

22.51 Quantum Theory of Radiation Interactions

Mid-Term Exam

November 2, 2011

Solution

For most of this midterm exam we are interested in describing the quantum behavior of a particle –such as an atom or a ion– which is confined in a double well potential. The particle can be in either the left or right well. It has energy $E_R = E_0 - \Delta E$ in the right well and energy $E_L = E_0 + \Delta E$ in the left well. The barrier between the two wells is not very high, so the particle can tunnel between the two wells at a rate Γ . We neglect all degrees of freedom and characteristics of the particle, except for its position (either left or right). You can set for simplicity $E_0 = 0$ and $\hbar = 1$.



Figure 1: Left: Double well system. Right: Double well with $\Delta E = 0$ and photons going through the left well.

Problem 1: Particle in a double well

15 points

a) What is the Hamiltonian describing the tunneling, \mathcal{H}_T ? What is the total Hamiltonian – describing the tunneling and the energy of the particle in the two wells?

Solution:

The Hamiltonian describing the various energy/interaction terms should be hermitian operators that act on the basis states $|L\rangle, |R\rangle$ as desired:

$$\mathcal{H}_T = \Gamma(|R\rangle\langle L| + |L\rangle\langle R|) \quad \mathcal{H} = E_R |R\rangle\langle R| + E_L |L\rangle\langle L| + \mathcal{H}_T$$

or

$$\mathcal{H}_T = \Gamma\sigma_x, \quad \mathcal{H} = E_0\mathbb{1} + \Delta E\sigma_z + \Gamma\sigma_x$$

b) The particle is at thermal equilibrium. At high temperature T , what is the particle state, to first order in $1/T$?

Solution:

Since the state is at thermal equilibrium, it means we are considering a canonical ensemble. Then the state is in a mixed state given by

$$\rho = \frac{e^{-\beta\mathcal{H}}}{Z} \approx \frac{\mathbb{1}}{2} + \frac{\beta}{2}\mathcal{H},$$

where $\beta = 1/(k_b T)$. Note that $1/Z = 1/\text{Tr}\{e^{-\beta\mathcal{H}}\} \approx 1/N$ to first order, where N is the dimension of the system. Explicitly we have:

$$\rho = \frac{\mathbb{1}}{2} + \frac{1}{2}(\Delta E\sigma_z + \Gamma\sigma_x)$$

c) At zero temperature, $T = 0$, what is the probability of finding the particle in the right well? What conditions on ΔE and Γ should be satisfied for this probability to be $P_{eq,T=0}(R) \approx 1$?

Solution:

At zero temperature, the system will be in the ground state of the Hamiltonian. Thus we need to diagonalize $\mathcal{H} = \Gamma\sigma_x + \Delta E\sigma_z$. With the formula given in class, the ground state is

$$|\psi\rangle_- = \frac{1}{\sqrt{2}} \begin{bmatrix} -\sqrt{1 - \frac{\Delta E}{E}\sqrt{\Delta E^2 + \Gamma^2}} \\ \sqrt{1 + \frac{\Delta E}{\sqrt{\Delta E^2 + \Gamma^2}}} \end{bmatrix}$$

with energy $-\sqrt{\Delta E^2 + \Gamma^2}$. Thus, the probability of being in the right well is $P_{eq,T=0}(R) = \frac{1}{2}(1 + \frac{\Delta E}{\sqrt{\Delta E^2 + \Gamma^2}})$.

In order for this to be almost one, we need $\Delta E \ll \Gamma$ and $\Delta E > 0$ (if $\Delta E < 0$ the state is in the left well at zero temperature and for $\Delta E \ll \Gamma$).

Problem 2: Evolution

15 points

a) The particle is prepared in the right well at time $t = 0$. At time $t = 0^+$ the energy difference of the two wells is removed ($\Delta E = 0$) and the particle is left to evolve. What is the particle state? What is the probability of finding the particle in the left well at a time t ?

Solution:

The evolution is $|\psi(t)\rangle = U(t)|\psi(0)\rangle$. The propagator is simply $U = e^{-i\Gamma\sigma_x t}$ and with the usual formula $U = \cos(\Gamma t)\mathbb{1} - i\sin(\Gamma t)\sigma_x$ we find $|\psi(t)\rangle = \cos(\Gamma t)|R\rangle - i\sin(\Gamma t)|L\rangle$. Thus the probability is $P_L(t) = \sin(\Gamma t)^2$.

b) Now assume that the energy difference between the two wells is periodically varied in time, $\Delta E(t) = \Delta E \cos(\omega t)$, with $\omega \gg \Delta E$. What is the equation of motion for the particle in an appropriate “picture”?

