22.51 Quantum Theory of Radiation Interactions

Mid-Term Exam

October 27, 2010 Solution

Problem 1: Electron Spin: Magnetization *20 points*

Consider an isolated electron with spin-$\frac{1}{2}$, placed in a large magnetic f eld $\vec{B} = B_z\vec{z}$ at zero temperature. The spin is in the state

$$|\psi\rangle = c_0|\uparrow\rangle + c_1|\downarrow\rangle$$

where $|\uparrow\rangle \equiv |S_z = +\frac{\hbar}{2}\rangle$ ($|\downarrow\rangle \equiv |S_z = -\frac{\hbar}{2}\rangle$) represents the spin state aligned (anti-aligned) with the vertical z-axis.

We now assume that we can measure the magnetic dipole $\vec{\mu} = \gamma\vec{S} = \gamma\hbar\frac{\vec{\sigma}}{2}$ of this single spin (with γ the gyromagnetic ratio of the electronic spin):

a) *What is the probability of f nding an outcome $\mu_z > 0$? What is the spin state immediately after the measurement?*

Solution:

The eigenstates of μ_z are $|\uparrow\rangle$ and $|1\rangle$ with eigenvalues $\pm\gamma\frac{\hbar}{2}$. Assuming $\gamma > 0$, the probability of f nding $\mu_z > 0$ in a measurement is simply $p(\mu_z > 0) = |c_0|^2$. The spin state is projected into the corresponding eigenstate $|\psi\rangle' = |\uparrow\rangle$. (For $\gamma < 0$ it would have been $p(\mu_z > 0) = |c_1|^2$ and $|\psi\rangle' = |\downarrow\rangle$)

b) *What is the average magnetization $\langle\mu_z\rangle$ in the z direction?*

Solution:

We need to calculate $\langle\psi|\mu_z|\psi\rangle = (c_0^*\langle\uparrow| + c_1^*\langle\downarrow|)\mu_z(c_0|\uparrow\rangle + c_1|\downarrow\rangle)$. Since μ_z is already diagonal in the basis $|\uparrow\rangle, |\downarrow\rangle$ this is simply $\langle\mu_z\rangle = \gamma\frac{\hbar}{2}(|c_0|^2 - |c_1|^2) = \gamma\hbar(|c_0|^2 - \frac{1}{2})$.

c) *If the magnetic f eld is aligned with the z-axis, $\vec{B} = B_z\vec{z}$ and the spin is in its ground state, what are c_0 and c_1? What is now $\langle\mu_z\rangle$?*
[Assume that the only interaction is the Zeeman interaction, $\mathcal{H}_Z = \hbar\gamma B_z\sigma_z/2$]

Solution:

The energy levels of the Zeeman interaction are simply the eigenvalues of $\mathcal{H}_Z$, $\pm\gamma B_z\frac{\hbar}{2}$ corresponding to the eigenvectors $|\uparrow\rangle$, $|\downarrow\rangle$. Assuming $B_z > 0$ and $\gamma > 0$ (or more generally $\gamma B_z > 0$), the eigenvector $|\downarrow\rangle$ has thus the lowest energy. Then the state is simply $|\psi\rangle = |\downarrow\rangle$, that is $|c_1| = 1$ (or $c_1 = 1$ up to an unimportant phase factor) and $c_0 = 0$. The average magnetization is then $\langle\mu_z\rangle = -\gamma\frac{\hbar}{2}$.
If $\gamma B_z < 0$ the ground state is instead $|\uparrow\rangle$ and $\langle\mu_z\rangle = \gamma\frac{\hbar}{2}$.

