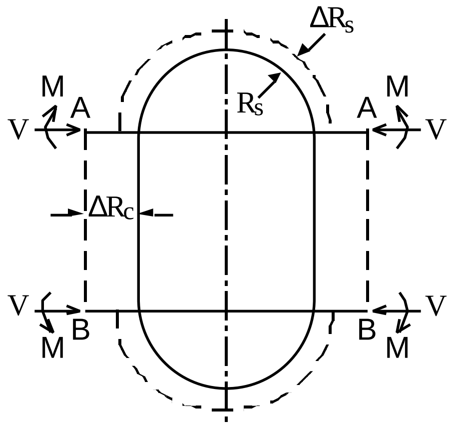
Figure 20 illustrates a cylinder closed by thin-walled hemispherical shells. They are joined together at AA and BB by rivets or welds. The dashed lines show the displacements due to internal pressure, p. These displacements are given by Eqs. 3.16 and 3.21 as:

$$\Delta R_c = \frac{pR^2}{2Et}(2 - \nu) \tag{3.22a}$$

$$\Delta R_s = \frac{pR^2}{2Et}(1 - \nu) \tag{3.22b}$$

The value of ΔR_C is more than twice that of ΔR_S for the same thickness, t, and as a result the deformed cylinder and hemisphere do not match boundaries. To match boundaries, rather large shearing forces, V, and moments, M, must develop at the joints as Fig. 20 depicts (only those on the cylinder are shown; equal but opposite shears and moments exist on the ends of the hemispheres). This shear force is considerably minimized in most reactor pressure vessels by sizing the hemisphere thickness much smaller than the cylinder thickness. For example, if the hemisphere thickness is 120 mm and the cylinder thickness is 220 mm, for $\nu = 0.3$, then the ratio ΔR_C to ΔR_S is

$$\frac{\Delta R_c}{\Delta R_s} = \frac{t_s}{t_c}\left(\frac{2 - \nu}{1 - \nu}\right) = 1.32$$



V ≡ Shearing Forces

Figure 20. Discontinuities in Strains for the Cylinder (ΔR_c) and the Sphere (ΔR_s).

The general solution of the discontinuity stresses induced by matching cylindrical and hemisphere shapes is not covered in these notes. However, a fundamental input to this solution is the matching of both displacements and deflections of the two cylinders of different wall thicknesses at the plane they join. Thus, the relationship between load, stress, displacement and deflection for cylinders and hemispheres are derived. These relations are presented in Tables 1 and 2, respectively.

Table 1. Relationship Between Load, Stresses, Displaced and Deflection for the Cylinder
(equations cited are from Ref. 2)

Load	Stresses σ_t, σ_ℓ, σ_r σ_θ, σ_z, σ_r	Displacement u_o	Deflection θ_o
p	$\frac{pR}{t}$, $\frac{pR}{2t}$, $-\frac{p}{2}$	$\frac{pR^2}{2tE}\left(2 - \nu + \nu \frac{t}{R}\right)$	0
M_o Edge Moment per Unit Perimeter Length	$\frac{EM_o}{2\beta^2 DR} \pm \gamma \frac{6M_o}{t^2}$, $\pm \frac{6M_o}{t^2}$, 0 (Eq. 7.3.22, p. 7.3-12) (Eq. 7.3.21, p. 7.3-11) Note: $u_o = \frac{M_o}{2\beta^2 D}$; so first term $= \frac{Eu_o}{R}$.	$\frac{M_o}{2\beta^2 D}$ (Eq. 7.3.18, p. 7.3.7) units $\frac{F \square L/L}{\frac{1}{L^2}\ \frac{F}{L^2}\ L^3} = L$	$-\frac{M_o}{\beta D}$ (Eq. 7.3.19, p. 7.3.8) dimensionless

Table 1 continued on next page

Table 1. Relationship Between Load, Stresses, Displaced and Deflection for the Cylinder (cont'd)

Q_o Edge Shear Force per Unit Perimeter Length	$\frac{EQ_o}{R2\beta^3 D}$, 0 , 0 (Eq. 7.3.22, p. 7.3-12) Note: $u_o = \frac{Q_o}{2\beta^3 D}$; so first term $= \frac{u_o E}{R}$.	$\frac{Q_o}{2\beta^3 D}$ (Eq. 7.3.18, p. 7.3.7) units $\frac{F/L}{\frac{1}{L^3} FL} = L$	$-\frac{Q_o}{2\beta^2 D}$ (Eq. 7.3.19, p. 7.3.8) dimensionless

where Plate Flexural Rigidity:

$$D = \frac{Et^3}{12\left(1 - \nu^2\right)}, \quad \text{units} \left(\frac{F}{L^2} L^3\right)$$

$$\beta^4 = \frac{Et}{4R^2 D} = \frac{3\left(1 - \nu^2\right)}{R^2 t^2} \quad \left(\frac{1}{L^2} \frac{1}{L^2}\right)$$

Equations and pages cited are from Ref. 2.

Table 2. Relationship Between Load, Stresses, Displaced and Deflection for the Hemisphere

Load	Stresses σ_t, σ_ℓ, σ_r	Displacement δ_o	Deflection θ_o
p	$\frac{pR}{2t}$, $\frac{pR}{2t}$, $-\frac{p}{2}$	$\frac{pR^2}{2tE}\left[1 - \nu + \nu\frac{t}{R}\right]$	0
M_o Edge Moment per Unit Perimeter Length	$\frac{2\lambda^2 M_o}{tR} \pm \gamma\frac{6M_o}{t^2}$, $\pm \frac{6M_o}{t^2}$, 0 Note: $\frac{2\lambda^2 M_o}{t} = \frac{\delta_o}{E}$; so first term $= \frac{E\delta_o}{R}$.	$\frac{2\lambda^2 M_o}{Et}$	$\frac{4\lambda^3 M_o}{REt}$
Q_o Edge Shear Force per Unit Perimeter Length	$\frac{2\lambda Q_0}{t}$, 0 , 0 Note: $\frac{2\lambda Q_0}{t} = \frac{E\delta_o}{R}$.	$\frac{2R\lambda Q_0}{Et}$	$\frac{2\lambda^2 Q_o}{Et}$

where

$$\lambda = \beta R \; ; \quad \beta_4 = \frac{3\left(1 - \nu^2\right)}{R^2 t^2}$$

4. THICK-WALLED CYLINDER UNDER RADIAL PRESSURE [Ref. 1, pp. 293-300]

- Long Cylinder Plane Strain

The cylinder is <u>axially restrained</u> ($\varepsilon_z = 0$) at either end and subjected to uniform radial pressure, not body forces. The cylinder cross section and stresses on an element are illustrated in Fig. 21.

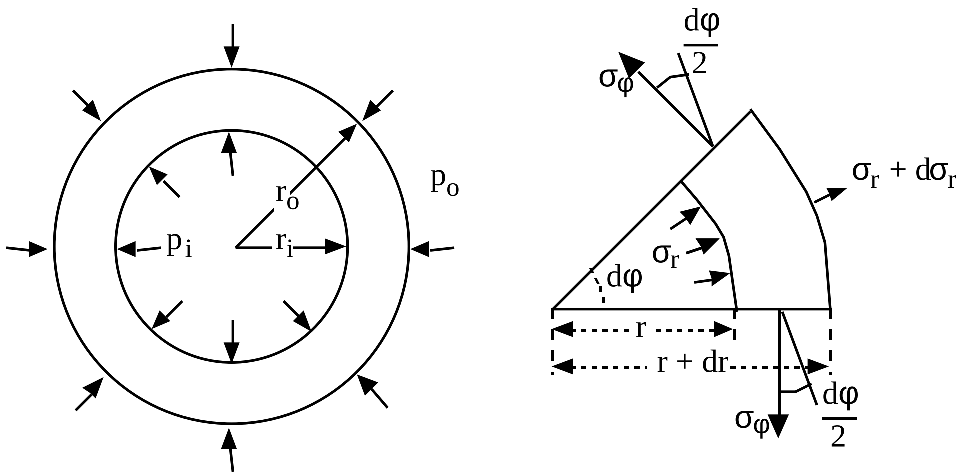


Figure 21. Force Balance and Displacements in a Pressurized Cylinder.

The relevant elastic equation set when azimuthal symmetry prevails is

- Strain–Displacement Relations

$$\varepsilon_r = \frac{du}{dr}, \qquad \varepsilon_\varphi = \frac{u}{r}, \qquad \gamma_{r\varphi} = \frac{dv}{dr} - \frac{v}{r} \tag{4.1a,b,c}$$

where u is the outward radial displacement and v is the φ direction displacement. Since the effect of internal pressure is to move the material in the radial direction without any rotation, both v and $\gamma_{r\varphi}$ should be zero. This will be demonstrated formally from the boundary conditions.

- Stress–Equilibrium Relations (for forces in the r and φ directions)

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\varphi}{r} = 0, \qquad \frac{d\tau_{r\varphi}}{dr} + \frac{2\tau_{r\varphi}}{r} = 0. \tag{4.2a,b}$$

- Stress–Strain Relations

$$\varepsilon_r = \frac{1}{E}\left[\sigma_r - \gamma\left(\sigma_\varphi + \sigma_z\right)\right] \tag{4.3a}$$

$$\varepsilon_\varphi = \frac{1}{E}\left[\sigma_\varphi - \gamma\left(\sigma_z + \sigma_r\right)\right] \tag{4.3b}$$

$$\varepsilon_z = 0 = \frac{1}{E}\left[\sigma_z - \gamma\left(\sigma_r + \sigma_\varphi\right)\right] \tag{4.3c}$$

The strain-displacement relations can, by elimination of the displacement variables, be combined as a compatibility (of strain) equation:

$$\frac{d\varepsilon_\varphi}{dr} + \frac{\varepsilon_\varphi - \varepsilon_r}{r} = 0 \qquad (4.4)$$

Finally the boundary conditions are:

$$\sigma_r(r_i) = -p_i ; \qquad \sigma_r(r_o) = -p_o \qquad (4.5a)$$

$$\tau_{r\varphi}(r_i) = \tau_{r\varphi}(r_o) = 0 \qquad (4.5b)$$

Note that the radial dimension is assumed to be free of displacement constraints.

