PS #3 Answer Key

Fall 2002

1a) $\lambda_{mfp} = \frac{v_{th}}{\nu} = v_{th}\tau$

- We'll take $v_{th} = \sqrt{\frac{T}{m}}$ (One degree of freedom)

$v_{th} = 2.65\times10^{7}\frac{m}{s} = 2.65\times10^{9}\frac{cm}{s}$ for 4 keV

Remember, they're in cgs!

- For τ, we've to be careful to use Molvig's v_e definition ($v_e = \sqrt{\frac{2T_e}{m_e}}$) when we use the formulas in the class notes.
- We'll take $\ln\Lambda = 16$
- The ν_i (and thus τ's) are weighted for momentum transfer (i.e. each e-e counts less than each e-i)

→ let's calculate ν:

$\nu = \nu_{ei} + \nu_{ee}$ for $e \rightarrow i$ & $e \rightarrow e$ collisions

from class notes pg. 6 & pg. 8 of

$$\nu_{ei} = \underbrace{\frac{m_i}{m_e+m_i}}_{\text{Weighting factor}} \approx 1 \; \frac{n_i 4\pi e^4 Z^2}{m_e^2 v_e^3}\ln\Lambda$$

$$\nu_{ee} = \frac{n_i 4\pi e^4 Z e^4}{m_e^2 v_e^3}\ln\Lambda \times \underbrace{\frac{m_e}{m_e+m_e}}_{\text{Weighting factor}} = \frac{1}{2}$$

for e → w/ ions & electron Maxwellians.

(in cgs)

1a) Con't

$n_i \sim \frac{10^{14}}{cm^3}$ $\quad v_e^3 = \left(\frac{2T_e}{m_e}\right)^{3/2} = 5.27 \frac{cm^3}{s^3} \times 10^{28}$

$\frac{m_i}{m_e+m_i} \approx 1$ $\quad \mu_{R_{e-i}} = \frac{m_e m_i}{m_i+m_e} \approx m_e$ (for ν_{ei})

$\mu_{R_{ee}} = \frac{m_e^{\cancel{2}}}{2\cancel{m_e}} = \frac{m_e}{2}$

$\nu_{ei} = \frac{2.44\times10^4}{s}$

$\nu_{ee} = \frac{1.22\times10^4}{s}$

so

$\nu = \frac{3.66\times10^4}{s}$.

$$\Rightarrow \boxed{\lambda_{mfp} = \frac{2.65\times10^9 cm/s}{3.66\times10^4 \frac{}{s}} = 7\times10^4\ cm}$$

1b) Using just ν_{ei} from above,

$$\lambda_{mfp\,e-i} = \frac{2.65\times10^9\ cm/s}{2.44\times10^4/s} = \boxed{1\times10^5\ cm}$$

where T is in eV, n in $\frac{1}{cm^3}$

$$\lambda_D = 7.43\times10^2\, T^{1/2} n^{-1/2}\, cm = \boxed{4.7\times10^{-3}\,cm}$$

hence, $\lambda_{mfp\,e-i} \gg \lambda_D$

this is always true! : $\frac{\lambda_{mfp}}{\lambda_D} \gg 1$

$$\frac{\lambda_{mfp\,e-i}}{\lambda_D} = \frac{v_{th}/\nu_{ei}}{\underbrace{\left(\frac{T_e}{4\pi n e^2}\right)^{\frac{1}{2}}}_{\text{in cgs}}} = \boxed{\frac{n\lambda_D^3}{\ln\Lambda} \gg 1 \quad \text{always!}}$$

$n\lambda_D^3 \gg 1$ (Definition of Plasma!)