Now assume that we cannot achieve zero temperature, but only a temperature T (as provided e.g. by liquid Nitrogen), so that the spin is at thermal equilibrium in the f eld $\vec{B} = B_z\vec{z}$.

d) *What is the state of the spin?*

Solution:

At f nite temperature, we expect to have a mixed state as given by the canonical ensemble. Thus the state is given by $\rho = e^{-\beta\mathcal{H}_Z}/Z$, where $Z = \text{Tr}\left\{e^{-\beta\mathcal{H}_Z}\right\}$. For this simple Hamiltonian we can calculate the state explicitly:

$$\rho = \frac{e^{-\beta\gamma B_z\hbar/2}|\uparrow\rangle\langle\uparrow| + e^{\beta\gamma B_z\hbar/2}|\downarrow\rangle\langle\downarrow|}{e^{-\beta\gamma B_z\hbar/2} + e^{\beta\gamma B_z\hbar/2}}$$

Notice that the partition function $Z = 2\cosh(\beta\gamma B_z\hbar/2)$. We can also write the state as:

$$\rho = e^{-\beta\mathcal{H}_Z}/Z = [\cosh(\beta\gamma B_z\hbar/2)\mathbb{1} - \sinh(\beta\gamma B_z\hbar/2)\sigma_z]/Z = \frac{1}{2}[\mathbb{1} - \tanh(\beta\gamma B_z\hbar/2)\sigma_z]$$

e) *What is the average magnetization? How many spins would you need in order to achieve the same magnetization as in question c?*
[Hint: you can assume a temperature $T = 77K$ which corresponds to $\approx 1600GHz$ and a magnetic f eld $B \approx 6$ Tesla which gives a Zeeman energy $\hbar\gamma B \approx 160GHz$.]

Solution:

For a mixed state, the expectation value of an observable is $\langle O\rangle = \text{Tr}\{\rho O\}$. Thus the magnetization is $\langle\mu_z\rangle = \text{Tr}\{\rho\mu_z\}$ or:

$$\langle\mu_z\rangle = \text{Tr}\left\{\frac{e^{-\beta\gamma B_z\hbar/2}|\uparrow\rangle\langle\uparrow| + e^{\beta\gamma B_z\hbar/2}|\downarrow\rangle\langle\downarrow|}{e^{-\beta\gamma B_z\hbar/2} + e^{\beta\gamma B_z\hbar/2}}\frac{\gamma\hbar}{2}(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|)\right\}$$
$$= \frac{\gamma\hbar}{2}\frac{e^{-\beta\gamma B_z\hbar/2} - e^{\beta\gamma B_z\hbar/2}}{e^{-\beta\gamma B_z\hbar/2} + e^{\beta\gamma B_z\hbar/2}} = -\frac{\gamma\hbar}{2}\tanh(\beta\gamma B_z\hbar/2)$$

This result could have been found also by remembering that all average properties of a system can be found from the partition function, in particular the internal energy is $\langle E\rangle = -\frac{\partial\ln Z}{\partial\beta}$. The magnetization μ_z is related to the internal energy by the simple relation $\mathcal{H}_z = \mu_z B_z$. Then

$$\langle\mu_z\rangle = -\frac{1}{B_z}\frac{\partial\ln Z}{\partial\beta} = -\frac{1}{B_z Z}\frac{\partial Z}{\partial\beta} = -\frac{\gamma\hbar B_z}{2}\frac{2\sinh(\beta\gamma B_z\hbar/2)}{B_z 2\cosh(\beta\gamma B_z\hbar/2)} = -\frac{\gamma\hbar}{2}\tanh(\beta\gamma B_z\hbar/2)$$

At the temperature given, $T \approx 77$K, $1/\beta = k_b T \approx 1600$GHz. Then $\beta\gamma B_z\hbar/2 \approx \frac{1}{2}\frac{160\text{GHz}}{1600\text{GHz}} = \frac{1}{20}$. To f rst order $\tanh(0.05) \approx 0.05$. Then the magnetization of one spin at 77K is $\langle\mu_z\rangle = -\frac{\gamma\hbar}{2}\frac{1}{20}$ and we need about 20 spins to have the same amount of magnetization as one spin at zero temperature.

Problem 2: Electronic Spin: Dynamics *30 points*

We consider again an electronic spin-1/2 subjected to an external f eld B_z via the Zeeman interaction.

a) *We consider two cases, where the initial state is either what you found in Problem 1, question c or in Problem 1, question d. The initial state is rotated by 90° by the operator $U_y = e^{i\pi/4\sigma_y}$, to be aligned with the x-axis and it then evolves under the Zeeman interaction.*
By choosing the most eff cient "picture" (Schrödinger , Heisenberg or interaction picture), calculate $\langle\mu_x(t)\rangle$ for the two initial states. [Hint: i) What is $Ue^AU^\dagger$ for U unitary? ii) $e^ABe^{-A} = B + [A,B] + \dots \frac{1}{n!}[A,[A,[\dots,B]]]$.]