4.1 Displacement Approach

When displacements are known, it is preferable to use this solution approach. In this case we solve for the displacements. Proceeding, solve Eq. 4.3 for stresses and substitute them into the stress–equilibrium relations of Eq. 4.2 to obtain:

$$(1 - \nu)\left(\frac{d\varepsilon_r}{dr} + \frac{\varepsilon_r - \varepsilon_\varphi}{r}\right) + \nu\left(\frac{d\varepsilon_\varphi}{dr} + \frac{\varepsilon_\varphi - \varepsilon_r}{r}\right) = 0 \qquad (4.6a)$$

$$\frac{d\gamma_{r\varphi}}{dr} + \frac{2\gamma_{r\varphi}}{r} = 0 \qquad (4.6b)$$

Substituting Eq. 4.1 in Eq. 4.6 and simplification show that the displacements u and v are governed by the equations:

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = \frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(ru)\right] = 0 \qquad (4.7a)$$

$$\frac{d^2v}{dr^2} + \frac{1}{r}\frac{dv}{dr} - \frac{v}{r^2} = \frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(rv)\right] = 0 . \qquad (4.7b)$$

Successive integration of these equations yields the displacement solutions

$$u = \frac{C_1 r}{2} + \frac{C_2}{r} ; \qquad v = \frac{C_3 r}{2} + \frac{C_4}{r} \qquad (4.8a,b)$$

where the constants of integration are to be determined by the stress boundary conditions (Eqs. 4.5a,b). With the use of Eqs. 4.8a,b, the strain and stress solutions are obtained from Eq. 4.1 and

$$\sigma_r = \frac{E}{(1 + \nu)(1 - 2\nu)}\left[(1 - \nu)\varepsilon_r + \nu\varepsilon_\varphi\right] \qquad (4.9a)$$

$$\sigma_\varphi = \frac{E}{(1 + \nu)(1 - 2\nu)}\left[(1 - \nu)\varepsilon_\varphi + \nu\varepsilon_r\right] \qquad (4.9b)$$

$$\tau_{r\varphi} = G\gamma_{r\varphi} \qquad (4.9c)$$

as

$$\varepsilon_r = \frac{C_1}{2} - \frac{C_2}{r^2} , \qquad \varepsilon_\varphi = \frac{C_1}{2} + \frac{C_2}{r^2} , \qquad \gamma_{r\varphi} = -\frac{2C_4}{r^2} \qquad (4.10a,b,c)$$

$$\sigma_r = \frac{E}{(1 + \nu)(1 - 2\nu)}\left[\frac{C_1}{2} - (1 - 2\nu)\frac{C_2}{r^2}\right] \qquad (4.11a)$$

$$\sigma_\varphi = \frac{E}{(1 + \nu)(1 - 2\nu)}\left[\frac{C_1}{2} + (1 - 2\nu)\frac{C_2}{r^2}\right] \qquad (4.11b)$$

$$\sigma_z = \nu\left(\sigma_r + \sigma_\varphi\right) \frac{\nu E C_1}{(1 + \nu)(1 - 2\nu)} \qquad (4.11c)$$

$$\tau_{r\varphi} = -\frac{2GC_4}{r^2} \,. \qquad (4.11d)$$

By virtue of the boundary conditions, Eqs. 4.5a,b, the first and last of Eq. 4.11 yield the following equations for the determination of the constants C_1 and C_2.

$$\frac{C_1}{2} - (1 - 2\nu)\frac{C_2}{r_i^2} = -\frac{(1 + \nu)(1 - 2\nu)}{E}\, p_i \qquad (4.12a)$$

$$\frac{C_1}{2} - (1 - 2\nu)\frac{C_2}{r_o^2} = -\frac{(1 + \nu)(1 - 2\nu)}{E}\, p_o \qquad (4.12b)$$

$$C_4 = 0 \qquad (4.12c)$$

The solutions for C_1 and C_2 are

$$\frac{C_1}{2} = \frac{(1 + \nu)(1 - 2\nu)\left(p_i r_i^2 - p_o r_o^2\right)}{E\left(r_o^2 - r_i^2\right)} \qquad (4.13a)$$

$$C_2 = \frac{(1 + \nu)(p_i - p_o)\, r_i^2 r_o^2}{E\left(r_o^2 - r_i^2\right)} \,. \qquad (4.13b)$$

As to the constant C_3, it remains undetermined. However, with $C_4 = 0$, Eq. 4.8 shows that $v = C_3 r/2$ which corresponds to a rigid-body rotation about the axis of the cylinder. Since this rotation does not contribute to the strains, C_3 is taken as zero. Hence Eqs. 4.11a,b,c and Eq. 4.8a become:

$$\sigma_r = \frac{1}{(r_o/r_i)^2 - 1}\left\{\left[1 - \frac{(r_o/r_i)^2}{(r/r_i)^2}\right] p_i - \left[1 - \frac{1}{(r/r_i)^2}\right]\left(\frac{r_o}{r_i}\right)^2 p_o\right\} \qquad (4.14a)$$

$$\sigma_\varphi = \frac{1}{(r_o/r_i)^2 - 1}\left\{\left[1 + \frac{(r_o/r_i)^2}{(r/r_i)^2}\right] p_i - \left[1 + \frac{1}{(r/r_i)^2}\right]\left(\frac{r_o}{r_i}\right)^2 p_o\right\} \qquad (4.14b)$$

$$\sigma_z = \frac{2\nu}{(r_o/r_i)^2 - 1}\left[p_i - \left(\frac{r_o}{r_i}\right)^2 p_o\right] \qquad \text{Plane Strain!} \qquad (4.14c)$$

$$u = \frac{(1 + \nu)(r/r_i)\, r_i}{(r_o/r_i)^2 - 1}\left\{\left[(1 - 2\nu) + \frac{(r_o/r_i)^2}{(r/r_i)^2}\right]\frac{p_i}{E} - \left[(1 - 2\nu) + \frac{1}{(r/r_i)^2}\right]\left(\frac{r_o}{r_i}\right)^2 \frac{p_o}{E}\right\} \qquad (4.14d)$$

4.2 Stress Approach

In this case we solve directly for the stresses and apply the boundary conditions as before. Thus, re-expressing the stress-strain relations (Eq. 4.3) directly as strain when $\varepsilon_z = 0$ we get:

$$\varepsilon_r = \frac{1 + \nu}{E}\left[(1 - \nu)\,\sigma_r - \nu\sigma_\varphi\right] \tag{4.15a}$$

$$\varepsilon_\varphi = \frac{1 + \nu}{E}\left[(1 - \nu)\,\sigma_\varphi - \nu\sigma_r\right] \tag{4.15b}$$

$$\gamma_{r\varphi} = \frac{\tau_{r\varphi}}{G}\,, \tag{4.15c}$$

and then substituting them into Eq. 4.4 upon simplification yields

$$(1-\nu)\frac{d\sigma_\varphi}{dr} - \nu\frac{d\sigma_r}{dr} + \frac{\sigma_\varphi - \sigma_r}{r} = 0\;. \tag{4.16}$$

By the first stress-equilibrium relation in Eq. 4.2, this equation is reduced to the simple form

$$\frac{d}{dr}\left(\sigma_r + \sigma_\varphi\right) = 0\;. \tag{4.17}$$

Integration of this equation and the second of Eq. 4.2 yields

$$\sigma_r + \sigma_\varphi = C_1\;; \qquad \tau_{r\varphi} = \frac{C_2}{r^2}\;. \tag{4.18a,b}$$

Substitution of the relation $\sigma_\varphi = C_1 - \sigma_r$ in Eq. 4.2, transposition, and integration yield

$$\sigma_r = \frac{C_1}{2} + \frac{C_3}{r^2}\,, \tag{4.19}$$

where C_1, C_2 and C_3 are constants of integration. Applying the boundary conditions we finally get

$$C_1 = \frac{p_i r_i^2 - p_o r_o^2}{r_o^2 - r_i^2}\,, \qquad C_2 = 0\,, \qquad C_3 = \frac{(p_i - p_o)\, r_i^2 r_o^2}{r_o^2 - r_i^2}\;. \tag{4.20a,b,c}$$

With these values for the constants, Eqs. 4.18, 4.19 and the relation $\sigma_z = \nu(\sigma_r + \sigma_\varphi)$, the complete stress solution is explicitly given. This solution can then be substituted in Eqs. 4.15a and 4.15b to determine ε_r and ε_φ and, finally, u is determined by the relation $u = r\varepsilon_\varphi$.

NOTE: Since the prescribed boundary conditions of the problem are stresses, it is evident that the stress approach in obtaining the solution involves simpler algebra than the displacement approach.

5. THERMAL STRESS

Let us consider the additional stresses if the cylinder wall is subjected to a temperature gradient due to an imposed internal wall temperature, T_1, and an external wall temperature, T_o. For this case the following stresses and stress gradients are zero,

$$\tau_{r\varphi}\,; \qquad \tau_{z\varphi} \qquad \text{and} \qquad \frac{\partial \sigma_\varphi}{\partial \varphi} = 0 \qquad \text{(5.1a,b,c)}$$

$$\tau_{rz} = 0 \qquad \text{and} \qquad \frac{\partial \sigma_z}{\partial z} = 0 \qquad \text{(5.1d, e)}$$

NOTE: All boundary planes are planes of principal stress.

For this case the governing equations become:

- Equilibrium

$$\frac{\partial \sigma_r}{\partial r} - \frac{\sigma_\varphi - \sigma_r}{r} = 0 \qquad \text{(5.2)}$$

- The strain equation can be applied to get

$$E\varepsilon_r = E\frac{\partial u}{\partial r} = \sigma_r - \nu\sigma_\varphi - \nu\sigma_z + E\alpha T \qquad \text{(5.3a)}$$

$$E\varepsilon_\varphi = E\frac{u}{r} = \sigma_\varphi - \nu\sigma_r - \nu\sigma_z + E\alpha T \qquad \text{(5.3b)}$$

$$E\varepsilon_z = E\frac{\partial w}{\partial z} = \sigma_z - \nu\sigma_r - \nu\sigma_\varphi + E\alpha T \qquad \text{(5.3c)}$$

- Compatibility

$$\frac{\partial}{\partial r}\left(\frac{u}{r}\right) = \frac{1}{r}\frac{\partial u}{\partial r} - \frac{1}{r}\frac{u}{r} \qquad \text{(5.4)}$$

or from this with the expression for u/r and $\partial u/\partial r$,

$$\frac{\partial}{\partial r}\left(\sigma_\varphi - \nu\sigma_r - \nu\sigma_z + E\alpha T\right) = \frac{1}{r}\left(\sigma_r - \nu\sigma_\varphi - \nu\sigma_z + E\alpha T\right) - \frac{1}{r}\left(\sigma_\varphi - \nu\sigma_r - \nu\sigma_z + E\alpha T\right) \qquad \text{(5.5)}$$

$$= \frac{1}{r}\left(\sigma_r - \sigma_\varphi\right) + \frac{\nu}{r}\left(\sigma_r - \sigma_\varphi\right) = \frac{1+\nu}{r}\left(\sigma_r - \sigma_\varphi\right)$$

and therefore,

$$\left(\sigma_\varphi - \sigma_r\right) = -\frac{r}{1+\nu}\frac{\partial}{\partial r}\left(\sigma_\varphi - \nu\sigma_r - \nu\sigma_z + E\alpha T\right) = \underbrace{r\frac{\partial \sigma_r}{\partial r}}_{\text{from the Equilibrium Condition}} \qquad \text{(5.6)}$$

or,

$$-\frac{\partial\sigma_\varphi}{\partial r} + \nu\frac{\partial\sigma_z}{\partial r} - E\alpha\frac{\partial T}{\partial r} = \frac{\partial\sigma_r}{\partial r} \tag{5.7}$$

or,

$$\frac{\partial\sigma_\varphi}{\partial r} + \frac{\partial\sigma_r}{\partial r} - \nu\frac{\partial\sigma_z}{\partial r} + E\alpha\frac{\partial T}{\partial r} = 0\ . \tag{5.8}$$

- Assumptions (ε_z = constant so that it is independent of r)

$$\frac{\partial\varepsilon_z}{\partial r} = 0 \quad \text{or} \quad \frac{\partial w^2}{\partial r\partial z} = 0 \tag{5.9}$$

and therefore, differentiating Eq. 5.3c with respect to r

$$\frac{\partial\sigma_z}{\partial r} - \nu\frac{\partial\sigma_\varphi}{\partial r} - \nu\frac{\partial\sigma_r}{\partial r} + E\alpha\frac{\partial T}{\partial r} = 0\ . \tag{5.10}$$

Eliminating $\partial\sigma_z/\partial r$ from Eqs. 5.8 and 5.10 we get

$$\frac{\partial\sigma_\varphi}{\partial r} + \frac{\partial\sigma_r}{\partial r} - \nu^2\frac{\partial\sigma_\varphi}{\partial r} - \nu^2\frac{\partial\sigma_r}{\partial r} + E\alpha\nu\frac{\partial T}{\partial r} + E\alpha\frac{\partial T}{\partial r} = 0 \tag{5.11a}$$

$$(1 - \nu)\frac{\partial}{\partial r}\left(\sigma_\varphi + \sigma_r\right) + E\alpha\frac{\partial T}{\partial r} = 0 \tag{5.11b}$$

By integration we get

$$\sigma_\varphi + \sigma_r + \frac{E\alpha}{(1 - \nu)}T = Z \qquad \text{(independent of r).} \tag{5.11c}$$

Let us solve the energy equation to get the wall radial temperature distribution:

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} = 0 \tag{5.12a}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = 0 \tag{5.12b}$$

$$\frac{\partial T}{\partial r} = \frac{b}{r} \tag{5.12c}$$

$$T = a + b\,\ell nr\ . \tag{5.13}$$

5.1 Stress Distribution

- Radial Stress

From Eq. 5.11c

$$\sigma_\varphi + \sigma_r + \frac{E\alpha T}{(1 - \nu)} = Z\ . \tag{5.14}$$

From Eq. 5.2

$$\sigma_\varphi - \sigma_r - r\frac{\partial \sigma_r}{\partial r} = 0 \qquad (5.15)$$

or by subtraction

$$2\sigma_r + r\frac{\partial \sigma_r}{\partial r} + \frac{E\alpha T}{(1 - \nu)} = Z \qquad (5.16)$$

or

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r^2\sigma_r\right) + \frac{E\alpha}{(1 - \nu)}(a + b\ell nr) = Z \qquad (5.17)$$

By integration we get

$$r^2\sigma_r + \frac{E\alpha ar^2}{2(1 - \nu)} + \frac{E\alpha b}{(1 - \nu)}\left(\frac{r^2\ell nr}{2} - \frac{r^2}{4}\right) = \frac{Zr^2}{2} + B \qquad (5.18)$$

or

$$\boxed{\sigma_r = A + \frac{B}{r^2} - \frac{E\alpha T}{2(1 - \nu)}} \qquad (5.19)$$

where $A = E\alpha b/4(1 - \nu) + Z/2$ and B is a constant with respect to r.