$\ln\Lambda \sim 1-20$

1c) for ions, we've

$$v_{th_i} = 9.79\times10^5\, T_i^{1/2}\, cm/s = 6.2\times10^7 \frac{cm}{s}$$

$$\nu_{ii} = \frac{m_i}{2m_i}\,\frac{4\pi n_i e^4 Z_i^4}{\mu_R^2\, v_i^3}\, \ln\Lambda$$

where $v_i^3 = \left(\frac{2T_i}{m_i}\right)^{3/2} = 6.72\times10^{23}\frac{cm^3}{s^3}$

$$\mu_R^2 = \frac{m_i^2}{2m_i} = \frac{m_i}{2} = \frac{1.67\times10^{-24}}{2} gm$$

take $\ln\Lambda = 16$

$$\nu_{ii} = \frac{1{,}140}{s}$$

$$\nu_{ie} = \overbrace{\frac{m_e}{m_i+m_e}}^{\approx\frac{1}{1837}} \times \frac{4\pi n_i e^4 Z_i^4}{\mu_R^2\, v_e^3}\ln\Lambda$$

where μ_R now is $\mu_R = \frac{m_e m_i}{m_e+m_i} \approx m_e$

$$\nu_{ie} = \left(\frac{1}{1837}\right)\frac{4\pi n_i e^4 Z_i^4}{m_e^2\, 5.27\times10^{28}}\ln\Lambda$$

$$= \frac{1.3\times10^1}{s} = \frac{13}{s}$$

so,

$$\nu \simeq \nu_{ii} + \nu_{ie} = \frac{1.15\times10^3}{s}$$

$$\boxed{\lambda_{mfp} = \frac{v_{th_i}}{\nu} = 7.6\times10^4\ cm}$$

$$\boxed{\lambda_{mfp_i} \sim \lambda_{mfp_e}}\,!$$

1d) $T \simeq 5eV$, $n \simeq \frac{5\times10^{12}}{cm^3}$, using the formulas in a) $\nu_{ei} = 2.8\times10^7/s$, $\nu_{ee} \sim 1.4\times10^7/s$

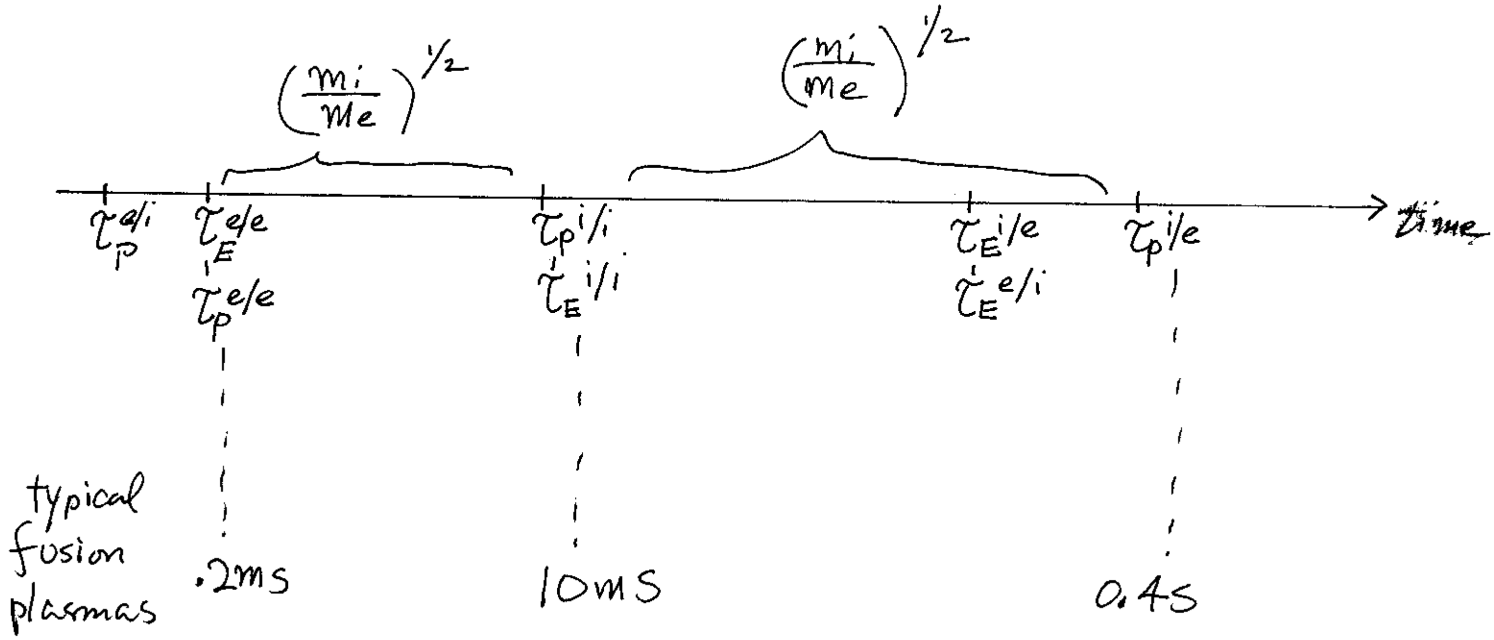
$v_{th} \sim 9.36\times10^7\, cm/s$: $\boxed{\lambda_{mfp} : 2.20\ cm}$

