22.51 Quantum Theory of Radiation Interactions

Final Exam

December 14, 2010 Name: . . . .

Problem 1: Electric Field Evolution *20 points*

Consider a single mode electromagnetic field in a volume $V = L^3$. Calculate the evolution of the expectation value of the electric field $E = \sqrt{\frac{2\pi\hbar\omega}{L^3}}(a + a^\dagger)$ in the following cases:

a) *The state of the e.m. field is a superposition of two coherent states:*

$$\psi(0) = [\cos(\vartheta)|\alpha\rangle + \sin(\vartheta)e^{i\varphi}|\beta\rangle]/\mathcal{N}$$

where $\mathcal{N}$ is a coefficient to normalize the state.

Solution:

In the Heisenberg picture we can calculate the evolution of the creation and annihilation operators: $a(t) = a(0)e^{i\omega t}$. Thus we obtain:

$$\langle E(t)\rangle = \sqrt{\frac{2\pi\hbar\omega}{L^3}}\left\langle ae^{i\omega t} + a^\dagger e^{-i\omega t}\right\rangle$$

$$\langle E(t)\rangle = \sqrt{\frac{2\pi\hbar\omega}{L^3}}\left[\cos(\vartheta)^2 Re[\alpha e^{i\omega t}] + \sin(\vartheta)^2 Re[\beta e^{i\omega t}] + \sin(2\vartheta) Re\left\{e^{i\varphi}\left(\alpha\langle\beta|\alpha\rangle e^{i\omega t} + \beta^*\langle\alpha|\beta\rangle e^{-i\omega t}\right)\right\}\right]$$

where $\langle\alpha|\beta\rangle = e^{-(|\alpha|^2+|\beta|^2)/2+\alpha^*\beta}$.

b) *The state of the e.m. field is a mixture of the two coherent states above:*

$$\rho(0) = \cos^2(\vartheta)|\alpha\rangle\langle\alpha| + \sin^2(\vartheta)|\beta\rangle\langle\beta|$$

Solution:

Still in the Heisenberg picture we can calculate the expectation value as:

$$\langle E\rangle = \text{Tr}\left\{\rho(0)E(t)\right\} = \sqrt{\frac{2\pi\hbar\omega}{L^3}}\text{Tr}\left\{\rho(0)(ae^{i\omega t} + a^\dagger e^{-i\omega t})\right\}$$

We note that $\text{Tr}\left\{a|\alpha\rangle\langle\alpha|\right\} = \langle\alpha|a|\alpha\rangle = \alpha$ and find:

$$\langle E\rangle = \text{Tr}\left\{\rho(0)E(t)\right\} = \sqrt{\frac{2\pi\hbar\omega}{L^3}}\left[\cos(\vartheta)^2\left(\alpha e^{i\omega t} + \alpha^* e^{-i\omega t}\right) + \sin(\vartheta)^2\left(\beta e^{i\omega t} + \beta^* e^{-i\omega t}\right)\right]$$

$$= \sqrt{\frac{2\pi\hbar\omega}{L^3}}\left[\cos(\vartheta)^2 Re[\alpha e^{i\omega t}] + \sin(\vartheta)^2 Re[\beta e^{i\omega t}]\right]$$

c) *What is the average photon number in the two cases?*

Solution:

We want to calculate $\left\langle a^\dagger a\right\rangle$ in the two cases.

In the first case, we find:

$$\langle n\rangle = \cos(\vartheta)^2|\alpha|^2 + \sin(\vartheta)^2|\beta|^2 + \sin(2\vartheta) Re\left\{e^{i\varphi}\alpha\beta^*\langle\beta|\alpha\rangle\right\}$$

while in the second case, the last term is zero:

$$\langle n\rangle = \cos(\vartheta)^2|\alpha|^2 + \sin(\vartheta)^2|\beta|^2$$

d) *Assuming for simplicity that $\alpha, \beta \in \mathbb{R}$ (are real), in what limit the two results found in a) and b) (and the two results in c) become equivalent?*

Solution:

If the coherent states where orthogonal, the coherent superposition and incoherent mixture would have given the same expectation values. Their overlap is $\langle\alpha|\beta\rangle = e^{-(|\alpha|^2+|\beta|^2)/2+\alpha^*\beta}$. For $\alpha, \beta \in \mathbb{R}$ we have $\langle\alpha|\beta\rangle = e^{-(\alpha^2+\beta^2-2\alpha\beta)/2} = e^{-(\alpha-\beta)^2}$ which goes to zero if $|\alpha - \beta| \gg 1$.