- Tangential Stress

From Eq. 5.11c

$$\sigma_\varphi = -\sigma_r - \frac{E\alpha T}{(1 - \nu)} + Z = -A - \frac{B}{r^2} + \frac{E\alpha T}{2(1 - \nu)} - \frac{E\alpha T}{(1 - \nu)} + 2A - \frac{E\alpha b}{2(1 - \nu)} \qquad (5.20)$$

$$\boxed{\sigma_\varphi = A - \frac{B}{r^2} - \frac{E\alpha T}{2(1 - \nu)} - \frac{E\alpha b}{2(1 - \nu)}} \qquad (5.21)$$

- Axial Stress

1) $\frac{\partial w}{\partial z} = 0$

2) no axial load so that $2\pi\int_{r_i}^{r_o} \sigma_z r dr = 0$.

From Eq. 5.3c

$$\sigma_z = \nu\left(\sigma_\varphi + \sigma_r\right) - E\alpha T + E\varepsilon_z \qquad (5.22a)$$

$$\sigma_z = \nu\left(2A - \frac{E\alpha T}{(1 - \nu)} - \frac{E\alpha b}{2(1 - \nu)}\right) - E\alpha T + E\varepsilon_z$$

$$= 2\nu A - \frac{E\alpha b\nu}{2(1 - \nu)} - \frac{E\alpha T}{(1 - \nu)} + E\varepsilon_z \qquad (5.22b)$$

$$= 2\nu A - \frac{E\alpha b\nu}{2(1 - \nu)} - \frac{E\alpha a}{(1 - \nu)} - \frac{E\alpha b\ell nr}{(1 - \nu)} + E\varepsilon_z$$

1) If $\varepsilon_z = 0$ then from Eq. 5.22b

$$\sigma_z = 2\nu A - \frac{E\alpha b\nu}{2(1-\nu)} - \frac{E\alpha T}{(1-\nu)} \tag{5.23}$$

NOTE: $T = a + b\ell nr$

2) If no axial load so that $2\pi\int_{r_i}^{r_o} \sigma_z r dr = 0$;

or

$$\int_{r_i}^{r_o}\left(Dr - \frac{E\alpha a}{(1-\nu)} - \frac{E\alpha b\ell nr}{(1-\nu)}\right) dr = 0 \tag{5.24}$$

where $D \equiv 2\nu A - E\alpha b\nu/2(1-\nu)$, and

$$\sigma_z = \left(D - \frac{E\alpha T}{(1-\nu)}\right) = \left(D - \frac{E\alpha a}{(1-\nu)} - \frac{E\alpha b\ell nr}{(1-\nu)}\right) \tag{5.25}$$

or

$$\int_{r_i}^{r_o}\left[Dr - \frac{E\alpha}{(1-\nu)}\frac{\partial}{\partial r}\left(\frac{ar^2}{2} + \frac{br^2\ell nr}{2} - \frac{br^2}{4}\right)\right] dr = 0 \tag{5.26}$$

or

$$\left|\frac{Dr^2}{2} - \frac{E\alpha}{2(1-\nu)}\left(ar^2 + br^2\ell nr - \frac{br^2}{2}\right)\right|_{r_i}^{r_o} = 0 \tag{5.27a}$$

$$\left|\frac{Dr^2}{2} - \frac{E\alpha}{2(1-\nu)}\left(r^2T - \frac{br^2}{2}\right)\right|_{r_i}^{r_o} = 0 \tag{5.27b}$$

$$D = \frac{E\alpha}{(1-\nu)}\left[\frac{\left(r_o^2T_o - r_1^2T_1\right)}{\left(r_o^2 - r_1^2\right)} - \frac{b}{2}\right] \tag{5.27c}$$

and

$$\sigma_z = \frac{E\alpha}{(1-\nu)}\left[\frac{\left(r_o^2T_o - r_1^2T_1\right)}{\left(r_o^2 - r_1^2\right)} - \frac{b}{2} - T\right] \tag{5.28}$$

5.2 Boundary Conditions

$r = r_o$, $\sigma_r = 0$, $T = T_o$

$r = r_1$, $\sigma_r = 0$, $T = T_1$

Therefore, from Eq. 5.19 at the inner and outer boundaries, respectively:

$$0 = A + \frac{B}{r_1^2} - \frac{E\alpha T_1}{2(1-\nu)} \tag{5.29a}$$

$$0 = A + \frac{B}{r_o^2} - \frac{E\alpha T_o}{2(1-\nu)} . \tag{5.29b}$$

Solving Eqs. 5.29a,b obtain

$$B = \frac{E\alpha r_o^2 r_1^2 (T_1 - T_o)}{2(1-\nu)\left(r_o^2 - r_1^2\right)} \tag{5.30a}$$

$$A = \frac{E\alpha\left(r_o^2 T_o - r_1^2 T_1\right)}{2(1-\nu)\left(r_o^2 - r_1^2\right)} . \tag{5.30b}$$

With regard to temperature, from Eq. 5.13, at the inner and outer boundaries, respectively:

$$T_1 = a + b\ell n r_1, \qquad T_o = 1 + b\ell n r_o . \tag{5.31a,b}$$

Solving Eqs. 5.31a,b obtain

$$b = \frac{T_1 - T_o}{\ell n \frac{r_1}{r_o}} \tag{5.32a}$$

$$a = \frac{T_o \ell n r_1 - T_1 \ell n r_o}{\ell n \frac{r_1}{r_o}} \tag{5.32b}$$

5.3 Final Results

$$\sigma_r = \frac{E\alpha}{2(1-\nu)}\left[\frac{\left(r_o^2 T_o - r_1^2 T_1\right) + \frac{r_o^2 r_1^2}{r^2}(T_1 - T_o)}{\left(r_o^2 - r_1^2\right)} - T\right] \tag{5.33a}$$

$$\sigma_\varphi = \frac{E\alpha}{2(1-\nu)}\left[\frac{\left(r_o^2 T_o - r_1^2 T_1\right) + \frac{r_o^2 r_1^2}{r^2}(T_1 - T_o)}{\left(r_o^2 - r_1^2\right)} - \frac{T_1 - T_o}{\ell n \frac{r_1}{r_o}} - T\right] \tag{5.33b}$$

$\varepsilon_z = 0$, from Eq. 5.23

1) $$\sigma_z = \frac{E\alpha}{2(1-\nu)}\left[\frac{2\nu\left(r_o^2 T_o - r_1^2 T_1\right)}{\left(r_o^2 - r_1^2\right)} - \frac{\nu(T_1 - T_o)}{\ell n \frac{r_1}{r_o}} - 2T\right] \tag{5.34}$$

2) No axial load

$$\sigma_z = \frac{E\alpha}{(1-\nu)}\left[\frac{\left(r_o^2 T_o - r_1^2 T_1\right)}{\left(r_o^2 - r_1^2\right)} - \frac{(T_1 - T_o)}{2\ell n \frac{r_1}{r_o}} - T\right] \tag{5.35}$$

intentionally left blank

6. DESIGN PROCEDURES

There are four main steps in a rational design procedure:

Step 1:

Determine the mode of failure of the member that would most likely take place if the loads acting on the member should become large enough to cause it to fail.

The choice of material is involved in this first step because the type of material may significantly influence the mode of failure that will occur.

NOTE: Choice of materials may often be controlled largely by general factors such as:

- availability
- cost
- weight limitations
- ease of fabrication

rather than primarily by the requirements of design for resisting loads.

Step 2:

The mode of failure can be expressed in terms of some quantity, for instance, the maximum normal stress.

Independent of what the mode of failure might be, it is generally possible to associate the failure of the member with a particular cross section location.

For the linearly elastic problem, failure can be interpreted in terms of the state of stress at the point in the cross section where the stresses are maximum.

Therefore, in this step, relations are derived between the loads acting on the member, the dimensions of the member, and the distributions of the various components of the state of stress within the cross section of the member.

Step 3:

By appropriate tests of the material, determine the maximum value of the quantity associated with failure of the member. An appropriate or suitable test is one that will produce the same action in the test specimen that results in failure of the actual member.

NOTE: This is difficult or even impossible. Therefore, theories of failure are formulated such that results of simple tests (tension and compression) are made to apply to the more complex conditions.

Step 4:

By use of experimental observations, analysis, experience with actual structures and machines, judgment, and commercial and legal considerations, select for use in the relation derived in Step 2 a working (allowable or safe) value for the quantity associated with failure. This working value is considerably less than the limiting value determined in Step 3.

The need for selecting a working value less than that found in Step 3 arises mainly from the following uncertainties:

1. uncertainties in the service conditions, especially in the loads,
2. uncertainties in the degree of uniformity of the material, and
3. uncertainties in the correctness of the relations derived in Step 2.

These considerations clearly indicate a need for applying a so-called safety factor in the design of a given load-carrying member. Since the function of the member is to carry loads, the safety factor should be applied to the loads. Using the theory relating the loads to the quantity associated with failure desired in Step 2 and the maximum value of the quantity associated with failure in Step 3, determine the failure loads which we will designate P_f. The safety factor, N, is the ratio:

$$N = \frac{P_f}{P_w} = \frac{\text{failure load}}{\text{working load}}$$

NOTE: If P_f and P_w are *each directly proportional to stress*, then

$$N = \frac{\sigma_f}{\sigma_w}$$

The magnitude of N may be as low as 1.4 in aircraft and space vehicle applications, whereas in other applications where the weight of the structure is not a critical constraint, N will range from 2.0 to 2.5.

6.1 Static Failure and Failure Theories

This section will treat the problem of predicting states of stress that will cause a particular material to fail—a subject which is obviously of fundamental importance to engineers.

Materials considered are crystalline or granular in nature. This includes metals, ceramics (except glasses) and high-strength polymers.

The reason for the importance of crystalline materials is their *inherent resistance* to deformation. This characteristic is due to the fact that the atoms are compactly arranged into a simple crystal lattice of relatively low internal energy.

In this section we neglect the following types of failures:

- creep failures which occur normally only at elevated temperature,
- buckling and excessive elastic deflection, and
- fatigue failure which is dynamic in nature.

Thus we limit ourselves to failures which are functions of the applied loads.

Definition: Failure of a member subjected to load can be regarded as any behavior of the member which renders it unsuitable for its intended function.