Solution:

We can make a transformation to the interaction picture given by the Hamiltonian $\mathcal{H}_0 = \omega\sigma_x$. Note that it is possible to define an infinite number of “interaction pictures”, the one which is more appropriate depends on what we want to describe or on what would give the simplest (mathematical) expression. Here one goal could be to find a time-independent Hamiltonian, since we know that solving for time-dependent ones is very difficult. Hence, it is more appropriate to choose $\mathcal{H}_0 = \omega\sigma_x$ than e.g. $\mathcal{H}_0 = \Gamma\sigma_x$, which would not remove the time dependency, although this last one is still a correct interaction picture transformation. In the interaction picture defined by $\mathcal{H}_0 = \omega\sigma_x$, we have:

$$\mathcal{H}_I = (\Gamma - \omega)\sigma_x + \Delta E/2\sigma_z + \Delta E/2[\sigma_z \cos(2\omega t) - i\sigma_x \sin(\omega t)]$$

The evolution in the interaction picture is simply given by the usual Schrödinger equation:

$$\frac{d|\psi_I\rangle}{dt} = i\mathcal{H}_I|\psi_I\rangle$$

c) Extra credits: Using a well-known approximation, solve for the evolution of the particle (either for $\omega = \Gamma$ or in the general case).

Solution:

We can simplify the Hamiltonian above by neglecting the time-dependent part (rotating wave approximation):

$$\mathcal{H}_I = (\Gamma - \omega)\sigma_x + \Delta E/2\sigma_z$$

for $\omega = \Gamma$ this is further simplified to $\mathcal{H}_I = \Delta E/2\sigma_z$. Thus we have,

$$|\psi_I(t)\rangle = e^{-i\Delta E\sigma_z/2t}|\psi_I(0)\rangle$$

If $|\psi_I(0)\rangle = |\psi(0)\rangle = |R\rangle$ then the system evolves as if $\Delta E = 0$.

Off resonance, we need to first diagonalize the Hamiltonian to find the evolution. We would then retrieve Rabi’s formula:

$$|\psi_I(t)\rangle = \left[\cos\left(\frac{\Omega t}{2}\right) + i\frac{\Delta E}{\Omega} \sin\left(\frac{\Omega t}{2}\right) \right] |R\rangle - i\frac{\Gamma}{\Omega} \sin\left(\frac{\Omega t}{2}\right) |L\rangle$$

with $\Omega = \sqrt{\Delta E^2 + \Gamma^2}$ (see also section 5.4 of lecture notes).

Problem 3: Measurement

20 points

To measure if the particle is in the left well, we send in a photon. The photon, initially in the ground state $|0\rangle$, interacts with the particle (only if the particle is in the left well) and is promoted to the state $|1\rangle$. (If the photon had been in the state $|1\rangle$ the scattering would have demoted it to the state $|0\rangle$).

a) What is the unitary operator U describing the combined transformation of the particle and photon? Is measuring the state of the photon after this interaction a good way of measuring the particle's position?

Solution:

The unitary operator is the CNOT operation that we saw in class:

$$U = |R\rangle\langle R| \otimes \mathbb{1} + |L\rangle\langle L| \otimes \sigma_x$$

If we assume to be able to prepare the photon in the ground state $|0\rangle$, then the photon is mapped to $|1\rangle$ only if the particle is in the left well. So measuring the state of the photon is a good way of measuring the particle position. As we see in question (b) the two systems become entangled after this interaction and there is a perfect (anti)correlation between their states.

Note that other similar operators could be found, but it is at least needed to check that they are indeed unitary. For example $V = |L\rangle\langle L| \otimes \sigma_x$ is not unitary, since $V^\dagger = V$ but $V^\dagger V = |L\rangle\langle L| \otimes \mathbb{1} \neq \mathbb{1} \otimes \mathbb{1}$.

A possible interaction that gives rise to this operator is: $\mathcal{H}_{int} = |L\rangle\langle L| \otimes \sigma_x$. Then we have:

$$\begin{aligned} U'(t) &= e^{-i\mathcal{H}_{int}t} = \sum_{n=0}^{\infty} (-it |L\rangle\langle L| \otimes \sigma_x)^n / n! = \mathbb{1} \otimes \mathbb{1} + |L\rangle\langle L| \otimes \sum_{n=1}^{\infty} (-it\sigma_x)^n / n! \\ &= |R\rangle\langle R| \otimes \mathbb{1} + |L\rangle\langle L| \otimes \sum_{n=0}^{\infty} (-it\sigma_x)^n / n! = |R\rangle\langle R| \otimes \mathbb{1} + |L\rangle\langle L| \otimes e^{-i\sigma_x t} \end{aligned}$$

setting $t = \pi/2$ we obtain

$$U' = |R\rangle\langle R| \otimes \mathbb{1} - i |L\rangle\langle L| \otimes \sigma_x$$

(which is indeed unitary and acts as desired).

b) The particle evolves for a time $t^* = \pi/(4\Gamma)$ as in Problem 2:a. If the photon (initially in the ground state $|0\rangle$) interacts with the particle at the time t^* , is the combined state of the photon and particle entangled? (assume the photon-particle interaction of question (a) happens instantaneously at the time t^*).