Problem 2: Atom observed via a quantum meter — *35 points*

Consider the experiment performed by Brune et al. (PRL 77(24) 4887, 1996). A Rydberg atom is prepared in an equal superposition of two states (its ground $|g\rangle$ and excited state $|e\rangle$), which are separated by an energy $\ \omega$.

This state is achieved e.g., by applying the operator: $U_H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ to the atom's ground state.

The atom interacts with an e.m. field, which is initially in a coherent state $|\alpha\rangle$ with average photon number $\langle n\rangle = \alpha^2$ (with α real). The e.m. field is inside a cavity and thus restricted to a single mode of frequency ν. The interaction can drive a transition between the two atom levels at a rate λ, exciting the atom from the ground to the excited state, while annihilating a photon; and creating a photon, while lowering the atom from the excited to the ground state.

a) *Write the Hamiltonian describing these two systems ($\mathcal{H}_0$) and their interaction (V).*

Solution:

$$\mathcal{H} = \ \omega\frac{\sigma_z}{2} + \ \nu(a^\dagger a + \frac{1}{2}) + \ \lambda(a\sigma^+ + a^\dagger\sigma^-)$$

or

$$\mathcal{H} = \ \omega|e\rangle\langle e| + \ \nu(a^\dagger a + \frac{1}{2}) + \ \lambda(a|e\rangle\langle g| + a^\dagger|g\rangle\langle e|)$$

b) *We now take the limit where $\lambda \ll \omega, \nu$. Thus the interaction can be considered as a perturbation. Further, we have $\lambda \ll \Delta = \omega - \nu$, i.e. the system is off-resonance. Then we can simplify the Hamiltonian as:*

$$\mathcal{H} \approx \tilde{\mathcal{H}} = \mathcal{H}_0 + \sum_n (\delta E_{g,n}|g,n\rangle\langle g,n| + \delta E_{e,n}|e,n\rangle\langle e,n|)$$

where $\delta E_{g/e,n}$ are the energy shifts due to the interaction, to the first non-zero order in time-independent perturbation theory. Write an explicit expression for $\tilde{\mathcal{H}}$.

Solution:

The zeroth order correction is zero, so we need to calculate the second order correction, which gives

$$\delta E^{(2)}_{g,n} = \frac{|\langle g,n|V|e,n-1\rangle|^2}{E_{g,n} - E_{e,n-1}} = -\frac{\lambda^2 n}{\Delta} \quad \rightarrow \quad E_{g,n} \approx \ \nu(n+\frac{1}{2}) - \frac{\lambda^2 n}{\Delta}$$

and

$$\delta E^{(2)}_{e,n} = \frac{|\langle e,n|V|g,n+1\rangle|^2}{E_{e,n} - E_{g,n+1}} = \frac{\lambda^2(n+1)}{\Delta} \quad \rightarrow \quad E_{e,n} \approx \ \omega + \ \nu(n+\frac{1}{2}) + \frac{\lambda^2(n+1)}{\Delta}$$

c) *What is the evolution of the initial state described above? Use the Hamiltonian found above to prove that the evolved state (in the interaction picture defined by $\mathcal{H}_0$) is given by $\frac{1}{\sqrt{2}}(|g, \alpha e^{-i\varphi(t)}\rangle + e^{i\varphi(t)}|e, \alpha e^{i\varphi(t)}\rangle$.*

Solution:

We write the initial state, $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|e\rangle + |g\rangle)|\alpha\rangle$ in terms of the Hamiltonian eigenstates:

$$|\psi\rangle = \frac{e^{-|\alpha|^2/2}}{\sqrt{2}} \sum_n \frac{\alpha^n}{\sqrt{n!}}(|e,n\rangle + |g,n\rangle)$$