Solution:

Since we want to calculate the evolution of an observable (and for different initial states) it is more convenient to adopt the Heisenberg picture, in which the observables are time-dependent and the states are constant.
First we calculate $\mu_x(t)$ in the Heisenberg picture under the action of the the Zeeman Hamiltonian $\mathcal{H}_Z$.

$$\mu_x(t) = U^\dagger(t)\mu)x(0)U(t) = e^{i\mathcal{H}_z t/\hbar}\left(\frac{\hbar\gamma}{2}\sigma_x\right)e^{-i\mathcal{H}_z t/\hbar}$$

Now $\mathcal{H}_Z = \hbar\gamma B_z\sigma_z/2$ and we can call $\omega = \gamma B_z$. Also remember that

$$e^{i\omega t\sigma_z/2}\sigma_x e^{-i\omega t\sigma_z/2} = \sigma_x\cos(\omega t) - \sigma_y\sin(\omega t).$$

This could have also been calculated from the formula above, $e^ABe^{-A} = B + [A,B] + \dots \frac{1}{n!}[A,[A,[\dots,B]]]$, with $A = i\omega t/2\sigma_z$ and $B = \sigma_x$ and the usual commutation relationships of the Pauli matrices. Then

$$\mu_z(t) = \frac{\hbar\gamma}{2}(\sigma_x\cos(\omega t) - \sigma_y\sin(\omega t))$$

and $\langle \mu_x(t) \rangle = \frac{\hbar\gamma}{2} (\langle \sigma_x \rangle \cos(\omega t) + \sin(\omega t) \langle \sigma_y \rangle)$. We thus need to calculate $\langle \sigma_x \rangle$ and $\langle \sigma_y \rangle$ with respect to the two initial states (pure state at zero temperature and mixed state). The initial states after the rotation U_y are :

$$U_y|\downarrow\rangle = |+\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle), \qquad U_y \left(e^{-\beta \mathcal{H}_Z}/Z\right) U_y^\dagger = \frac{e^{\beta\hbar\gamma B_z \sigma_x/2}}{2\cosh(\beta\gamma B_z \hbar/2)}$$

which follows from $Ue^A U^\dagger = e^{UAU^\dagger}$ and $U_y \sigma_z U_y^\dagger = -\sigma_x$. Also notice that this state can be written as:

$$\rho_x = \frac{e^{\beta\hbar\gamma B_z \sigma_x/2}}{2\cosh(\beta\gamma B_z \hbar/2)} = \frac{1}{2}[\mathbb{1} + \tanh(\beta\gamma B_z \hbar/2)\sigma_x]$$

From these states we can calculate $\langle \sigma_x \rangle = \langle + |\sigma_x| + \rangle = 1$ and $\mathrm{Tr}\{\rho_x \sigma_x\} = \tanh(\beta\gamma B_z \hbar/2)$, while $\langle \sigma_y \rangle = 0$ for both states. Notice that these results could have been obtained even by simply noting that after rotating the states, the expectation value of σ_x should have been equal to the expectation value of σ_z (calculate in Problem 1) before the rotation.

Finally, we have $\langle \mu_x(t) \rangle = \frac{\gamma\hbar}{2} \cos(\gamma B_z t)$ for the ground state at zero temperature and
$\langle \mu_x(t) \rangle = \frac{\gamma\hbar}{2} \tanh(\beta\gamma B_z \hbar/2) \cos(\gamma B_z t)$ for the thermal state.