Eliminating creep, buckling and excessive elastic deflection, and fatigue, we are left with the following two basic categories of static failure:

1. Distortion, or plastic strain—failure by distortion is defined as having occurred when the plastic deformation reaches an arbitrary limit. The standard 0.2% offset yield point is usually taken as this limit.
2. Fracture—which is the separation or fragmentation of the member into two or more parts.

I. Distortion is always associated with shear stress.

II. Fracture can be either brittle or ductile in nature (or a portion of both).

As tensile loading acts on an atomic structure and is increased, one of two events must eventually happen:

- Either the shear stress acting in the slip planes will cause slip (plastic deformation), or
- The strained cohesive bonds between the elastically separated atoms will break down (brittle fracture) with little if any distortion. The fractured surfaces would be normal to the applied load and would correspond to simple crystallographic planes or to grain boundaries.

NOTE: The stress required for fracture ranges from about 1/5 to as little as 1/1000 of the theoretical cohesive strength of the lattice structure because of sub-microscopic flaws or dislocations.

Many fractures are appropriately described as being partially brittle and partially ductile, meaning that certain portions of the fractured surface are approximately aligned with planes of maximum shear stress and exhibit a characteristic fibrous appearance, while other portions of the fractured surface appear granular as in the case of brittle fracture and are oriented more toward planes of maximum tensile stress.

NOTE: Tensile fractures accompanied by less than 5% elongation are often classed as brittle. If the elongation is > 5% elongation, then the fracture is classed as ductile.

Brittle fractures often occur suddenly and without warning. They are associated with a release of a substantial amount of elastic energy (integral of force times deflection) which for instance may cause a loud noise. Brittle fractures have been known to propagate great distances at velocities as high as 5,000 fps.

Primary factors promoting brittle fracture are:

a. low temperature increases the resistance of the material to slip but not to cleavage,

b. relatively large tensile stresses in comparison with the shear stresses,

c. rapid impact – rapid rates of shear deformation require greater shear stresses, and these may be accompanied by normal stresses which exceed the cleavage strength of the material,

d. thick sections – this "size effect" has the important practical implication that tests made with small test samples appear more ductile than thick sections such as those used in pressure vessels. This is because of extremely minute cracks which are presumably inherent in all actual crystalline materials.

6.2 Prediction of Failure under Biaxial and Triaxial Loading

Engineers concerned with the design and development of structural or machine parts are generally confronted with problems involving biaxial (occasionally triaxial) stresses covering an infinite range or ratios of principal stresses.

However, the available strength data usually pertain to uniaxial stress, and often only to uniaxial tension.

As a result, the following question arises: If a material can withstand a known stress in uniaxial tension, how highly can it be safety stressed in a specific case involving biaxial (or triaxial) loading?

The answer must be given by a failure theory. The philosophy that has been used in formulating and applying failure theories consists of two parts:

1. Postulated theory to explain failure of a standard specimen. Consider the case involving a tensile specimen, with failure being regarded as initial yielding. We might theorize that tensile yielding occurred as a result of exceeding the capacity of the materials in one or more respects, such as:

   a) capacity to withstand normal stress,

   b) capacity to withstand shear stress,

   c) capacity to withstand normal strain,

   d) capacity to withstand shear strain,

   e) capacity to absorb strain energy (energy associated with both a change in volume and shape),

   f) capacity to absorb distortion energy (energy associated with solely a change in shape).

2. The results of the standard test are used to establish the magnitude of the capacity chosen sufficient to cause initial yielding. Thus, if the standard tensile test indicates a yield strength of 100 ksi, we might assume that yielding will always occur with this material under any combination of static loads which results in one of the following:

   a) a maximum normal stress greater than that of the test specimen (100 ksi),

   b) a maximum shear stress greater than that of the test specimen (50 ksi),

   c–f) are defined analogously to a and b.

Hence, in the simple classical theories of failure, it is assumed that the same amount of whatever caused the selected tensile specimen to fail will also cause any part made of the materials to fail *regardless of the state of stress involved*.

When used with judgment, such simple theories are quite usable in modern engineering practice.

6.3 Maximum Normal Stress Theory (Rankine)

In a generalize form, this simplest of the various theories states merely that a material subjected to any combination of loads will:

1. Yield whenever the greatest positive principal stress exceeds the tensile yield strength in a simple uniaxial tensile test of the same material or whenever the greatest negative principal stress exceeds the compressive yield strength.

2. Fracture whenever the greatest positive (or negative) principal stress exceeds the tensile (or compressive) ultimate strength in a simple uniaxial tensile (or compressive) test of the same material.

NOTE: Following this theory, the strength of the material depends upon only *one* of the principal stresses (the largest tension or the largest compression) and is entirely *independent of the other two.*

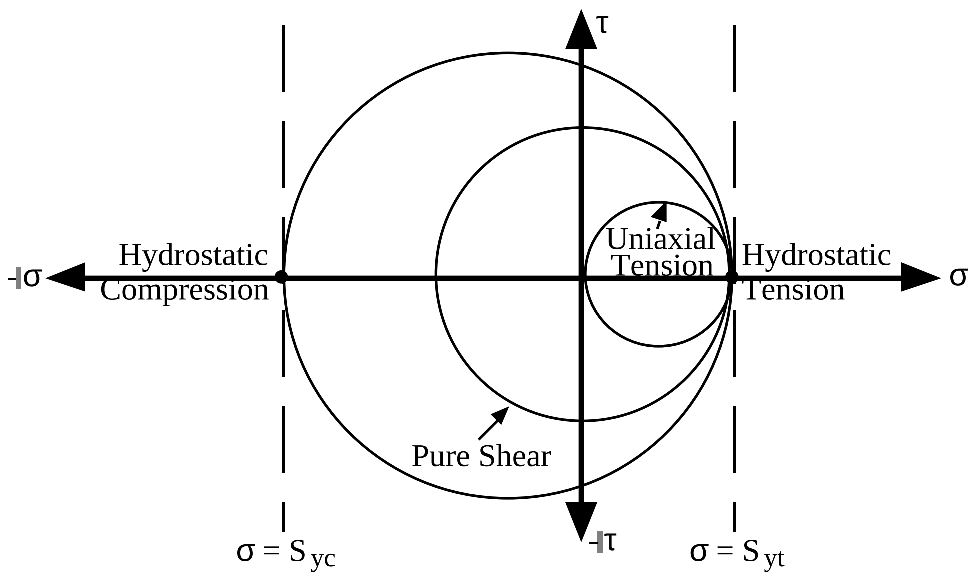


Figure 22. Principal Mohr's Circles for Several Stress States Representing Incipient Yielding According to Maximum Normal Stress Theory (Note, for Pure Shear $\sigma_1 = |\sigma_2| = \tau$).

NOTE: Each of the circles is a principal circle for the state of stress which it represents. This theory implies that failure (in this case yielding) occurs when and only when the principal Mohr's circle extends outside the dashed vertical lines given by S_{yt} and S_{yc}.

The failure locus for the biaxial stress state for yield according to the maximum normal-stress theory is to be illustrated. $\sigma_{1,2}$ are the principal stresses. Yield will occur if either the compressive yield strength, S_{yc}, or the tensile yield strength, S_{yt}, is exceeded by either of the principle stresses σ_1 or σ_2. Hence, the maximum value of $+\sigma_1$ is S_{yt}, $+\sigma_2$ is S_{yt}, $-\sigma_1$ is S_{yc}, and $-\sigma_2$ is S_{yc}. These maxima are plotted in Fig. 23, thus defining the failure locus as a rectangle. Failure is predicted for those states that are represented by points falling *outside* the rectangle.

NOTE: If use is made of S_{ut} and S_{uc} instead of S_{yt} and S_{yc}, the theory would have predicted failure by fracture.

NOTE: In the 3D case we have to deal with a cube.

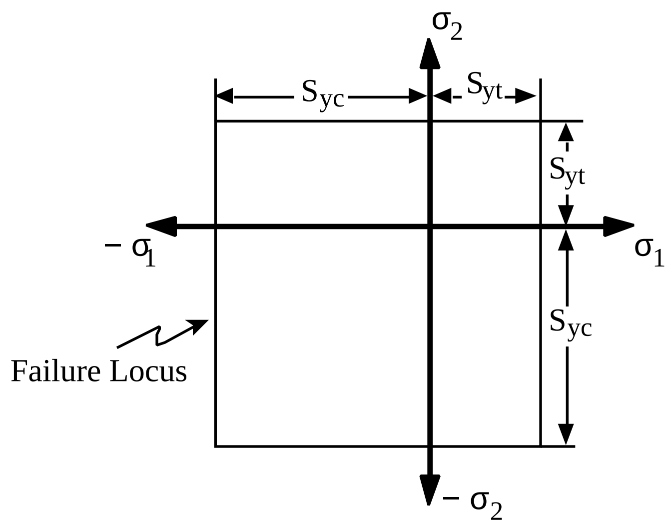


Figure 23. Failure Locus for the Biaxial Stress State for the Maximum Normal-Stress Theory.

- Failure for Brittle Material

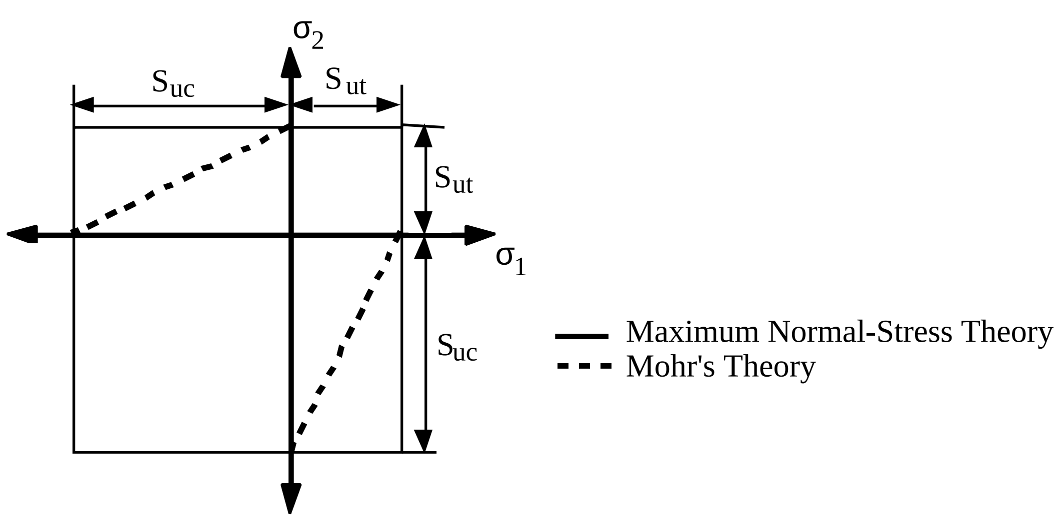


Figure 24. Failure Locus for Mohr's Theory

NOTE: For most brittle materials the ultimate compressive strength exceeds the ultimate tensile strength. The locus of biaxial stress states of incipient failure will be a square as shown above, and "safe" stress will lie within this square. Mohr's theory (denoted by dashed lines) is more conservative.

It is often convenient to refer to an equivalent stress, S_e (σ_e), as calculated by some particular theory.

NOTE: The equivalent stress may or may not be equal to the yield strength.

Mathematically, the equivalent stress based on the maximum stress theory is given by:

$$\boxed{S_e = (\sigma_i)_{max} \qquad i = 1, 2, 3} \tag{6.1}$$

Applicability of Method –

Reasonably accurate for materials which produce brittle fracture both in the test specimen and in actual service such as: Cast iron, concrete, hardened tool steel, glass [Ref. 3, Fig. 6.8].

It cannot predict failure under hydrostatic compression (the state of stress in which all three principle stresses are equal). Structural materials, including those listed above, can withstand hydrostatic stresses many times S_{uc}.

It cannot accurately predict strengths where a ductile failure occurs.

