solution Key

22.611J, 6.651J, 8.613J Introduction to Plasma Physics I

Problem Set #4

1) Derive the plasma adiabatic equation of state given in class via the following path:

(a) Take the energy moment of the kinetic equation and show that the result is:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}mnV^2+\frac{3}{2}nT\right)+\overline{\nabla}\cdot\left(\frac{1}{2}mnV^2+P+\frac{3}{2}nT\right)\overline{V}-qn\overline{E}\cdot\overline{V}=0$$

Where V is the fluid velocity, T the temperature, n the density, P the pressure, m the mass, q the charge, and E the electric field.

(b) Reduce the above to the standard hydrodynamic form of the equation advancing temperature in time. The key is to take V(dot)(momentum equation) to obtain a separate equation for $\delta V^2/\delta t$ which then can be subtracted from the above equation and manipulated, giving the adiabatic equation of state:

$\vec{F}=m\vec{a}=q\vec{E}+q\vec{v}\times\vec{B}$

$$\frac{D}{Dt}\left(\frac{p}{\rho^{5/3}}\right)=0$$

2) The derivation of the MHD equations form the ion and electron fluid equations were given in class. Review this derivation and answer the following question adapted from the Fall 2001 NE qualifying exam:

1. The equations governing a plasma approximated as a single fluid, in addition to Maxwell's equations, can be written:

$$\rho\left(\frac{\partial\overline{v}}{dt}+\overline{v}\cdot\nabla\overline{v}\right)=\overline{J}x\overline{B}-\nabla P$$

$$\overline{E}+\overline{v}x\overline{B}=\eta\overline{J}+\frac{m_i}{\rho q_i}(\overline{J}x\overline{B}-\nabla P_e)-\frac{m_i m_e}{q_e q_i}\frac{\partial}{\partial t}\left(\frac{\overline{J}}{\rho}\right)$$

(a) Which of these terms are omitted in the ideal MHD description of a plasma?

(b) Discuss the circumstances that would justify omitting those terms for the plasma in a typical tokamak plasma; i.e. numerically show that these terms are small.

3) A Z-pinch is a cylindrically symmetric plasma column with current in the z direction only. Consider a static ideal MHD Z-pinch equilibrium with

$$J = c_1 \frac{r^2/a^2}{(1+r^2/a^2)^3}$$

where c_1 is a constant.

(a)Calculate $B_\theta(r)$ and $p(r)$. Express your answers in terms of I, the total current. Sketch the fields and the currents.

(b) (Extra credit) Since J(r) vanishes for large r the Z pinch is apparently confined by its own current. Doesn't this violate the virial theorem(a plasma can not be confined by its own currents)? Explain.

(From Problem 5.2, Ideal Magnetohydrodynamics, J.P. Freidberg)

PS#4 Answer Key

V. Tang

22.611J, 6.651J, 8.613J Intro to Plasma (One)

1) Kinetic equation:

$$\frac{\partial f}{\partial t} + \nabla_v \cdot (\vec{a} f) + \nabla \cdot (\vec{v} f) = 0$$

Now, take the energy moment:

$$\underbrace{\int \frac{1}{2} m v^2 \frac{\partial f}{\partial t} d^3v}_{1} + \underbrace{\int \frac{1}{2} m v^2 \nabla_v \cdot (\vec{a} f) d^3v}_{2} + \underbrace{\int \frac{1}{2} m v^2 \nabla \cdot (\vec{v} f) d^3v}_{3} = 0$$

Terms: 1 2 3

→ Let's start w/ 1:

$$\int \frac{1}{2} m v^2 \frac{\partial f}{\partial t} d^3v = \frac{\partial}{\partial t} \int \frac{1}{2} m v^2 f \, d^3v$$ (since v & t are independent variables)

$$= \boxed{\frac{\partial}{\partial t}\left(\frac{1}{2} m \langle v^2 \rangle n\right) = 1}$$, where $\langle v^2 \rangle$ is the average velocity.