b) *We now want to describe the thermalization process that gives rise to the state you found in Problem 1.d. When we raise the temperature from $T = 0$ to $T \approx 77$K, the electronic spin will undergo an evolution to reach a new thermal state under the action of the Zeeman Hamiltonian and the coupling with a reservoir. We can represent this thermalization process by the Lindblad operators $L_1 = \sqrt{\alpha}\sigma_-$, $L_2 = \sqrt{1-\alpha}\sigma_+$where $\sigma_+ = |0\rangle\langle 1|$ ($\sigma_- = |1\rangle\langle 0|$) and $\alpha = \frac{1}{2}[1 - \tanh(\beta\hbar\gamma B_z/2)]$. Write out the Lindblad equation describing the total evolution of the system.*

Solution:

The Lindblad equation is the differential equation:

$$\dot{\rho}(t) = -i[\mathcal{H}, \rho(t)] + \sum_k \left[L_k \rho(t) L_k^\dagger - \frac{1}{2}(L_k^\dagger L_k \rho(t) + \rho(t) L_k^\dagger L_k) \right]$$

In the specif c case of the problem this becomes:

$$\dot{\rho} = -i\frac{1}{2}\hbar\omega[\sigma_z, \rho] + \alpha \left[\langle 1|\rho|1\rangle |0\rangle\langle 0| - \frac{1}{2}\left(|1\rangle\langle 1|\rho + \rho|1\rangle\langle 1|\right) \right] + (1-\alpha)\left[\langle 0|\rho|0\rangle |1\rangle\langle 1| - \frac{1}{2}\left(|0\rangle\langle 0|\rho + \rho|0\rangle\langle 0|\right) \right]$$

where we used the fact that $(\sigma_-)^\dagger \sigma_- = (|1\rangle\langle 0|)(|0\rangle\langle 1|) = |1\rangle\langle 1|$ and $\sigma_+^\dagger \sigma_+ = |0\rangle\langle 0|$.

c) *What is $\dot{\rho_{00}}$, where $\rho_{00} = \langle 0|\rho|0\rangle$? What is $\dot{\rho}_{11}$? Take the steady-state (SS) limit of the system of equations you found and calculate ρ_{00}^{SS} and ρ_{11}^{SS}. Compare the result with the state you found in Problem 1.d.*

Solution:

We need to project out $\langle 0|\dot{\rho}|0\rangle$:

$$\dot{\rho_{00}} = \langle 0| \left(-i\frac{1}{2}\hbar\omega[\sigma_z, \rho] \right) |0\rangle + \alpha\langle 0| \left[\rho_{11}|0\rangle\langle 0| - \frac{1}{2}\left(|1\rangle\langle 1|\rho + \rho|1\rangle\langle 1|\right) \right] |0\rangle$$

$$+(1-\alpha)\langle 0| \left[\rho_{00}|1\rangle\langle 1| - \frac{1}{2}\left(|0\rangle\langle 0|\rho + \rho|0\rangle\langle 0|\right) \right] |0\rangle$$

The f rst term is zero, since $\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|$ yielding $\langle 0|[\sigma_z, \rho]|0\rangle = \langle 0|(\sigma_z\rho - \rho\sigma_z)|0\rangle = \langle 0|\rho|0\rangle - \langle 0|\rho|0\rangle$. The second term yields $\alpha\rho_{11}$ and the third term $-(1-\alpha)\rho_{00}$. Thus we have:

$$\dot{\rho_{00}} = \alpha\rho_{11} - (1-\alpha)\rho_{00}$$

We can calculate $\dot{\rho}_{11}$ in a similar way, or remember that $\rho_{11} + \rho_{00} = 1$ so that $\dot{\rho}_{11} = -\dot{\rho}_{00}$:

$$\dot{\rho}_{11} = -\alpha\rho_{11} + (1-\alpha)\rho_{00}$$

We then have the system of equations:

$$\begin{cases} \dot{\rho_{00}} = \alpha - \rho_{00} \\ \dot{\rho_{11}} = (1-\alpha) - \rho_{11} \end{cases}$$

At the steady state, $\dot{\rho} = 0$ we obtain $\rho_{00} = \alpha = \frac{1}{2}[1 - \tanh(\beta\hbar\gamma B_z/2)] = \frac{e^{-\beta\hbar\gamma B_z/2}}{2\cosh(\beta\hbar\gamma B_z/2)}$ and $\rho_{11} = \frac{1}{2}[1 + \tanh(\beta\hbar\gamma B_z/2)] = \frac{e^{\beta\hbar\gamma B_z/2}}{2\cosh(\beta\hbar\gamma B_z/2)}$. These are the same values as for the thermal state in Problem 1. If we can prove that at the steady state $\rho_{10} = \rho_{01}^* = 0$ then we have recovered the thermal state.