The evolved state is then:

$$|\psi(t)\rangle = \frac{e^{-|\alpha|^2/2}}{\sqrt{2}} \sum_n \frac{\alpha^n}{\sqrt{n!}} \left(e^{-i(n+1)\lambda^2 t/\Delta}|e,n\rangle + e^{in\lambda^2 t/\Delta}|g,n\rangle\right)$$

$$|\psi(t)\rangle = \frac{e^{-|\alpha|^2/2}}{\sqrt{2}} \sum_n \frac{(\alpha e^{-i\lambda^2 t/\Delta})^n}{\sqrt{n!}} e^{-i\lambda^2 t/\Delta}|e,n\rangle + \frac{(\alpha e^{i\lambda^2 t/\Delta})^n}{\sqrt{n!}}|g,n\rangle)$$

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}(|g,\alpha e^{i\lambda^2 t/\Delta}\rangle + e^{-i\lambda^2 t/\Delta}|e,\alpha e^{-i\lambda^2 t/\Delta}\rangle)$$

d) *The atom leaves the cavity after a time T, and it is then rotated back by the propagator U_H. What is the probability $P_e(T)$ of finding the atom in the excited state?*
What does this probability becomes in the limit $\langle n\rangle \to \infty$? What about the limit $\langle n\rangle \to 0$?

Solution:

We set $\beta = \alpha e^{i\lambda^2 T/\Delta}$. The state becomes:

$$|\psi(T)\rangle = \frac{1}{2}\left[|g,\beta\rangle + |e,\beta\rangle + e^{i\varphi}|g,\beta^*\rangle - e^{i\varphi}|e,\beta^*\rangle\right]$$

$$|\psi(T)\rangle = \frac{1}{2}\left[|g\rangle\left(|\beta\rangle + e^{i\varphi}|\beta^*\rangle\right) + |e\rangle\left(|\beta\rangle - e^{i\varphi}|\beta^*\rangle\right)\right]$$

The probability of being in the excited state is

$$P_e(T) = \frac{1}{4}\mathrm{Tr}\left\{\left(|\beta\rangle - e^{i\varphi}|\beta^*\rangle\right)\left(\langle\beta^*| - e^{-i\varphi}\langle\beta|\right)\right\} = \frac{1}{4}(\langle\beta|\beta\rangle + \langle\beta^*|\beta^*\rangle - e^{-i\varphi}\langle\beta|\beta^*\rangle - e^{i\varphi}\langle\beta^*|\beta\rangle) = \frac{1}{2}(1 - Re[e^{i\varphi}\langle\beta^*|\beta\rangle])$$

From the value of $\langle\beta^*|\beta\rangle = e^{-\alpha^2(1-e^{-2i\varphi})}$ we have $Re[e^{i\varphi}\langle\beta^*|\beta\rangle] = e^{-\alpha^2\sin(\varphi/2)^2/2}\cos(\varphi + \alpha^2\sin\varphi)$.
In the limit $\langle n\rangle = \alpha^2 \to 0$ the probability is

$$P_e = \frac{1}{2}(1 - \cos\varphi) = \sin^2\left(\frac{\lambda^2 T}{2\Delta}\right)$$

thus the atom oscillates between its ground and excited state as if performing Rabi oscillations with Rabi frequency $\Omega = \lambda^2/\Delta$.
In the opposite limit $\langle n\rangle = \alpha^2 \to \infty$ the exponential term goes to zero provided that $\alpha^2\sin(\varphi/2)^2 \gg 1$. When the distance between the two states of the cavity e.m. field becomes large enough we have $P_e = P_g = \frac{1}{2}$: the reduced state of the atom (neglecting the e.m. field) decays to an incoherent (classical) mixture of the two levels.

Problem 3: Transition Rate *15 points*

Consider the same system as in the previous problem: a two-level atom (with energy separation ω) interacting with a single mode e.m. field of energy ν by an interaction of strength λ. Now we consider the case where the atom is initially in the ground state, while the field is still in a coherent state with $\alpha = \sqrt{n}$. At time $t = 0$ we turn on the interaction between the atom and the field.

a) *To first order approximation, what is the transition rate to a state $|e,\beta\rangle$, with $\beta = \alpha e^{i\psi}$?*

Solution:

Since the interaction is time-independent, we can use Fermi's Golden rule. The transition rate is given by:

$$W = \frac{2\pi}{\hbar}|V_{if}|^2\delta(\omega_{fi})$$

For the system at hand, since the initial and final states of the field are not eigenstates of the Hamiltonian, we have to find the correct ω_{fi} from the perturbation V in the interaction picture. We find $\tilde{V} = \hbar\lambda(a\sigma^+ e^{-i\Delta t} + a^\dagger \sigma^- e^{i\Delta t})$, thus $\omega_{fi} = \omega - \nu = \Delta$, since the transition from ground to excited state will involve also the exchange of a photon of energy ν. The matrix element is given by:

$$V_{if} = \langle e, \beta | \hbar\lambda(a\sigma^+ + a^\dagger\sigma^-)|g, \alpha\rangle = \langle\beta|\alpha\rangle\hbar\lambda\alpha$$

With $\beta = \alpha e^{i\psi}$ we have $|\langle\beta|\alpha\rangle|^2 = e^{-4\alpha^2 \sin(\psi/2)^2}$. Thus the rate is:

$$W = 2\pi\hbar\lambda^2 \langle n\rangle e^{-4\langle n\rangle \sin(\psi/2)^2}\delta(\Delta)$$

b) *Compare this result to what you found in problem 2. What would be the transition rate in problem 3.a if $\Delta \gg \lambda$? What would have been the probability $P_e(t)$ of the atom being in the excited state (problem 2.d) if the initial state were $|g, \alpha\rangle$ as in problem 3?*

Solution:

If $\Delta \gg \lambda$ or more generally $\Delta \not\approx 0$ the transition rate becomes zero. This is consistent with what found in the previous problem. There, we saw that for $\Delta \gg \lambda$ the perturbation only acts as a phase shift for the atom. Thus if the initial state is $|g, \alpha\rangle$ the probability of a transition to the excited state would be zero.

Problem 4: Resonant Scattering *30 points*

Consider light scattering from an atom. The system of interest is described by an atom (with eigenstates $|m_k\rangle$ and energies $\mathcal{E}_k$) and the e.m. radiation field.
For convenience the system is enclosed in a cavity of volume $V = L^3$. The interaction between the radiation field and the atom is described by the hamiltonian $\mathcal{V} = -\vec{d} \cdot \vec{E}$ in the dipole approximation, where

$$\vec{E} = \sum_{h,\xi} \sqrt{\frac{2\pi \ \ \omega_h}{V}} \left(a_{h\xi} e^{i\vec{h}\cdot\vec{R}} + a^\dagger_{h\xi} e^{-i\vec{h}\cdot\vec{R}}\right) \vec{\epsilon}_{h\xi}$$

with R the position of the center of mass of the atom.
You can use the following steps to calculate the scattering cross section $d\sigma = \frac{W_{fi}}{\Phi_{inc}}$, with $W_{fi} = \frac{2\pi}{\hbar}|\langle f|T|i\rangle|^2 \rho(E_f)$, where T is the transition matrix and $\rho(E_f)$ the final density of states.

a) *What is the flux of incoming photons and the density of states of the outgoing photons?*

Solution:

$$\Phi_{inc} = c/L^3$$

and

$$\rho(E_f) = \left(\frac{L}{2\pi}\right)^3 \frac{\omega_k^2}{\hbar c^3} d\Omega$$

b) *What are the possible intermediate (virtual) states that we need to consider in this scattering process?*

Solution:

The initial state is $|m_i, 1_{k,l\lambda}, 0_{k',\lambda'}\rangle$ and final state $|m_i, 0_{k,l\lambda}, 1_{k',\lambda'}\rangle$. Intermediate states are such that there is only 1-photon transition, either $|m_l, 0_{k,l\lambda}, 0_{k',\lambda'}\rangle$ or $|m_l, 1_{k,l\lambda}, 1_{k',\lambda'}\rangle$. Thus:

$$\langle f|T|i\rangle = \sum_l \frac{\langle m_f, 0_{k,l\lambda}, 1_{k',\lambda'}|\mathcal{V}|m_l, 0_{k,l\lambda}, 0_{k',\lambda'}\rangle\langle m_l, 0_{k,l\lambda}, 0_{k',\lambda'}|\mathcal{V}|m_i, 1_{k,l\lambda}, 0_{k',\lambda'}\rangle}{(\mathcal{E}_i + \hbar\omega_k) - \mathcal{E}_l}$$

$$+ \sum_l \frac{\langle m_f, 0_{k,l\lambda}, 1_{k',\lambda'}|\mathcal{V}|m_l, 1_{k,l\lambda}, 1_{k',\lambda'}\rangle\langle m_l, 1_{k,l\lambda}, 1_{k',\lambda'}|\mathcal{V}|m_i, 1_{k,l\lambda}, 0_{k',\lambda'}\rangle}{(\mathcal{E}_i + \hbar\omega_k) - (\mathcal{E}_l + \hbar\omega_k + \hbar\omega'_k)}$$