6.4 Maximum Shear Stress Theory (The Coulomb, later Tresca Theory)

The theory states that a material subjected to any combination of loads will fail (by yielding or fracturing) whenever *the maximum shear* stress exceeds the shear strength (yield or ultimate) in a simple uniaxial stress test of the same material.

The shear strength, in turn, is usually assumed to be determined from the standard uniaxial tension test. Principle Mohr's circles for several stress states representing incipient yielding according to maximum shear stress theory are shown in Fig. 25.

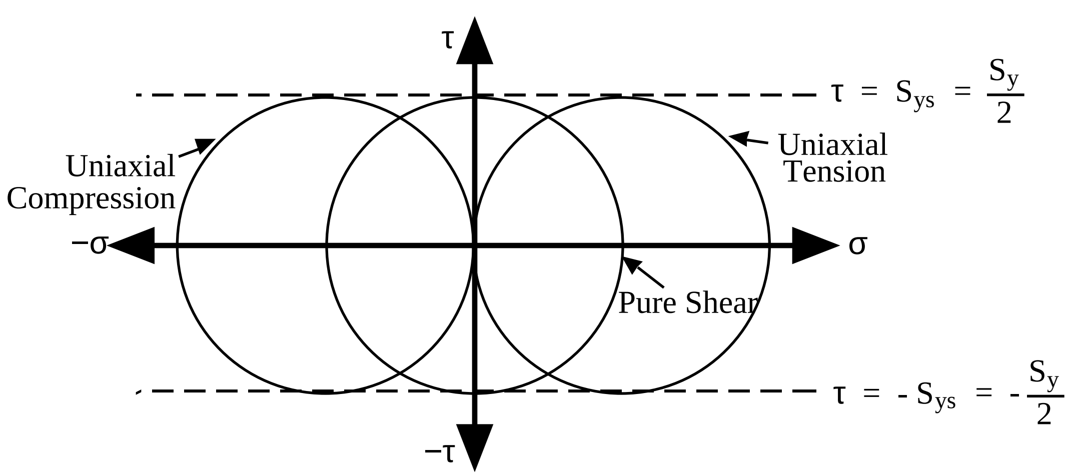


Figure 25. Stress States for Yielding According to the Maximum Shear Stress Theory

It was shown in connection with the Mohr's circle that,

$$\tau_{max} = \frac{1}{2}|\sigma_1 - \sigma_2|, \tag{6.2a}$$

where τ_{max} occurs on faces inclined at 45˚ to faces on which the maximum and minimum principle stresses act. Hence, in this failure theory, it is important to recognize σ_1 and σ_2 are the maximum and minimum principle stresses, or

$$\tau_{max} = \frac{1}{2}|\sigma_{max} - \sigma_{min}|. \tag{6.2b}$$

In the tensile test specimen, $\sigma_1 = S_y, \sigma_2 = \sigma_3 = 0$, and thus:

$$\boxed{\tau_{max} = \frac{1}{2}S_y}. \tag{6.3}$$

The assumption is then made that this will likewise be the limiting shear stress for more <u>complicated combined stress loadings</u>, i.e.,

$$\tau_{max} = \frac{1}{2}S_y = \frac{1}{2}|\sigma_{max} - \sigma_{min}| \tag{6.4}$$

or

$$\boxed{S_y = |\sigma_{max} - \sigma_{min}|} \tag{6.5}$$

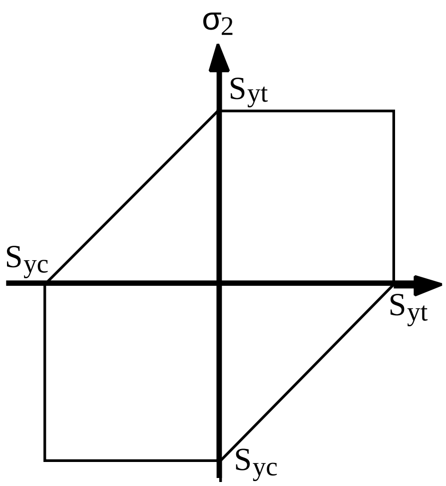
The failure locus for the biaxial stress state for the maximum shear stress theory is shown in Fig. 26. This locus is determined as follows. There are three principal stresses involved, σ_1, σ_2 and σ_3 (σ_3 is always equal to zero). In the first quadrant, along the vertical line, $\sigma_1 > \sigma_2 > \sigma_3$, which means that $\sigma_1 = \sigma_{max}$ and $\sigma_3 .= \sigma_{min}$. Thus, the value of σ_2 is free to be any value between σ_1 and σ_3, yielding the vertical line. Similarly, along the horizontal line in the first quadrant, $\sigma_2 > \sigma_1 > \sigma_3$, which means that $\sigma_2 .= \sigma_{max}$.and $\sigma_3 .= \sigma_{min}$. Thus, in this situation, σ_1 is free to be any value between σ_2 and σ_3, yielding the horizontal line. In Quadrant II, σ_1.is a compressive stress. Hence, this stress is now σ_{min}. Thus, one now has the situation:

$$\sigma_2 = \sigma_{max} > \sigma_3 = 0 > \sigma_1 = \sigma_{min} \tag{6.6}$$

and direct application of the criterion

$$\boxed{S_y = |\sigma_{max} - \sigma_{min}| = |\sigma_2 - (-\sigma_1)|} \tag{6.7}$$

yields the diagonal line in Quadrant II. Similar arguments apply to Quadrants III and IV.



Yield strength is S_y in tension and compression

Figure 26. The Failure Locus for the Biaxial Stress State for the Maximum Shear-Stress Theory

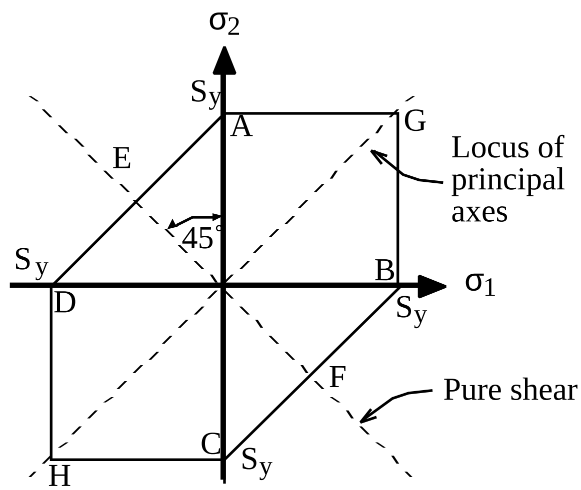


Figure 27. Pure Shear State Representation on the Failure Locus for the Maximum Shear-Stress Theory

NOTE: When σ_1 and σ_2 have like signs, the failure locus is identical to that of the maximum stress theory.

NOTE: The boundaries of all principal Mohr circles not representing failure are the two *horizontal* lines $\pm S_{ys}$ (or $\pm S_{us}$). This theory predicts that failure cannot be produced by pure hydrostatic stress.

The failure locus for the biaxial stress state is shown in Fig. 27. EF represents the shear diagonal of the σ_1 - σ_2 plot, since it corresponds to the equation $\sigma_1 = -\sigma_2$ which yields Mohr's circle with $\sigma_1 = |\sigma_2| = \tau$ which represents pure shear in the 1-2 plane. GH corresponds to $\sigma_1 = \sigma_2$, which yields Mohr's circle as a point with $\tau = 0$. Hence, GH represents the locus of principal axes.

Applicability of Method –

For ductile failure (usually yielding) – steel, aluminum, brass. 15% error on the conservative side.

6.5 Mohr Theory and Internal-Friction Theory

This theory suggests that Mohr's circles be drawn representing every available test condition and that the envelope of these circles be taken as the envelope of any and all principal Mohr circles representing stress states on the verge of failure.

Figure 28 represents what might be called Mohr's theory in its simplest form where only uniaxial tension and compression data are available, and where the envelope is assumed to be represented by the two tangent straight lines.

In this form, the Mohr theory is seen to be a modification of the maximum shear-stress theory. If both compression and tension data are available, the Mohr theory is obviously the better of the two.

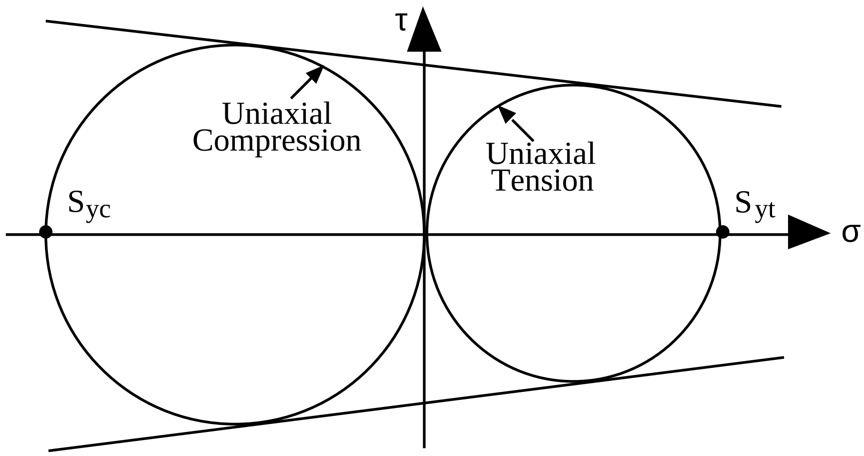


Figure 28. Mohr's Failure Theory for Uniaxial Tension and Compression

6.6 Maximum Normal-Strain Theory (Saint-Vanent's Theory)

Failure will occur whenever a principal normal strain reaches the maximum normal strain in a simple uniaxial stress test of the same material.

The principal normal strains have been written as follows in Eq. 2.4:

$$\varepsilon_i = \frac{1}{E}\left[\sigma_i - \nu(\sigma_j + \sigma_k)\right] \tag{6.8}$$

which for a biaxial stress state are

$$\varepsilon_1 = \frac{1}{E}(\sigma_1 - \nu\sigma_2) \tag{6.9a}$$

$$\varepsilon_2 = \frac{1}{E}(\sigma_2 - \nu\sigma_1) \tag{6.9b}$$

For failure in a simple tensile test, Eq. 6.9 reduces to

$$\varepsilon_f = \frac{\sigma_f}{E} \tag{6.10}$$

where ε_f and σ_f are taken in the uniaxial loading direction.

Hence, taking σ_f as S_y, the failure criteria are

$$\sigma_1 - \nu\sigma_2 \geq S_y \quad (6.11a)$$

$$\sigma_2 - \nu\sigma_1 \geq S_y \quad (6.11b)$$

and

$$\sigma_1 - \nu\sigma_2 \leq -S_y \quad (6.11c)$$

$$\sigma_2 - \nu\sigma_1 \leq -S_y \quad (6.11d)$$

where failure is predicted if any one of the relations of Eq. 6.11 are satisfied.

NOTE: Unlike the previously considered theories, the value of the intermediate principal stress influences the predicted strength.

The graphical representation of this failure theory is presented in Fig. 29.

This theory predicts failure in hydrostatic states of stress, i.e., ductile, which is not in agreement with experimental evidence plus does not work well for brittle material failures. It is of historical but not current importance.