Solution:

The state of the particle at time t^* is $|\psi\rangle = e^{-i\Gamma\sigma_x t^*} |L\rangle = (\cos(\pi/4)\mathbb{1} - i\sin(\pi/4)\sigma_x) |L\rangle = \frac{1}{\sqrt{2}}(|L\rangle - i|R\rangle)$. Considering the photon as well, we have:

$$|\psi(t^*)\rangle = \frac{1}{\sqrt{2}}(|L\rangle - i|R\rangle)|0\rangle = \frac{1}{\sqrt{2}}(|L0\rangle - i|R0\rangle)$$

Then, applying the unitary found above we have

$$U |\psi(t^*)\rangle = \frac{1}{\sqrt{2}}(|L1\rangle - i|R0\rangle)$$

which is locally equivalent to a Bell state and thus it is entangled.

c) Now assume that the photon detector we used in (a-b) is faulty: it absorbs the photon but does not record the state of the photon. What is the reduced state of the particle after the measurement? (here again, assume that the particle had evolved for a time t^* before the interaction with the photon).

Solution:

The reduced state is just the identity, $\rho = \frac{1}{2}(|R\rangle\langle R| + |L\rangle\langle L|) = \frac{\mathbb{1}}{2}$, since with probability $\frac{1}{2}$ the photon was measured in the $|0\rangle$ state (and the particle is projected to the $|R\rangle$ state) and with probability $\frac{1}{2}$ the photon was measured in the $|1\rangle$ state (and the particle is projected to the $|L\rangle$ state).

Problem 4: Dephasing

20 points

As our photon detector is faulty, we decide not to measure the particle. Unfortunately, we cannot completely block the photons from entering the left well: there is still a flux of photons with intensity I . Each photon, initially in the ground state $|0\rangle$ interacts with the particle (only if the particle is in the left well) and with a probability $p = I\delta t$ is scattered by the particle and is promoted to the state $|1\rangle$. As we cannot measure the state of the photons, we are interested in the evolution of the particle only.

a) Describe the particle's evolution (due to the interaction with the photon only, neglect the particle Hamiltonian) in terms of a Kraus operator sum.

Solution:

We have the possible evolutions:

$$|R\rangle|0\rangle \rightarrow |R\rangle|0\rangle$$

and

$$|L\rangle|0\rangle \rightarrow \sqrt{1-p}|L\rangle|0\rangle + \sqrt{p}|L\rangle|1\rangle$$

Thus the two Kraus operators are:

$$M_0 = \langle 0|U|0\rangle = \sqrt{1-p}|L\rangle\langle L| + |R\rangle\langle R|, \quad M_1 = \langle 1|U|0\rangle\sqrt{p}|L\rangle\langle L|$$

The evolution is the Kraus sum:

$$\rho_1 = M_0\rho_0M_0^\dagger + M_1\rho_0M_1^\dagger$$

we can also check that these are good Kraus operators, since

$$M_k^\dagger M_k = M_0^\dagger M_0 + M_1^\dagger M_1 = \mathbb{1}$$

b) If the initial state of the particle is given by $\rho_0 = \begin{pmatrix} \rho_{LL} & \rho_{LR} \\ \rho_{RL} & \rho_{RR} \end{pmatrix}$, what is the particle's state ρ_n after the interaction with n photons? (Assume that photons interact one at a time).

Solution:

Applying the Kraus sum once, we have:

$$\rho_1 = \begin{pmatrix} (1-p)\rho_{LL} & \sqrt{1-p}\rho_{LR} \\ \sqrt{1-p}\rho_{RL} & \rho_{RR} \end{pmatrix} + \begin{pmatrix} p\rho_{LL} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \rho_{LL} & \rho_{LR}\sqrt{1-p} \\ \rho_{RL}\sqrt{1-p} & \rho_{RR} \end{pmatrix}$$

Repeating the same reasoning n times we obtain:

$$\rho_n = \begin{pmatrix} \rho_{LL} & \rho_{LR}(1-p)^{n/2} \\ \rho_{RL}(1-p)^{n/2} & \rho_{RR} \end{pmatrix}$$

c) In the limit $\delta t \rightarrow 0$, write ρ_n as a function of the time $t = n\delta t$ (remember $p = I\delta t$).

Solution:

Using the limit $\lim_{\delta t \rightarrow 0} (1 + a\delta t)^{t/\delta t} = e^{at}$ we have

$$\rho(t) = \begin{pmatrix} \rho_{LL} & \rho_{LR}e^{-It/2} \\ \rho_{RL}e^{-It/2} & \rho_{RR} \end{pmatrix}$$

d) Assume ρ_0 is given by the state of the particle at the time t^* (see Problem 3:b). What is the state of the particle for $t \rightarrow \infty$? How does it compare to what you found in Problem 3:c? Comment.