Notice that in the Exam I had written $\alpha = \frac{1}{2}[1 - \tanh(\beta\hbar\gamma B_z/2)]$ which would have given a result off by a factor 2

d) *Now calculate as well $\dot{\rho_{01}}$ and the relative steady-state ρ_{01}^{SS} to prove that the steady-state is indeed the thermal state found in Problem 1.d.*

Solution:

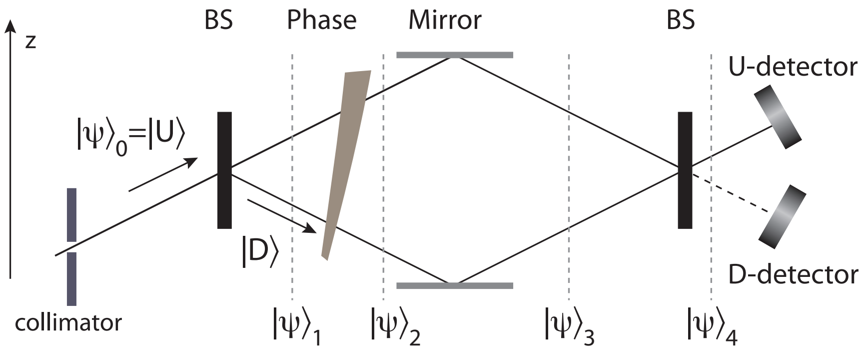
By taking the projection $\langle 0|\dot{\rho}|1\rangle$ we obtain :

$$\dot{\rho}_{01} = -i\hbar\omega\rho_{01} - \frac{1}{2}\alpha\rho_{01} - \frac{1}{2}(1-\alpha)\rho_{01} = -(i\hbar\omega + 1)\rho_{01}$$

Then, at the steady-state $\rho_{01} = 0$. Thus the equilibrium state (such that $\dot{\rho} = 0$) reached under the action of this Lindbladian process is indeed the thermal equilibrium.

Problem 3: Neutron interferometer *20 points*

Consider a Mach-Zehnder neutron interferometer such as the one seen in class and in the problem sets. The possible states of the neutrons are described by its momentum, either $|U\rangle$ for neutrons moving upward or $|D\rangle$ for neutrons moving downward.



The interferometer components act on the neutrons passing through with the following unitary operators:

$$U_{\text{BS}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad U_{\text{mirror}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad U_{\text{phase}} = \begin{pmatrix} e^{i\varphi_U} & 0 \\ 0 & e^{i\varphi_D} \end{pmatrix}$$

where BS stands for beam-splitter. U_{phase} represents a phase shift that neutrons acquire when passing through a material sample: the phase is φ_U for neutrons crossing the material while going upward and φ_D for neutrons going downward.

a) *We send in a beam of neutrons moving upward, $|\psi_0\rangle = |U\rangle$. What is the neutron state at each step, 1-4? (see f g.)*

Solution:

With simple matrix multiplications we have:

$$\psi_1 = \frac{1}{\sqrt{2}}(|U\rangle + |D\rangle)$$

$$\psi_2 = \frac{1}{\sqrt{2}}(e^{i\varphi_U}|U\rangle + e^{i\varphi_D}|D\rangle)$$

$$\psi_3 = \frac{1}{\sqrt{2}}(e^{i\varphi_U}|D\rangle + e^{i\varphi_D}|U\rangle)$$

$$\psi_4 = \frac{1}{2}[e^{i\varphi_U}(|U\rangle - |D\rangle) + e^{i\varphi_D}(|U\rangle + |D\rangle)]$$

b) *What is the measured contrast? [Hint: we define the contrast as the difference in the number of neutrons measured at the detector U and the detector D that measure neutrons moving upward and downward respectively.]*