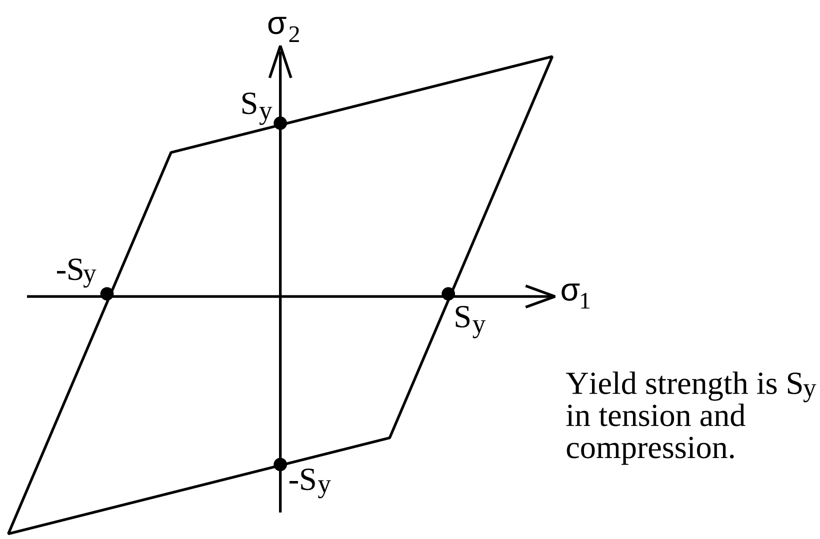


Figure 29. The Failure Locus for the Maximum Normal Strain Theory (for fixed ν).

6.7 Total Strain-Energy Theory (Beltrami Theory)

The total amount of elastic energy absorbed by an element of material is the proper criterion for its yielding. It is a forerunner to the important maximum distortion-energy theory discussed next.

6.8 Maximum Distortion-Energy Theory (Maximum Octahedral-Shear-Stress Theory, Van Mises, Hencky)

Given a knowledge of only the tensile yield strength of a material, this theory predicts ductile yielding under combined loading with greater accuracy than any other recognized theory. Where the stress involved is triaxial, this theory takes into account the influence of the third principal stress.

NOTE: Its validity is limited to materials having similar strength in tension and compression.

Equations can be developed from at least five different hypotheses! The most important of these relate to octahedral shear stress and distortion energy. [see Ref. 3, p. 139 for a derivation based on direct evaluation of distortion energy.]

We consider this theory as the maximum octahedral-shear-stress theory, i.e., yielding will occur whenever the shear stress acting on octahedral planes exceed a critical value. This value is taken as the octahedral shear existing in the standard tensile bar at incipient yielding.

The maximum octahedral-shear-stress theory is closely related to the maximum shear-stress theory but may be thought of as a refinement in that it considers the influence of all *three principal stresses*.

From

$$\tau_{oct} = \frac{1}{3}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{1/2} \qquad (1.13b)$$

the octahedral shear stress produced by uniaxial tension, i.e., only $\sigma_1 \neq 0$, is

$$\tau_{oct} = \frac{\sqrt{2}}{3}\sigma_1 \, . \qquad (6.12)$$

According to the theory, yielding always occurs at a value of octahedral shear stress established by the tension test as

$$\tau_{oct}\ (\text{limiting value}) = \frac{\sqrt{2}}{3} S_y \, . \qquad (6.13)$$

Thus, the octahedral shearing stress theory of failure can be expressed as follows by utilizing Eqs. 6.13 and 1.13b:

$$S_y = \frac{\sqrt{2}}{2}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{1/2} . \qquad (6.14)$$

Equation 6.14 implies that any combination of principal stresses will cause yielding if the right side of this equation exceeds the tensile test value of S_y. This may be written alternatively as

$$2S_y^2 = (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \, . \qquad (6.15)$$

A variation of Eq. 6.14 which is sometimes useful involves the concept of an equivalent uniaxial tensile stress, σ_e, where σ_e is the value of uniaxial tensile stress which produces the same level of octahedral shear stress as does the actual combination of existing principal stresses, thus

$$\sigma_e = \frac{\sqrt{2}}{2}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{1/2} \tag{6.16}$$

Obviously, if the loads are such that $\sigma_e > S_y$, yielding would be predicted. For design purposes, σ_e should be made equal to the allowable working uniaxial stress.

- Case of Pure Biaxial Shear-Stress

Recalling Mohr's circle for this case, we have the principal stresses

$$\sigma_1 = \tau, \quad \sigma_2 = -\tau, \quad \sigma_3 = 0.$$

Substituting these values into Eqs. 6.14 or 6.16 gives

$$S_y \ (\text{or } \sigma_e) = \sqrt{3}\ \tau , \tag{6.17}$$

This means that if $\tau > \frac{S_y}{\sqrt{3}} = 0.577\ S_y$, the material will yield. Hence, according to the maximum octahedral-shear-stress theory, a material is 57.7% as strong in shear as it is in tension.

- General Case of Biaxial Stress ($\sigma_3 = 0$)

Equation 6.16 reduces to

$$\sigma_e = \left(\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2\right)^{1/2} \tag{6.18}$$

In many biaxial-stress problems it is more convenient to work directly with stress σ_x, σ_y and τ_{xy}, because these can be determined more readily than principal stresses. Equation 6.18 can be modified for this purpose by application of Eq. 1.8 to yield Eq. 6.19:

$$\sigma_1, \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\tau_{xy}^2 + \left(\frac{\sigma_x - \sigma_y}{2}\right)^2} \tag{1.8}$$

$$\sigma_e = \left(\sigma_x^2 + \sigma_y^2 - \sigma_x\sigma_y + 3\tau_{xy}^2\right)^{1/2} \tag{6.19}$$

Equation 6.19 can also be derived by superposition of Eqs. 6.17 and 6.18.

The locus of failure conditions for this failure theory is illustrated by the ellipse in Fig. 30.

NOTE: The theory can be independently developed from the maximum distortion-energy theory, which postulates that failure (yielding) is caused by the elastic *energy* associated with this distortion.

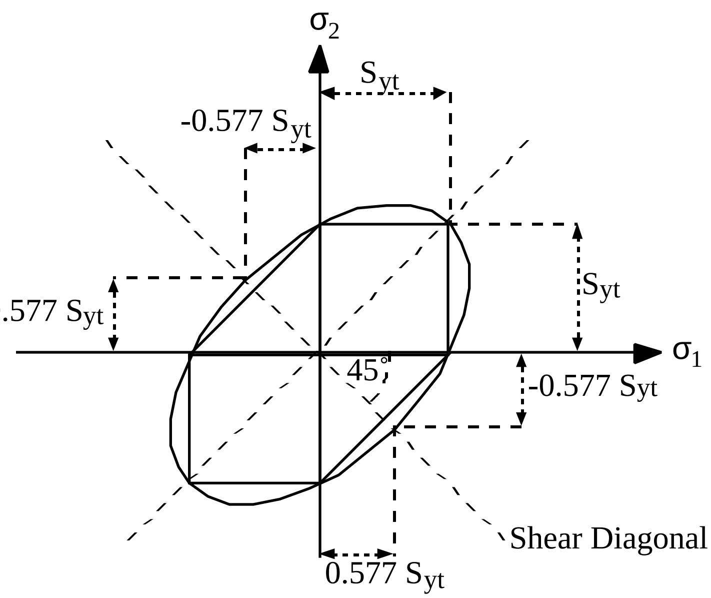


Figure 30. Failure Locus for the Biaxial Stress State for the Maximum Distortion Energy Theory

6.9 Comparison of Failure Theories

The failure theories are compared graphically for a biaxial state of stress in Fig. 31. From this figure, it can be seen that:

- The distortion energy and maximum shear stress theories predict similar results with the shear stress theory being more conservative.
- The maximum normal stress and maximum shear stress theories agree in the first and third quadrants where the signs of the principal stresses are the same but not in the second and fourth quadrants.
- Biaxial strength data for a variety of ductile and brittle materials are shown in Fig. 32 with several failure theory limits. From this figure it can be seen that experimental data supports:
  - Maximum normal stress theory is appropriate for brittle behavior.
  - Distortion energy or maximum shear stress theories is appropriate for ductile failure.

6.10 Application of Failure Theories to Thick-Walled Cylinders

An examination of the cases

a) internally pressurized cylinder, and

b) externally pressurized cylinder

indicate that in both cases failure would be expected at the *innermost* fibers. Moreover, this statement is true with respect to each of the aforementioned failure theories. Assuming zero axial

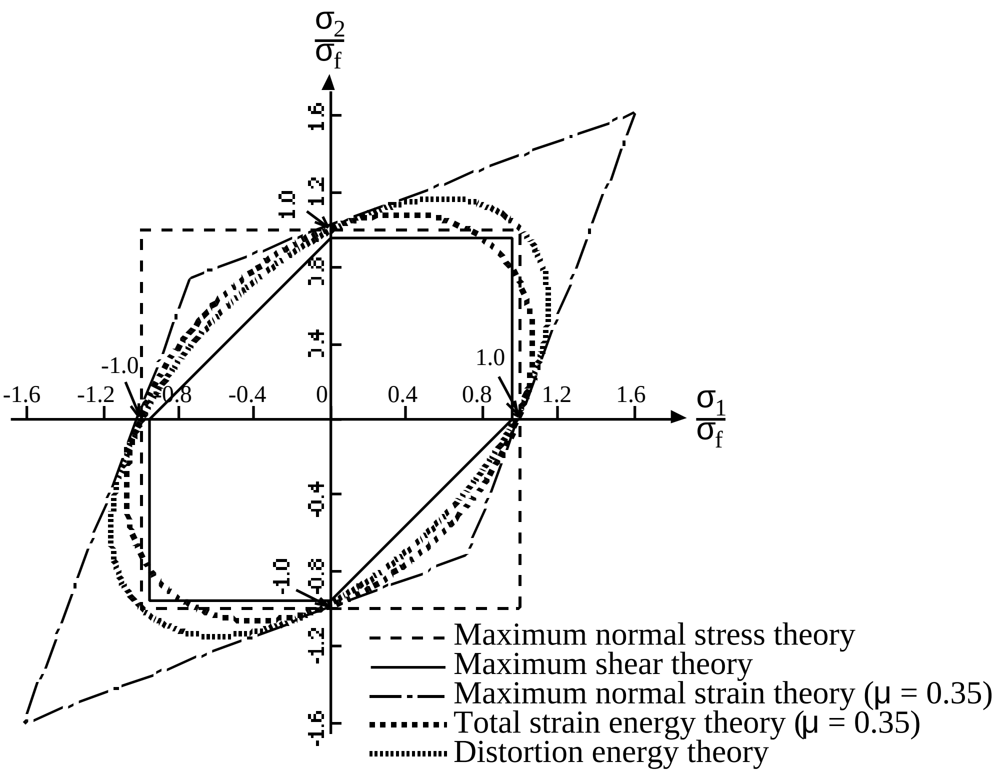


Figure 31. Comparison of Failure Theories for a Biaxial State of Stress. (From Ref. 4, p. 123.)

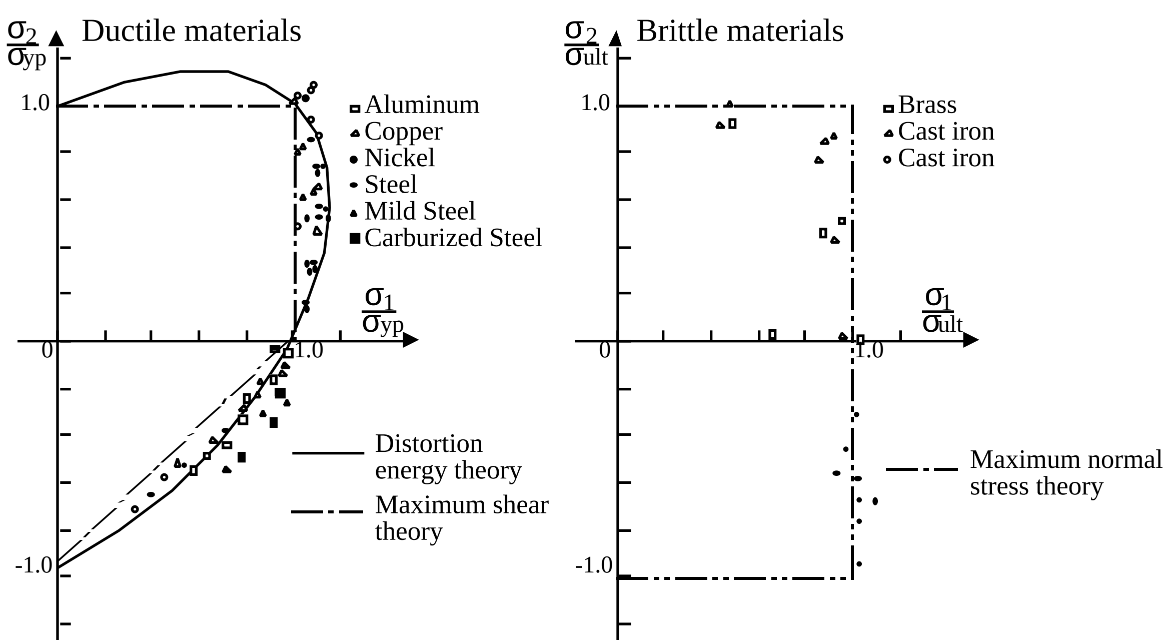


Figure 32. Comparison of Biaxial Strength Data with Theories of Failure for a Variety of Ductile and Brittle Materials. (From Ref. 3, p. 144.)

stress (plane state of stress), the critical inner surfaces are subjected to uniaxial stress. For these cases, the failure theories are, of course, in complete agreement as to the load intensity causing failure.