Solution:

We can write the state $|\psi(t^*)\rangle$ as

$$\rho(t^*) = \frac{1}{2} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$$

and for $t \rightarrow \infty$ this becomes the maximally mixed state, $\rho_\infty = \frac{1}{2}\mathbb{1}$, as the information about the phase coherence is lost.

Problem 5: Irreversible Evolution

20 points

a) In the previous problem you found the evolution for the particle, $\rho(t)$. Is it possible to find a differential equation describing this evolution? Explain why.

Solution:

The system environment is Markovian, since the photon is lost and any information about its state is lost as well. Thus we expect that a Master equation in the Lindblad form well describes the system evolution.

We can check that indeed taking the derivative of $\rho(t)$ we find a first order differential equation:

$$\frac{d\rho(t)}{dt} = -\frac{I}{2} \begin{pmatrix} 0 & \rho_{LR}e^{-It/2} \\ \rho_{RL}e^{-It/2} & 0 \end{pmatrix}$$

b) A differential equation for the particle's evolution is given by the Lindblad equation, with Lindblad jump operator $L = \sqrt{I}/2\sigma_z$. Is this consistent with the result you found in the previous problem? (You could for example check by taking the derivative of $\rho(t)$).

Solution:

The Lindblad equation is

$$\frac{d\rho(t)}{dt} = L\rho(t)L^\dagger - \frac{1}{2}(L^\dagger L\rho(t) + \rho(t)L^\dagger L) = -\frac{I}{4}[\rho(t) - \sigma_z\rho(t)\sigma_z]$$

which indeed gives the differential equation written above.

c) Consider now the full evolution of the system, with the Lindblad operator found above and the tunneling Hamiltonian of Problem 1:a (i.e. set $\Delta E = 0$). What is the differential equation describing the evolution? What is the (long-time) steady-state solution to this equation?

Solution:

The differential equation now contains the Hamiltonian term (given by the usual Liouville equation) in addition to the Lindblad operator above:

$$\frac{d\rho(t)}{dt} = -i[\mathcal{H}, \rho(t)] + \sum_k \left[L_k\rho(t)L_k^\dagger - \frac{1}{2}(L_k^\dagger L_k\rho(t) + \rho(t)L_k^\dagger L_k) \right]$$

and explicitly:

$$\frac{d\rho(t)}{dt} = -i\Gamma \begin{pmatrix} \rho_{LR} - \rho_{RL} & \rho_{RR} - \rho_{LL} \\ \rho_{RL} - \rho_{LR} & \rho_{LL} - \rho_{RR} \end{pmatrix} - \frac{I}{2} \begin{pmatrix} 0 & \rho_{LR}e^{-It/2} \\ \rho_{RL}e^{-It/2} & 0 \end{pmatrix}$$

The steady state condition corresponds to setting $\frac{d\rho(t)}{dt} = 0$. Solving the equation, this imposes $\rho_{RL} = \rho_{LR} = 0$ and $\rho_{RR} = \rho_{LL} = \frac{1}{2}$ (using the fact that $\text{Tr}\{\rho\} = 1$). Then the steady state is just $\rho_{SS} = \frac{\mathbb{1}}{2}$.

Problem 6: Short questions

10 points

a) Is the state $\frac{1}{4} \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$ pure? Describe two possible physical situation from where this state could have arisen.

Solution:

The purity of the state is $\text{Tr}\{\rho^2\} = \frac{3}{4} < 1$, thus the state is mixed. Indeed, we can write it as $\frac{1}{2}\mathbb{1} + \frac{1}{4}(\sigma_x - \sigma_z)$. Then we can verify that the Bloch vector is $\vec{n} = \frac{1}{2}[1, 0, -1]$ and $|\vec{n}| = \frac{1}{\sqrt{2}} < 1$.

We can think of this state as being the thermal state you found in Problem 1:b, for $\Gamma = \Delta E$.

Or if we write it as $\rho = \frac{1}{4}(\mathbb{1} + \sigma_x) + \frac{1}{4}(\mathbb{1} - \sigma_z)$ we could think it as a state prepared with 50% probability in the positive x direction and with 50% probability in the negative z direction.

Another alternative is to consider this state as arising from an interaction with the environment.

b) Consider two density operators, ρ_1 and ρ_2 describing e.g. an n -level system. What are the general conditions for there to be a unitary operations that connects them? If ρ_1 represents a pure state, what is the purity of ρ_2 ?

Solution:

A general requirement is that the eigenvalues of the two operators are the same. Thus, if one state is pure, the second is as well pure, if they are connected by a unitary transformation.

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