Using the explicit expression for $\mathcal{V}$, we have:

$$\langle f|T|i\rangle = \frac{2\pi}{V}\sqrt{\omega_k\omega_k'}e^{i(k-k)\cdot R)}\sum_l \frac{(\vec{\epsilon}_k' \cdot \vec{d}_{fl})(\vec{\epsilon}_k \cdot \vec{d}_{li})}{\mathcal{E}_i - \mathcal{E}_l + \hbar\omega_k} + \frac{(\vec{\epsilon}_k \cdot \vec{d}_{fl})(\vec{\epsilon}_k' \cdot \vec{d}_{li})}{\mathcal{E}_i - \mathcal{E}_l - \hbar\omega_k'}$$

d) *Find an expression for the differential cross section $\frac{d\sigma}{d\Omega}$ (where $d\Omega$ is the solid angle into which the photon is scattered)*

Solution:

From $d\sigma = \frac{W_{fi}}{\Phi_{inc}} = \frac{2\pi}{\hbar}\frac{|\langle f|T|i\rangle|^2}{\Phi_{inc}}\rho(E_f)$ we find:

$$\frac{d\sigma}{d\Omega} = \frac{2\pi}{\hbar}\frac{L^3}{c}\left(\frac{L}{2\pi}\right)^3\frac{\omega_k^2}{\hbar c^3}\frac{4\pi^2 \ \ 2}{L^6}(\omega_k\omega_{k'})\left|\frac{(\vec{\epsilon}_{k'} \cdot \vec{d}_{fl})(\vec{\epsilon}_k \cdot \vec{d}_{li})}{\mathcal{E}_i - \mathcal{E}_l + \hbar\omega_k} + \frac{(\vec{\epsilon}_k \cdot \vec{d}_{fl})(\vec{\epsilon}_{k'} \cdot \vec{d}_{li})}{\mathcal{E}_i - \mathcal{E}_l - \hbar\omega_{k'}}\right|^2$$

simplifying the expression:

$$\frac{d\sigma}{d\Omega} = \frac{\omega_k^3\omega_{k'}}{c^4}\left|\frac{(\vec{\epsilon}_{k'} \cdot \vec{d}_{fl})(\vec{\epsilon}_k \cdot \vec{d}_{li})}{\mathcal{E}_i - \mathcal{E}_l + \hbar\omega_k} + \frac{(\vec{\epsilon}_k \cdot \vec{d}_{fl})(\vec{\epsilon}_{k'} \cdot \vec{d}_{li})}{\mathcal{E}_i - \mathcal{E}_l - \hbar\omega_{k'}}\right|^2$$

e) *We now consider resonant scattering. This occurs when the incoming photon energy is almost equal to the transition energy to one intermediate level: $\omega \approx \mathcal{E}_l - \mathcal{E}_i$ (for the virtual state l with energy $\mathcal{E}_l$).*
Write an expression for the cross section assuming that only the dominant term is important.

Solution:

$$\frac{d\sigma}{d\Omega} = kk'^3\left|\frac{(d_{fh}\cdot\epsilon_{k'})(d_{hi}\cdot\epsilon_k)}{\epsilon_h - \epsilon_i - \hbar\omega_k}\right|^2_{\hbar\omega_k\approx\epsilon_h-\epsilon_i}$$

f) *A more realistic expression is obtained if one assumes a finite linewidth of the atomic level, so that $\mathcal{E}_l - \mathcal{E}_i$ is replaced by $\mathcal{E}_l - \mathcal{E}_i - i\ \ \Gamma/2$. What is the resonant scattering cross section as a function of $\Delta = (\mathcal{E}_l - \mathcal{E}_i) - \hbar\omega_k$ and Γ?*

Solution:

$$\frac{d\sigma}{d\Omega} = kk'^3\left[\frac{|(d_{fh}\cdot\epsilon_{k'})(d_{hi}\cdot\epsilon_k)|^2}{\Delta^2 + \hbar^2\Gamma^2/4}\right]_{\Delta\approx 0}$$