· General Notes about collisions and class notes:

→ In Molvig's class notes, under summary, $\nu_{ij} \propto v_i^{-3}$ .... v_i should actually be the faster of the i or j species.... (i.e, for ν_{ie}, it should be v_e, not v_i)

→ When in doubt, use the formulary to check your answers.... might not be exact w/ class notes, but order of magnitude should be the same

→ I found this "timeline" extremely helpful:



where τ_E is for energy transfer
τ_p is for momentum transfer

→ so energy equilibrium between ions & e's take a while! (i.e T_e doesn't have to equal T_i all the time!)

2) for slowing down, (i.e. momentum loss)

$$\frac{dv_\alpha}{dt} = -\nu_s^{\alpha\backslash\beta} v_\alpha$$, where β is the field particles

(Plasma formulary)

at $E = 3.5$ MeV,

$v_\alpha \sim 1.3 \times 10^7 \frac{m}{s}$

Plasma $T = 10$ keV

$v_e \sim 4.2 \times 10^7 \frac{m}{s}$

$v_D \sim 6.9 \times 10^5 \frac{m}{s}$

$v_T \sim 5.67 \times 10^5 \frac{m}{s}$

thermal velocities, $\sim \left(\frac{kT}{m}\right)^{1/2}$

hence, $v_\alpha >> v_D$ & v_T, but $v_e > v_\alpha$

so, for $\alpha \rightarrow e^-$

$$\frac{\nu_s^{\alpha\backslash e}}{(n_e Z^2 \ln\Lambda)} \approx 1.6 \times 10^{-9} \mu^{-1} T_e^{-3/2}$$ (slow, T_e in eV, $\mu_\alpha \sim 4 = \frac{m_i}{m_p}$)

$(\ln\Lambda \sim 16)$

$$\nu_s^{\alpha\backslash e} \cong 2.56 \ \frac{1}{s}$$

$\boxed{\tau_s^{\alpha\backslash e} \approx 3.9 \times 10^{-1} \text{ s}}$ for $\alpha \rightarrow e$ collisions at $\alpha = 3.5$ MeV and below (since $v_e > v_\alpha$ always)

for $\alpha \rightarrow D$ & T

$$\frac{\nu_s^{i\backslash i'}}{n_{i'} Z^2 Z'^2 \ln\Lambda_{ii'}} \rightarrow 9.0 \times 10^{-8} \left(\frac{1}{\mu} + \frac{1}{\mu'}\right) \frac{\mu^{1/2}}{\epsilon^{3/2}}$$

(fast, ϵ in eV, primes = field particles, and $\mu_\alpha = \frac{m_i}{m_p} = 4$)

then, for D,

$\frac{1}{2} n_T + \frac{1}{2} n_D = n_e = \frac{10^{14}}{cm^3}$

$$\nu_s^{\alpha\backslash D} \rightarrow 9 \times 10^{-8} \left(\frac{1}{4} + \frac{1}{2}\right) \frac{(4)^{1/2}}{(3.5 \times 10^6)^{3/2}} \cdot \left(\frac{10^{14}}{2}\right) 4 \cdot 1 \cdot 16$$

$$\rightarrow 6.6 \times 10^{-2} \left(\frac{1}{s}\right)$$

$\boxed{\tau_s^{\alpha\backslash D} = 15.2 \text{ sec}}$ for $\alpha \rightarrow D$ collisions when $v_\alpha > v_D$

