Massachusetts Institute of Technology

Subject 6.651J/8.613J/22.611J R. Parker
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**Problem Set 4**

**Problem 1.**

A beam of ions with energy E is passing through a plasma of density $n_e = n_i = n_0$. The velocity of the beam $v_0$ is large compared to the thermal velocities of the electrons and ions, so the latters' motion can be neglected.

i) Calculate the effective frequency of transverse diffusion of the beam $\nu_\perp$ defined as

$$\frac{d\langle v_\perp^2 \rangle}{dt} = -\nu_\perp v_0^2$$

Express your answer in terms of a numerical constant, the energy of the beam in eV, the plasma density, the Coulomb logarithm and the mass and charge of the beam and plasma ions relative to the mass and charge of a proton. (Ignore terms of order $1/\ln\Lambda$.)

ii) Compare your answer with the result given in the NRL formulary, pp 31-32 (See link under Resources on Course Home Page.)

**Problem 2.**

In this problem, we will use the plasma fluid equations to describe electrostatic waves that can exist in a one-dimensional electron beam. The relevant equations are:

Conservation of particles/mass/charge:

$$\frac{\partial n}{\partial t} + \nabla \bullet n\vec{v} = 0$$

Conservation of momentum:

$$mn\frac{d\vec{v}}{dt} = -en\vec{E}$$

Gauss' law for the electric field:

$$\nabla \bullet \varepsilon_0 \vec{E} = -e(n_e - n_i)$$

Assume that only the electrons participate in the wave and that the ions only provide a constant neutralizing background.

a) Writing the variables in the form

$$n_e = n_{e0} + n_1(x,t)$$

$$n_i = n_{io} = n_{e0}$$

$$\vec{v}_e = V_{e0}\hat{x} + v_1(x,t)\hat{x}$$

$$\vec{E} = \hat{x}E_1(x,t),$$

where subscripts containing a 0 denote constant terms, develop a set of linear equations relating $n_1, v_1$ and $E_1$. Note that since $n_{e0} = n_{io}$, there is no equilibrium electric field.

b) Assuming a wave solution with all variables proportional to $\exp(i\omega t - ikx)$, determine and sketch $k(\omega)$. Explain why the two waves that you found are referred to as the slow and fast wave.

c) For each of the two waves, compute $J_x = \varepsilon_0 \frac{\partial E_1}{\partial t} + J_b$, the sum of displacement plus beam current. Note that this term is the source of the magnetic field in Ampère's law. Why can the magnetic field be ignored in analyzing the beam wave motion?

**Problem 3.**

A cold homogeneous plasma supports oscillations at the plasma frequency $\omega = \omega_p$. Perhaps surprisingly, the oscillation occurs at the same frequency regardless of the wavenumber $k = 2\pi/\lambda$. However, when pressure is included in the equations describing a plasma oscillation, the situation changes and the frequency of oscillation depends on *k*. In this problem you are asked to find the $\omega - k$ relationship for a plasma in which pressure plays a role.

a) The electron fluid equations are:

$$\frac{\partial n}{\partial t} + \nabla \bullet n\vec{v} = 0$$
$$m_e n \frac{d\vec{v}}{dt} = -en\vec{E} - \nabla p$$
$$pn^{-\gamma} = p_0 n_0^{-\gamma}$$
$$\nabla \bullet \varepsilon_0 \vec{E} = -e(n - n_{0)}$$

Let $n = n_0 + n_1, \vec{v} = \vec{v}_1, \ p = p_0 + p_1$ and $\vec{E} = \vec{E}_1$ where quantities with subscript 0 refer to the spatially homogeneous equilibrium and those with subscript 1 indicate small perturbations. Develop a set of linear equations sufficient to solve for the perturbed variables.

b) Assume that all variables have time-space dependence proportional to $\exp(-i\omega t + ikx)$. Determine the relation between $\omega$ and *k* that permits a nontrivial solution to the equations found in part a).