Example 1:

Internally pressurized cylinder. Determine the internal pressure required to yield the inner surface of a cylinder, where r_i = 1 in., r_o = 2 in., (t = 1 in.), and the material is steel with the properties S_y = 100 ksi and ν = 0.3. Assume plane stress.

Maximum stresses located at the inner surface are

$$\sigma_t = p_i \frac{r_o^2 + r_i^2}{r_o^2 - r_i^2} = p_i \frac{4 + 1}{4 - 1} = \frac{5}{3} p_i$$

$$\sigma_r = - p_i \,.$$

The predictions of the various failure theories are displayed as numbered curves on Fig. 33 for a spectrum of geometries of internally pressurized cylinders. The specific results for several failure theories for this example are as follows:

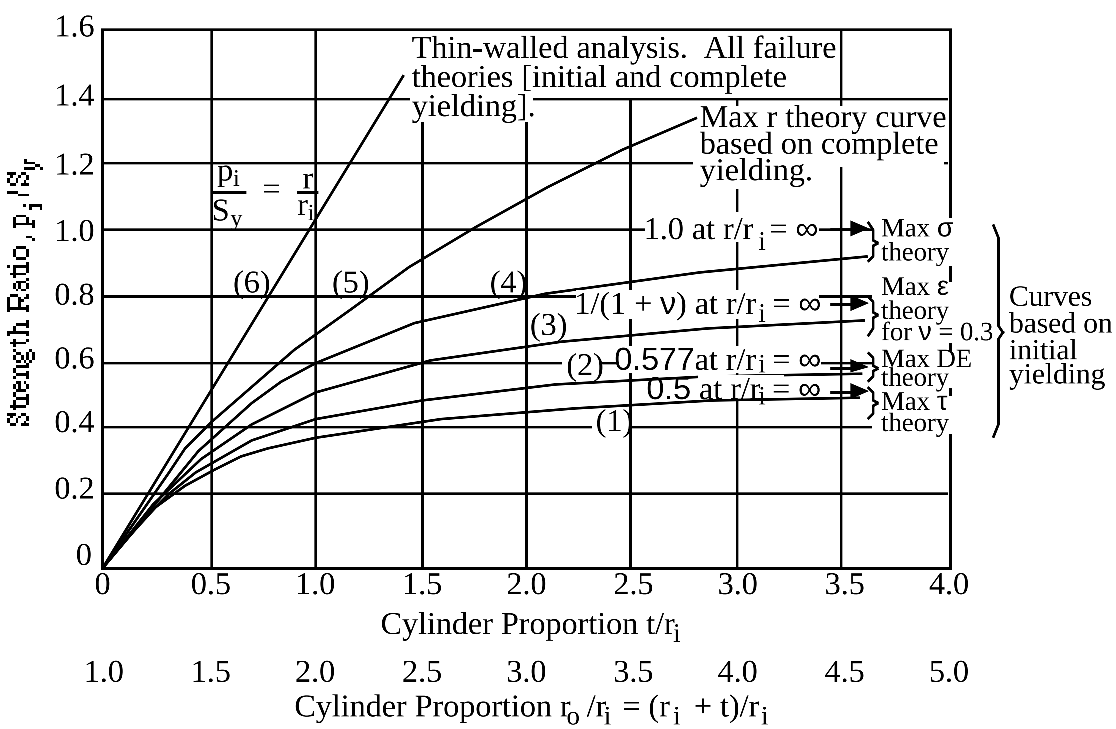


Figure 33. Failure Predictions for Internally Pressurized Cylinder

Curve (1)

According to the maximum shear-stress theory, yielding will begin when the highest shear stress in the cylinder reaches that in the standard tensile test at initial yielding, which is $S_y/2$. Thus,

$$\tau_{max} = \frac{S_y}{2}$$

or

$$\frac{\sigma_t - \sigma_r}{2} = 50 \text{ ksi},$$

$$\frac{4}{3} p_i = 50 \text{ ksi}$$

or

$$p_i = 37.5 \text{ ksi}.$$

Hence, for this case where $t/r_i = 1.0$, $p_i/S_y = 0.375$. This value appears on Fig. 33.

Curve (2)

According to the maximum distortion-energy theory, yielding will begin when the equivalent tensile stress at the inner cylinder wall reaches the tensile stress in the standard tensile test at initial yielding, which is S_y. Hence, using Eq. 6.18 and the known principal stresses:

$$\sigma_e = S_y$$

$$\left(\sigma_t^2 + \sigma_r^2 - \sigma_t\sigma_r\right)^{1/2} = 100 \text{ ksi}$$

$$\left[\left(\frac{5}{3} p_i\right)^2 + (-p_i)^2 - \left(-\frac{5}{3} p_i^2\right)\right]^{1/2} = 100 \text{ ksi}$$

$$2.33\, p_i = 100 \text{ ksi}$$

or

$$p_i = 43 \text{ ksi} \qquad \text{for } t/r_i = 1.0$$

Curve (4)

According to the maximum normal-stress theory, yielding will begin when the highest normal stress in the cylinder reaches the highest normal stress in the standard tensile test at initial yielding, which is S_y. Hence,

$$\sigma_t = S_y$$

$$5/3\, p_i = 100 \text{ ksi}$$

or the internal pressure required to yield the inner surface is

$$p_i = 60 \text{ ksi for } t/r_i = 1.0.$$

For the full range of cylindrical geometries, p_i can be determined for each theory and is presented on Fig. 33.

Example 2:

Strain-gage tests on the surface of a steel part indicate the stress state to be biaxial with principal stresses of 35 ksi tension and 25 ksi compression. The steel has been carefully tested in tension, compression and shear, with the results that $S_{yt} = S_{yc} = 100$ ksi and $S_{ys} = 60$ ksi.

- Estimate the safety factor with respect to initial yielding using the following failure theories:

  Maximum normal-stress,
  Maximum shear-stress,
  Distortion-energy, and
  Mohr's theory.

- Evaluate briefly the relative merits of the four approaches.

Solution:

We assume that the safety factor should be computed on the basis of all stresses increasing proportionally as the load is increased to failure. On this basis the load line has been extended outward from the nominal load point until it intersects the limiting lines corresponding to each failure theory.

NOTE: By simple proportion, we obtain the safety factor as the ratio by which the nominal stresses can be increased before yielding is predicted.

a) Maximum Normal-Stress Theory

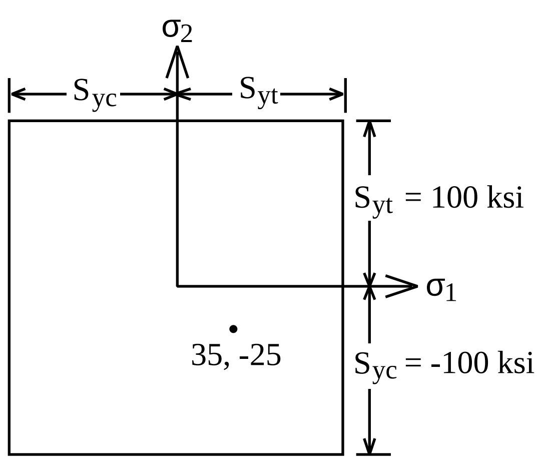


Figure 34. Maximum Normal-Stress Theory

Yield will occur if either the compressive yield strength, S_{yc}, or the tensile yield strength, S_{yt}, is exceeded by either of the principal stresses, σ_1 or σ_2. Hence, the maximum value of $+\sigma_1$ is S_{yt}, $+\sigma_2$ is S_{yt}, $-\sigma_1$ is S_{yc}, and $-\sigma_2$ is S_{yc}. Thus, for this problem:

$$\text{Safety Factor} = \min\left\{\frac{S_{yt}}{\sigma_1},\ \frac{S_{yc}}{\sigma_2}\right\}$$

$$= \min\left\{\frac{100}{35},\ \frac{-100}{-25}\right\}$$

$$= \min\{2.86,\ 4.00\}$$

$$= 2.86 \approx 2.9$$

b) Maximum Shear-Stress Theory

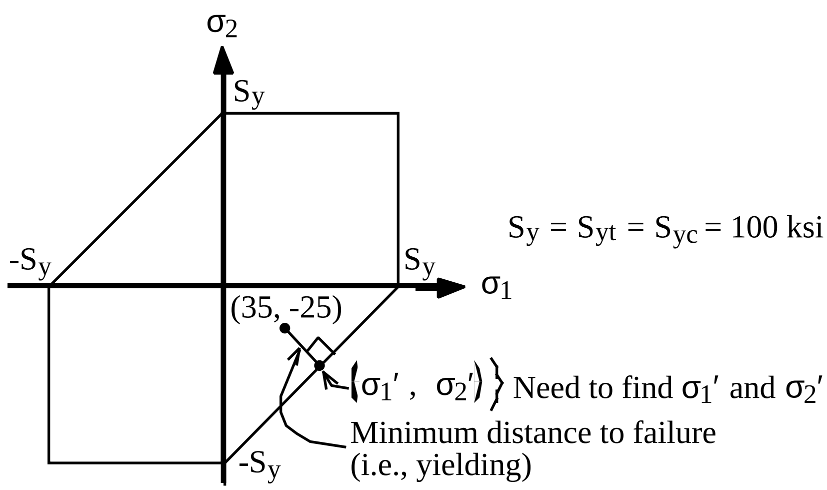


Figure 35. Maximum Shear-Stress Theory

For the failure locus in the fourth quadrant, $\sigma_2 = \sigma_1 - S_y$ □ slope = 1. For the minimum distance to failure line □ slope = -1.

NOTE: $(m)(m_{□}) = -1$ □ $m_{□} = \frac{-1}{m} = \frac{-1}{1} = -1$

Thus, $-1 = \frac{\Delta y}{\Delta x} = \frac{-25 - \sigma_2'}{35 - \sigma_1'}$

At the point of intersection between the failure locus line and the minimum distance to failure line (i.e., at σ_1', σ_2'), $\sigma_2' = \sigma_1' - S_y$. So:

$$-1 = \frac{-25 - (\sigma_1' - S_y)}{35 - \sigma_1'}$$

$$-35 + \sigma_1' = -25 - \sigma_1' + S_y$$

$$2\sigma_1' = 35 - 25 + S_y \qquad S_y = 100$$

$$\sigma_1' = \frac{35 - 25 + 100}{2} = 55$$

$\sigma_2' = \sigma_1' - S_y = 55 - 100 = -45$

$$\tau'_{max} = \frac{|\sigma'_{max} - \sigma'_{min}|}{2} = \frac{|55 - (-45)|}{2} = \frac{100}{2} = 50$$

For $\sigma_1' = 35$ ksi, $\sigma_2' = -25$ ksi □ $\tau_{max} = 30$

$$\text{Safety Factory} = \frac{\tau'_{max}}{\tau_{max}} = \frac{50}{30} = 1.67 \approx 1.7$$

Solutions for the distortion energy theory and Mohr theory are shown together with the foregoing solutions in Fig. 36. Satisfactory accuracy can be obtained by drawing the partial ellipse representing the distortion-energy theory as a careful freehand curve through the three known points. The curve representing the Mohr theory is drawn similar to the distortion energy curve except that it passes through a value of 60 on the shear diagonal instead of through 57.7.