for T,

$$\nu_s^{\alpha|T} \rightarrow 9\times10^{-8}\left(\frac{1}{4}+\frac{1}{3}\right)\left(\frac{4^{1/2}}{3.5E6^{3/2}}\right)\cdot\frac{10^{14}}{2}\cdot 4\cdot 1\cdot 16$$

$$\nu_s^{\alpha|T} \rightarrow 5.13\times10^{-2}\left(\frac{1}{s}\right)$$

$$\boxed{\tau_s^{\alpha|T} = 1.95\times10^{1}\ s = 19.5\,\text{sec}}$$ for $\alpha \rightarrow T$ when $v_\alpha > v_T$

(for 3.5 MeV α)

- So, Initially, the electrons will be most effective at slowing down the α's.
- The reason is that the α's & e's share similiar velocities when $\alpha \sim$ MeV range.
- This doesn't change until $v_\alpha \ll v_e$, or $v_\alpha \sim v_D$ & v_T. When that happens, we've

$$\frac{m_D v_\alpha^2}{2T_D} < 1 \quad \& \quad \frac{m_T v_\alpha^2}{2T_T} < 1$$

$(v_\alpha \sim 1\times10^6 \frac{m}{s})$

$T_\alpha \sim 20$ keV

Using the "slow" formulas, we've

$$\frac{\nu_s^{i|i'}}{n_{i'}Z^2Z'^2\ln\Lambda} \approx 6.8\times10^{-8}\,\frac{\mu'^{1/2}}{\mu}\left(1+\frac{\mu'}{\mu}\right)^{-1/2}T^{-3/2}$$

or, for $\alpha \rightarrow D$,

$$\nu_s^{\alpha|D} \approx 6.8\times10^{-8}\,\frac{2^{1/2}}{4}\left(1+\frac{2}{4}\right)^{-1/2}1\times10^4\,\text{eV}^{-3/2}\left(5\times10^{13}\right)(4)(1)\,16$$

$$= 62.8\ \frac{1}{s}$$

$$\boxed{\left.\begin{array}{l}\tau_s^{\alpha|D} \sim 0.016\ s \\ \text{Similarly, } \tau_s^{\alpha|T} = 0.014\ s\end{array}\right\} \text{for } m_\alpha v_\alpha^2 \lesssim T_D,\ T_T}$$

→ So, electrons are most effective at slowing down MeV α's, while D & T are most effective when $T_\alpha \lesssim 100$ keV

2b) Energy loss!

$$\frac{d v_\alpha^2}{dt} = -\nu_\epsilon^{\alpha\backslash\beta} v_\alpha^2 \quad , \quad \text{where} \quad \beta = e^-, D, T$$

for e^-, we use the $v_\alpha < v_e$ "slow" formulas again:

$$\nu_\epsilon = 2\nu_s - \nu_\perp - \nu_\parallel$$

$$\nu_s^{i\backslash e}/n_e Z^2 \lambda_{ie} \approx 1.6\times10^{-9}\mu^{-1}T^{-3/2}$$
$$\nu_\perp^{i\backslash e}/n_e Z^2 \lambda_{ie} \approx 3.2\times10^{-9}\mu^{-1}T^{-1/2}\epsilon^{-1}$$
$$\nu_\parallel^{i\backslash e}/n_e Z^2 \lambda_{ie} \approx 1.6\times10^{-9}\mu^{-1}T^{-1/2}\epsilon^{-1}$$

We may be able to neglect these when $\epsilon \gg 1$

$$\frac{\nu_\epsilon^{\alpha/e}}{n_e Z^2 \ln\Lambda} = \frac{2(1.6\times10^{-9})}{\mu T^{3/2}} - \frac{3.2\times10^{-9}}{\mu T^{1/2}\epsilon_\alpha} - \frac{1.6\times10^{-9}}{\mu T^{1/2}\epsilon_\alpha}$$

where $\epsilon_\alpha = 3.5$ MeV, $T = 10{,}000$ eV
$\mu = 4$, $n_e = 10^{14}/\text{cm}^3$, $Z = 2$, $\ln\Lambda \sim 16$

$$\nu_\epsilon^{\alpha/e} = 10^{14}(4)(16)\frac{(1.6\times10^{-9})}{\mu T^{1/2}}\left(\frac{2}{T} - \frac{3}{\epsilon_\alpha}\right)$$

$$\nu_\epsilon^{\alpha/e} = 2.56\times10^4\left(\frac{2}{10{,}000} - \frac{3}{\epsilon_\alpha}\right)$$

$$\nu_\epsilon^{\alpha/e} \approx 5.12 - \frac{3}{\epsilon_\alpha}$$ (Remember, ϵ_α in eV is 3.5×10^6!)