Solution:

The contrast operator is defined as $C = |U\rangle\langle U| - |D\rangle\langle D|$. We can rewrite $|\psi\rangle_4$ as

$$|\psi\rangle_4 = \frac{1}{2}e^{i(\varphi_U+\varphi_D)/2}\left[e^{i(\varphi_U-\varphi_D)/2}(|U\rangle - |D\rangle) + e^{-i(\varphi_U-\varphi_D)/2}(|U\rangle + |D\rangle)\right]$$

$$= e^{i(\varphi_U+\varphi_D)/2}\left[\cos\left[(\varphi_U-\varphi_D)/2\right]|U\rangle - i\sin\left[(\varphi_U-\varphi_D)/2\right]|D\rangle\right]$$

From this expression, it is easy to find the contrast as

$$\langle C\rangle = \cos\left[(\varphi_U-\varphi_D)/2\right]^2 - \sin\left[(\varphi_U-\varphi_D)/2\right]^2 = \cos(\varphi_U-\varphi_D).$$

c) *We now change the sample inside the interferometer so that the neutron will acquire a phase $\varphi'_U = \varphi_U - \varphi_D$ when traveling upward and $\varphi'_D = 0$ otherwise. How do your answers to questions a-b change?*

Solution:

With this new sample, the state $|\psi\rangle_4$ is

$$|\psi\rangle_4 = \frac{1}{2}[e^{i\varphi'_U}(|U\rangle - |D\rangle) + (|U\rangle + |D\rangle)] = \frac{1}{2}e^{i\varphi'_U/2}[e^{i\varphi'_U/2}(|U\rangle - |D\rangle) + e^{-i\varphi'_U/2}(|U\rangle + |D\rangle)]$$

Notice that this state is exactly the same as written above to calculate the contrast. Since a global phase $e^{i\varphi'_U/2}$ is unimportant when calculating expectation values, we find the same contrast: $\langle C\rangle = \cos(\varphi'_U) = \cos(\varphi_U - \varphi_D)$.

Problem 4: Faulty neutron interferometer *30 points*

Consider the same neutron interferometer as in Problem 3.c, but now assume that the mirrors are faulty: Instead of reflecting all the neutrons, they let pass some of them. We can describe the process as follow:

The mirrors are initially in their ground state, $|\psi\rangle_{\text{mirror}} = |0\rangle$. When one neutron traveling upward impacts the mirror, it is reflected with probability p (leaving the mirrors in the ground state) or it continues upward with probability $1-p$, leaving the mirrors in the state $|1\rangle$. When one neutron traveling downward impacts the mirror, it is reflected with probability p (leaving the mirrors in the ground state) or it continues downward with probability $1-p$, leaving the mirrors in the state $|2\rangle$.

a) *Describes formally this process giving the rules for the transitions $|U\rangle|0\rangle_{mirror} \to \dots$ and $|D\rangle|0\rangle_{mirror} \to \dots$.*

Solution:

The possible transitions described in this process are given by the propagator U_{nm} acting on both neutron and mirror:

$$|U\rangle|0\rangle_{\text{mirror}} \to U_{nm}|U\rangle|0\rangle_{\text{mirror}} = \sqrt{p}|D\rangle|0\rangle_{\text{mirror}} + \sqrt{1-p}|U\rangle|1\rangle_{\text{mirror}}$$

$$|D\rangle|0\rangle_{\text{mirror}} \to U_{nm}|D\rangle|0\rangle_{\text{mirror}} = \sqrt{p}|U\rangle|0\rangle_{\text{mirror}} + \sqrt{1-p}|D\rangle|2\rangle_{\text{mirror}}$$

b) *For which values of φ'_U is the mirror+neutron system entangled (at the step 3)?*

Solution:

Entanglement between the mirror and the neutron does not depend on φ'_U (which defines only a phase of the neutron state), but it can depend on p.