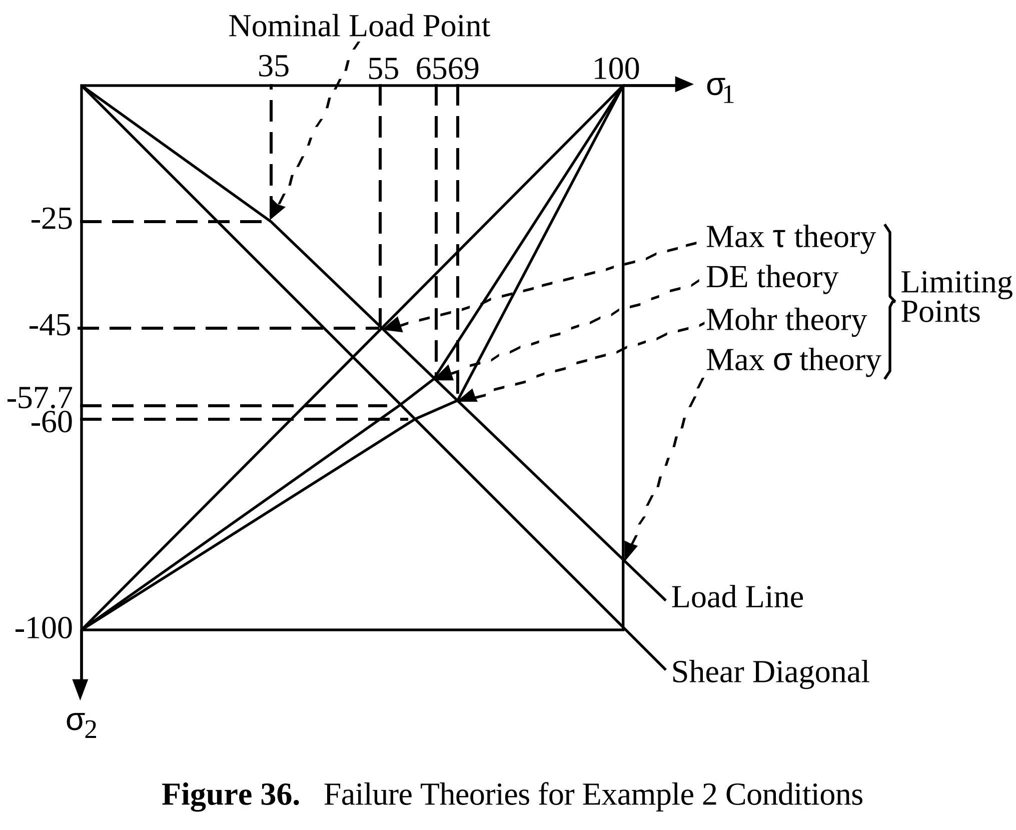


Figure 36. Failure Theories for Example 2 Conditions

NOTE: Only the lower right quadrant is needed, because of one tensile and one compressive stress.

Hence, the resulting safety factors for the failure theories examined are:

THEORY	SAFETY FACTOR	
Maximum Shear Stress	50/30 = 1.7	conservative
Distortion-Energy	65/35 = 1.9	
Mohr	69/35 = 2.0	preferable
Maximum Normal-Stress	100/35 = 2.9	no basis for believing this factor

6.11 Prediction of Failure of Thin-Walled Cylinders

- Maximum Normal-Stress Theory

  For a thin-walled cylinder

$$pR/t = \sigma_f$$

  where σ_f is as determined from a tension test.
  Rearranging yields

$$\frac{p}{\sigma_f} = \frac{1}{R/t} \quad (6.20)$$

- Maximum Shear-Stress Theory

  For a thin-walled pressure vessel

$$\tau_{max} = \frac{\sigma_{max} - \sigma_{min}}{2} = \frac{1}{2}\sigma_f .$$

  For a thin-walled pressure vessel, $\sigma_{max} = pR/t$, $\sigma_{min} = -p/2$, thus

$$pR/t + p/2 = \sigma_f, \text{ or}$$

$$\frac{p}{\sigma_f} = \frac{1}{R/t + 1/2} \quad (6.21)$$

- Maximum Distortion-Energy Theory

  According to this theory, failure occurs as expressed by Eq. 6.13 when the maximum octahedral shear stress, τ_{oct}, becomes equal to $\frac{\sqrt{2}}{3}\sigma_f$. Hence, applying Eq. 1.13b we can write

$$\tau_{oct} = \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} = \frac{\sqrt{2}}{3}\sigma_f \quad (6.22)$$

Now $\sigma_1 = pR/t$, $\sigma_2 = pR/2t$, $\sigma_3 = -p/2$

Hence

$$\frac{1}{3}\sqrt{\left(\frac{pR}{t} - \frac{pR}{2t}\right)^2 + \left(\frac{pR}{2t} + \frac{p}{2}\right)^2 + \left(-\frac{p}{2} - \frac{pR}{t}\right)^2} = \frac{\sqrt{2}}{3}\sigma_f \qquad (6.23)$$

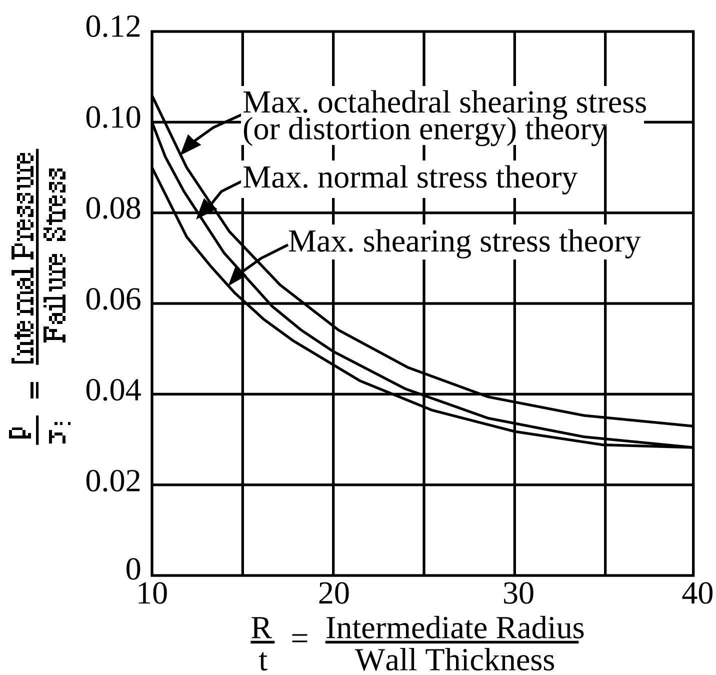
When the terms on the left side of Eq. 6.23 are squared and like terms collected, the following equation is obtained:

$$\frac{p}{\sigma_f} = \sqrt{\frac{2}{\frac{3}{2}\left(\frac{R}{t}\right)^2 + \frac{3}{2}\left(\frac{R}{t}\right) + \frac{1}{2}}} \qquad (6.24)$$

Figure 37 compares the foregoing results for p/σ_f for various thin-walled cylinders, and shows that the maximum shear-stress theory predicts the lowest value of pressure to cause failure and the maximum distortion-energy theory the largest. The maximum normal-stress theory predicts intermediate values.

NOTE: The maximum difference between the lowest and largest values of the failure pressure is about 15%.

NOTE: For cylinders of brittle materials, the maximum normal-stress theory (curve (4) of Fig. 33 with S_y replaced by S_u) may be applied. It is obvious that brittle cylinders must be designed with much larger safety factors than ductile cylinders, as failure involves complete fracture, whereas ductile cylinders have a substantial plastic reserve between initial (or small) yielding and fracture.



Maximum Internal Pressure in Closed-Ended Thin-Walled Circular Cylindrical Pressure Vessel Predicted by Various Theories of Failure

Figure 37. Failure of Closed-End Circular Cylinder Pressure Vessels

6.12 Examples for the Calculation of Safety Factors in Thin-Walled Cylinders

Example:

The most stressed volume element of a certain load-carrying member is located at the free surface. With the xy plane tangent to the free surface, the stress components were found to be

$$\sigma_x = 15{,}000 \text{ psi}$$
$$\sigma_y = -2{,}000 \text{ psi}$$
$$\tau_{xy} = 7{,}000 \text{ psi}$$

The load-stress relations for the member are linear so that the safety factor, N, can be applied either to loads or stresses. Determine N if the member is made of a ductile metal with a tensile yield stress of $\sigma_e = 44{,}000$ psi.

NOTE: In general $N \equiv \frac{\text{Yield Stress}}{\text{Load Stress}}$. Hence, we can write N(Load Stress) ≡ Yield Stress.

1) Maximum Distortion-Energy Theory – From Eq. 6.12 together with the definition of N we obtain:

$$N\,\tau_{oct} = \frac{\sqrt{2}}{3} S_y$$

For a uniaxial tensile test $\sigma_e = S_y$. Also, τ_{oct} is given by Eq. 1.14b for stresses in the general x, y, z directions. Hence, the above equation becomes:

$$\frac{N}{3}\left[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6\left(\tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2\right)^2 \right]^{1/2} = \frac{\sqrt{2}}{3}(44{,}000)$$

For the given stress components obtain:

$$\frac{N}{3}\sqrt{(17{,}000)^2 + (-2{,}000)^2 + (-15{,}000)^2 + 6\,(7{,}000)^2} = \frac{\sqrt{2}}{3}(44{,}000)$$

$$N\sqrt{812 \square 10^6} = \sqrt{2}\,(44{,}000)$$

$$N = 2.18$$

NOTE: This result can also be obtained directly from the result for a biaxial stress state, Eq. 6.19 as:

$$N\left(\sigma_x^2 + \sigma_y^2 - \sigma_x\sigma_y + 3\tau_{xy}^2\right)^{1/2} = \sigma_e$$

$$N\left[(15{,}000)^2 + (-2{,}000)^2 + \left(30 \square 10^6\right) + 3\,(7{,}000)^2 \right]^{1/2} = 44{,}000$$

$$N\sqrt{406 \square 10^6} = 44{,}000$$

$$N = 2.18$$

2) Maximum Shear-Stress Theory – In order to use this theory it is necessary to determine the three principal stresses. From Mohr's circle:

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= \frac{1}{2}(13{,}000) + \sqrt{\frac{1}{4}(17{,}000)^2 + (7{,}000)^2}$$

$$= 6{,}500 + 11{,}010$$

$$= 17{,}510 \text{ psi}$$

$$\sigma_2 = 6{,}500 - 11{,}010$$

$$= -4{,}510 \text{ psi}$$

$$\sigma_3 = 0$$

At failure the three principal stresses are:

$$\sigma_1 = 17{,}510 \text{ N psi}$$

$$\sigma_2 = -4{,}510 \text{ N psi}$$

$$\sigma_3 = 0$$

Now, from Eq. 6.5, which expresses the failure criterion for the maximum shear stress theory:

$$17{,}520 \text{ N} + 4{,}510 \text{ N} = 44{,}000$$

$$\text{N} \approx 2$$

If the maximum distortion-energy theory is used, the loads can be increased by a factor of 2.18 before failure by general yielding. Assuming this is correct, the maximum shear-stress theory of failure is conservative since it predicts smaller failure loads.

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