for $\epsilon = 3.5$ MeV,

$$\nu_\epsilon^{\alpha/e} \sim 5.12\left(\frac{1}{s}\right)$$

$$\boxed{\tau_e^{\alpha/e} = 0.195\ s}$$, quite independent of ϵ_α, as long as $\epsilon_\alpha > T_e$

Now let's do the same thing for D & T

for fast α's colliding w/ slow D & T's,

$$\frac{\nu_\epsilon}{n_i' Z^2 Z'^2 \ln\Lambda} \approx 2\left(9.0\times10^{-8}\left(\frac{1}{\mu}+\frac{1}{\mu'}\right)\frac{\mu^{1/2}}{\epsilon_\alpha^{3/2}}\right) - 1.8\times10^{-7}\mu^{-1/2}\epsilon_\alpha^{-3/2} - \frac{9.0\times10^{-8}\mu^{1/2}T}{\mu'\epsilon^{5/2}}$$

$$\approx \frac{1.8\times10^{-7}}{\epsilon_\alpha^{3/2}}\left(\left(\frac{1}{\mu}+\frac{1}{\mu'}\right)\mu^{1/2} - \frac{1}{\mu^{1/2}} - \frac{(2)^{-1}\mu^{1/2}T}{\mu'\epsilon}\right)$$

for D,

$$\nu_\epsilon^{\alpha/D} \approx \frac{5\times10^{13}(4)(1)(16)\,1.8\times10^{-7}}{\epsilon_\alpha^{3/2}}\left(\left(\frac{1}{4}+\frac{1}{2}\right)2 - \frac{1}{2} - \frac{2(10,000)}{4\,\epsilon}\right)$$

$$\approx \frac{5.76\times10^{8}}{\epsilon_\alpha^{3/2}}\left(1 - \frac{5,000}{\epsilon_\alpha}\right)$$

so, for $\epsilon_\alpha \simeq 3.5\ \text{MeV} \sim 3.5\times10^6\,\text{eV}$

$$\nu_\epsilon^{\alpha/D} \sim 8.9\times10^{-2}\ \frac{1}{s},$$

$\boxed{\tau_\epsilon^{\alpha/D} \simeq 11.4\ s}$ as long as $\epsilon_\alpha \gg 5,000$

for T,

$$\nu_\epsilon^{\alpha/T} \approx \frac{5\times10^{13}(4)(1)(16)\,1.8\times10^{-7}}{\epsilon_\alpha^{3/2}}\left(\left(\frac{1}{4}+\frac{1}{3}\right)2 - \frac{1}{2} - \frac{\cancel{2}(10,000)}{\cancel{2}\cdot3\,\epsilon_\alpha}\right)$$

$$\approx \frac{5.76\times10^{8}}{\epsilon_\alpha^{3/2}}\left(\frac{2}{3} - \frac{3,333}{\epsilon_\alpha}\right)$$