We calculated $|\psi\rangle_2 = \frac{1}{\sqrt{2}}(e^{i\varphi'_U}|U\rangle + |D\rangle)$. Considering now the mirror system as well, we have:

$$|\Psi\rangle_2 = \frac{1}{\sqrt{2}}(e^{i\varphi'_U}|U\rangle|0\rangle_{\text{mirror}} + |D\rangle|0\rangle_{\text{mirror}})$$

After the interaction with the mirror, we have:

$$|\Psi\rangle_3 = \frac{1}{\sqrt{2}}[e^{i\varphi'_U}(\sqrt{p}|D\rangle|0\rangle_{\text{mirror}} + \sqrt{1-p}|U\rangle|1\rangle_{\text{mirror}}) + |D\rangle(\sqrt{p}|U\rangle|0\rangle_{\text{mirror}} + \sqrt{1-p}|D\rangle|2\rangle_{\text{mirror}})]$$

$$= \frac{1}{\sqrt{2}}[\sqrt{p}(e^{i\varphi'_U}|D\rangle + |U\rangle)|0\rangle_{\text{mirror}} + \sqrt{1-p}(e^{i\varphi'_U}|U\rangle|1\rangle_{\text{mirror}} + |D\rangle|2\rangle_{\text{mirror}})]$$

This state cannot be written as $|\psi\rangle_{\text{neutron}} \otimes |\varphi\rangle_{\text{mirror}}$ thus it is entangled. To conf rm this, we can take the partial trace over the mirror:

$$\rho_3 = \text{Tr}_{\text{mirror}}\{|\Psi_3\rangle\langle\Psi_3|\} = p|\psi_3\rangle\langle\psi_3| + \frac{1}{2}(1-p)\mathbb{1}$$

(where $|\psi_3\rangle = (e^{i\varphi'_U}|D\rangle + |U\rangle)$ is the state found in Problem 3 for perfect mirrors.) This reduced state is a mixed state, as conf rmed by Tr $\{\rho_3^2\} = [p^2 + (1-p)^2/2 + p] = \frac{1}{2}(p^2+1) < 1$.

Notice that in all this calculation, the value of φ'_U is unimportant and the state is always entangled (unless $p = 1$ or $p = 0$).

c) *Write the Kraus operators that describe the faulty mirrors and the evolution $|\psi_2\rangle \to \rho_3$. What is ρ_3? What type of process is this Kraus sum describing for $p \to 0$?*

Solution:

The Kraus operators are $M_k = \langle k|U_{nm}|0\rangle$:

$$M_0 = \sqrt{p}\sigma_x, \qquad M_1 = \sqrt{1-p}|U\rangle\langle U|, \qquad M_2 = \sqrt{1-p}|D\rangle\langle D|$$

and $\rho_3 = \sum_{k=0}^{2} M_k \rho_2 M_k$, with $\rho_2 = |\psi\rangle\langle\psi|_2 =$. We obtain:

$$\rho_3 = p\sigma_x\rho_2\sigma_x + (1-p)(\langle 0|\rho_2|0\rangle\, |0\rangle\langle 0| + \langle 1|\rho_2|1\rangle\, |1\rangle\langle 1| = p|\psi_3\rangle\langle\psi_3| + \frac{1}{2}(1-p)\mathbb{1}$$

Notice that the diagonal terms of the density operator are swapped but not reduced in intensity, while the off-diagonal terms are reduced by an amount p. Thus, as the quality of the mirror decreases, $p \to 0$ and the off-diagonal terms go to zero; since the phase coherence of the state is lost, the process can be classif ed as a dephasing process. Notice that it is different then what seen in class, since it is combined with the σ_x operator, inverting the populations.

d) *What is the contrast obtained in this faulty interferometer?*

Solution:

The contrast is now given by $\langle C\rangle = \text{Tr}\{\rho_4 C\}$, where $\rho_4 = U_{BS}\rho_3 U_{BS}^\dagger$. By linearity,

$$\rho_4 = p|\psi_4\rangle\langle\psi_4| + \frac{1}{2}(1-p)\mathbb{1}$$

Then $\langle C\rangle = \text{Tr}\{Cp|\psi_4\rangle\langle\psi_4|\} = p\cos(\varphi'_U)$.

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