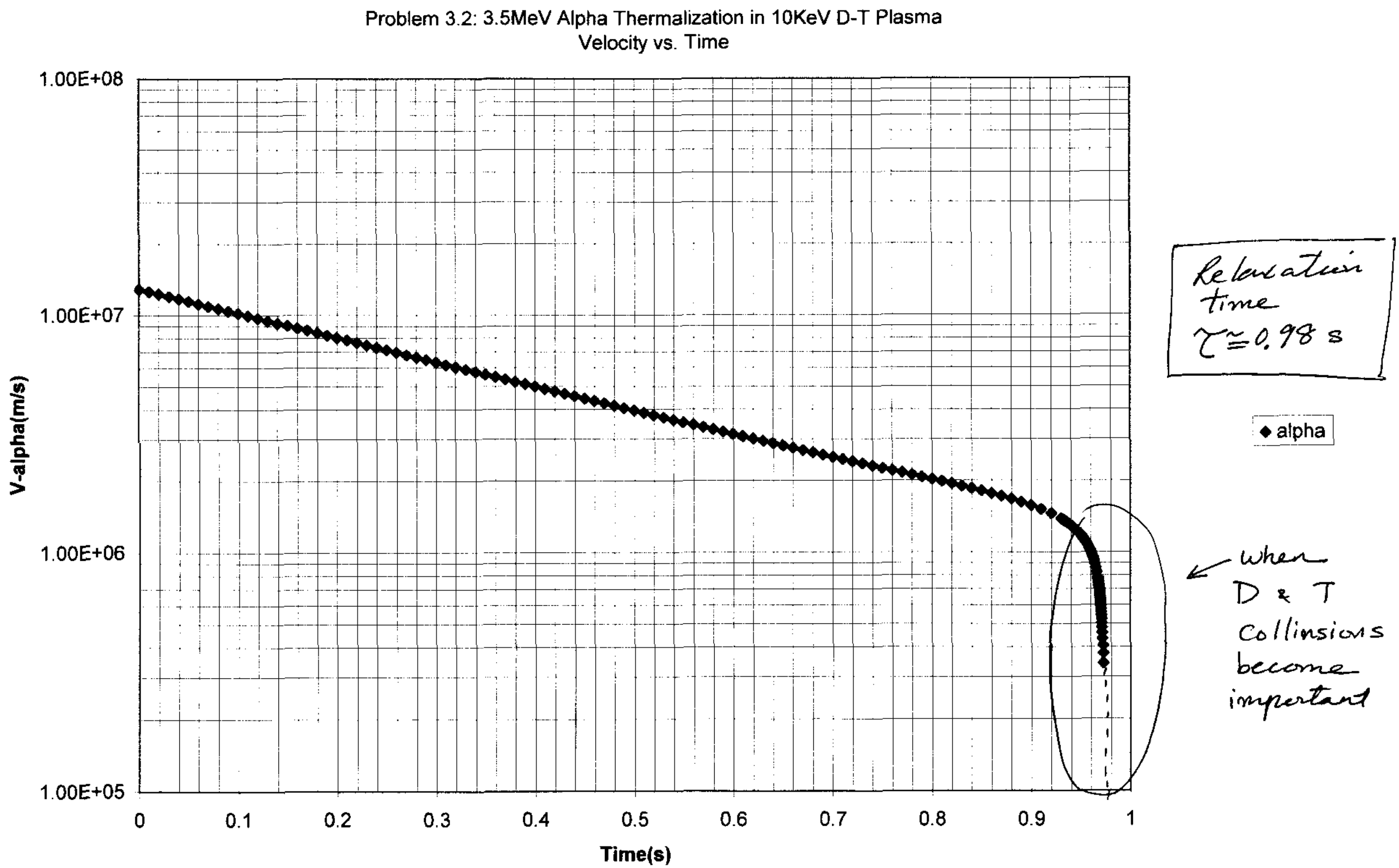
so, if $\epsilon_\alpha \gg 3,333\ \text{eV}$,

$$\nu_\epsilon^{\alpha/T} \sim 5.86\times10^{-2}\left(\frac{1}{s}\right)$$

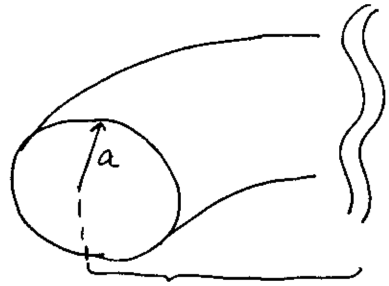
$\boxed{\tau_\epsilon^{\alpha/T} \sim 17\ s}$ as long as $\epsilon_\alpha \gg 3,333\,\text{eV}$

2) Con't

- So in general, for energy loss, the α's heat the electrons!
- the relaxation time is on the order of $\tau_e^{\alpha/e}$, OR $\sim 10^{-1}$ to 1 sec.
- to do this properley, the $\frac{m}{v_\alpha^2}\frac{dv_\alpha^2}{dt} = -m(\nu_E^{\alpha|e} + \nu_E^{\alpha|D} + \nu_E^{\alpha|T})$ must be solve numerally. (Very tedious!)
- See attached page for full solution
- In general, spieces slow down effectively when they're collidong w/ other spieces that're roughly the same order of velocity .... (when everything else is equal)


Problem 3.2: 3.5MeV Alpha Thermalization in 10KeV D-T Plasma
Velocity vs. Time
1.00E+08
1.00E+07
1.00E+06
1.00E+05
V-alpha(m/s)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Time(s)
alpha
Relaxation time
τ≅0.98 s
when D & T collinsions become important

3.



$R = 1.2\,m$,
$a = 0.3\,m$

By definition,

$$\vec{J} = \sigma \vec{E} = \frac{\vec{E}}{\eta}$$

We have $I = 4 \times 10^5 A$

$$\Rightarrow j = \frac{I}{\pi a^2} = 1.41 \times 10^6 \frac{A}{m^2}$$

a) · The resistivity is characterized by e^- momentum loss, or collisions, w/ ions.

· A rough estimate would be:

$$e\, n_e m \frac{d\vec{v}_e}{dt} = n_e e^2 \vec{E} - \frac{m_e n_e (\vec{v}_e - \vec{v}_i) e}{\tau_{ei}}$$

$$m_e \frac{d\vec{J}}{dt} = n_e e^2 \vec{E} - \frac{m_e n_e e (\vec{v}_e)}{\tau_{ei}}$$, assume $v_e > v_i$, and J is carried by electrons only.

so, for $\frac{d\vec{J}}{dt} = 0$,

$$n_e e^2 \vec{E} = \frac{m_e n_e e v_e}{\tau_{ei}} = \frac{m_e}{\tau_{ei}} \vec{J}$$

$$\vec{J} = \left(\frac{m_e}{\tau_{ei}}\right)^{-1} e^2 \vec{E} n_e, \quad \eta = \frac{m_e}{\tau_{ei} e^2 n_e}$$

for $v_e > v_i$, we've from the Plasma formulary,

$$\nu_s^{e\backslash i} / n_i Z^2 \ln \Lambda_{ei} \approx \frac{3.9 \times 10^{-6}}{E^{3/2}}$$, E_e in eV, $n_i\, [=]\, \frac{1}{cm^3}$, $\ln \Lambda_{ei} \sim 16$

hence, $$\nu_s^{e\backslash i} \approx \frac{3.9 \times 10^{-6}}{(1000)^{3/2}} (10^{13})(1)^2 16$$

$$\approx 1.97 \times 10^4 / s$$

$$\tau_{e/i} \sim 5.1 \times 10^{-5} \text{ sec}$$

hence, $$\eta \approx \frac{9.11 \times 10^{-31} kg}{5.1 \times 10^{-5} s\, (1.6 \times 10^{-19} C)^2 \left(\frac{10^{19}}{m^3}\right)} \cong 6.9 \times 10^{-8} \text{ ohm m}$$

then,

$$J \simeq \frac{E}{\eta} \Rightarrow 1.41\times10^{6}\frac{A}{m^2}(6.9\times10^{-8})\,\text{ohm-m} = E$$

$$\boxed{\begin{aligned} E &= 9.84\times10^{-2}\ \text{V/m (in SI)} \\ &= 3.3\times10^{-6}\ \frac{\text{stat-V}}{\text{cm}}\ \text{(in cgs)} \end{aligned}}$$

→ The real value (η) requires taking into account e-e collisions and also a maxwellian distribution. This η is called the Spitzer resistivity:

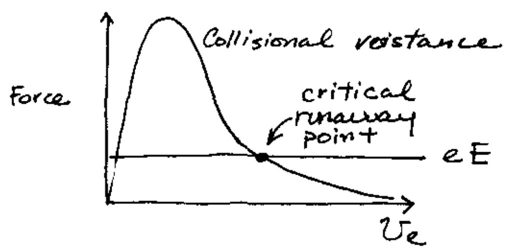
$$\eta_s = 1.65\times10^{-9}\frac{\ln\Lambda}{T_e^{3/2}}\ \text{ohm m}, \quad T_e \text{ is in keV}$$

$$\approx 2.8\times10^{-8}\ \text{ohm-m, which gives } E \approx 4.0\times10^{-2}\ \text{V/m} = 1.3\times10^{-6}\ \frac{\text{stat-V}}{\text{cm}}$$

b) Runaway electrons

- Runaway occurs when the resistance to acceleration from the electric field due to collisions is insufficient:

$$eE > \frac{m_e(v_e - v_i)}{(\tau_s)} \sim \frac{m_e v_e}{\left((\tau_{se}^{-1} + \tau_{si}^{-1})\right)^{-1}}$$



- We can make an estimate by letting $\tau_{se} \sim \tau_{e/e}$ and $\tau_{si} \sim \tau_{/}$; from the formulary,

$\tau_{e/e}$ for a fast e^- test particle is:

$$\nu_s^{e\backslash e'}/n_{e'}\lambda_{ee'} \rightarrow 7.7\times10^{-6}\,E_e^{-3/2}, \qquad \text{(where } e' \text{ is the background } e\text{)}$$

$\lambda_{ee'} = \ln\Lambda_{ee'}$

$$E_e^- = \tfrac{1}{2} m_e v_e^2 \rightarrow \text{(in eV)}$$

Similarly, for $\tau_{e/i}$ (also from pt. a)):

$$\nu_s^{e\backslash i}/n_i \ln\Lambda_{ei} \approx 3.9\times10^{-6}\,E_e^{-3/2}$$

→ (effectively, $\nu_s^{e|e'} = 2\nu_s^{e\backslash i}$)

taking $\ln\Lambda \sim 16$, we've

$$\nu_s^{e|e'} \to \frac{7.7\times10^{-6}}{\underbrace{\left(\frac{1}{2}m_e v_e^2\right)^{3/2}}_{(\text{in eV})}}\,(10^{13}\,cm^{-3})(16) = \frac{1.232\times10^{9}}{\left(\frac{1}{2}m_e v_e^2\right)^{3/2}}$$

for $\frac{v_e^2 m_e}{2}$ in eV, we've

$$\frac{511\,keV\ v_e^2}{2\left(3\times10^8\frac{m}{s}\right)^2}\cdot\frac{1000\,eV}{1\,keV} \Rightarrow v_e^2\ 2.84\times10^{-12}\ [=]\ eV \quad \text{if } v\ [=]\ \frac{m}{s}$$

$$\nu_s^{e|e'} \to \frac{1.232\times10^{9}}{\left(2.84\times10^{-12}v_e^2\right)^{3/2}} \to \frac{2.58\times10^{26}}{v_e^3}\ [=]\ \frac{1}{s} \quad \text{if } v\ [=]\ \frac{m}{s}$$

$$\nu_s^{e|i} = \frac{\nu_s^{e|e'}}{2}$$

then, we've

$$\frac{1}{\tau_s} = \left(\nu_s^{e|e'} + \nu_s^{e|i}\right) = \frac{3.87\times10^{26}}{v_e^3}$$

$$eE > \frac{m_e v_e}{(3.87\times10^{26})^{-1} v_e^3} > \frac{3.525\times10^{-4}}{v_e^2}$$

$$E > \frac{2.20\times10^{15}}{v_e^2}$$

$$\boxed{v_{e\,crit} = 2.35\times10^{8}\,\frac{m}{s}\ !}$$ (can't ignore relativity!)

→ from Wesson's <u>Tokamaks</u>, eq. 2.17.3:

$\mathcal{E}_c$ = critical energy = $6.6\times10^{-19}\,\frac{n}{E}$ keV, $n\ [=]\ \frac{1}{m^3}$, $E\ [=]\ V/m$

$$\mathcal{E}_c \approx 165\,keV, \text{ or } 0.32\,m_e c^2$$

$$v_c \approx 2.4\times10^{8}\,\frac{m}{s}$$