→ Term 2:

$$\int \frac{1}{2} m v^2 \nabla_v \cdot (\vec{a} f) d^3v = \int \frac{1}{2} m v^2 (\underbrace{\vec{a} \cdot \nabla_v f}_{2a} + \overbrace{f \nabla_v \cdot \vec{a}}^{2b}) d^3v$$

where $\vec{a} = \frac{q}{m}(\vec{E} + \vec{v} \times \vec{B})$

2a: $$\int \frac{1}{2} m v^2 \vec{a} \cdot (\nabla_v f) = \int \cancel{\nabla_v \cdot (\vec{a} \tfrac{1}{2} m v^2 f)} d^3v - \int f \nabla_v \cdot (\tfrac{1}{2} m v^2 \vec{a}) d^3v$$

0, since at $v \pm \infty$, the divergences is zero!

$$= -\int f \nabla_v \cdot (\tfrac{1}{2} m v^2 \vec{a}) \, d^3v$$

$$= -\int f \left(\vec{a} \cdot \nabla_v (\tfrac{1}{2} m v^2) + \cancel{\tfrac{1}{2} m v^2 \nabla_v \cdot \vec{a}}\right) d^3v$$

0, since $E \neq E(\vec{v})$ and $\hat{v} \cdot \vec{v} \times \vec{B} = 0$

$$= -\int \frac{q}{\cancel{m}} \vec{E} f \cdot \left(\frac{\cancel{2}^{\,1}}{\cancel{2}} \cancel{m} \vec{v}\right) = \boxed{-\int q f \vec{E} \cdot \vec{v} = 2a}$$

→ Term 2 Con't

Note: $\left(\nabla_v v^2 = \nabla_v(\vec{v}\cdot\vec{v}) = 2\vec{v}\nabla_v\vec{v} = 2\vec{v} \right)$

(2b) ∴ $\int f \nabla_v \cdot \vec{a} = 0$, since $\nabla_v \cdot \vec{a} = 0$ $\left(\begin{array}{l} \vec{E} \neq E(\vec{v}) \\ \hat{v}\cdot \vec{v}\times\vec{B} = 0 \end{array} \right)$

then,

$$(2) = -\int q f \vec{E}\cdot\vec{v}\, d^3v$$

$$= -q\vec{E}\cdot\int \vec{v} f\, d^3v = \boxed{-q\vec{E}\cdot\vec{V}n = (2)}$$

→ Term 3 :

$$\int \frac{1}{2}mv^2\,\nabla\cdot(\vec{v}f) = \nabla\cdot\left(\int\frac{1}{2}mv^2\vec{v}f\right)$$ (since v^2 doesn't depend on space coordinates) i.e v & x,y,z are independent)

then,

$$\int\frac{1}{2}mv^2\,\nabla\cdot(\vec{v}f) = \boxed{\nabla\cdot\left(\frac{1}{2}mn\langle v^2\vec{v}\rangle\right) = (3)}$$

So, we now 've :

$$\frac{\partial}{\partial t}\left(\frac{1}{2}mn\langle v^2\rangle\right) + \nabla\cdot\left(\frac{1}{2}mn\langle v^2\vec{v}\rangle\right) - qn\vec{V}\cdot\vec{E} = 0$$

→ this is so far completely general; we've only neglected the collisions terms.

→ Now, let us solve for $\langle v^2\rangle$ & $\langle v^2\vec{v}\rangle$ and see what we get.

→ Now, we'll also take f = maxwellian

Define $\vec{v} = \underset{\substack{\uparrow \\ \text{fluid velocity}}}{\vec{V}(\vec{r},t)} + \underset{\substack{\uparrow \\ \text{random velocity} \\ \text{(independent variable)}}}{\vec{w}}$

then, $d\vec{v} = d\vec{w}$ and $\langle \vec{w} \rangle = 0$

→ then, for $\langle v^2 \rangle$, we've

$$\langle v^2 \rangle = \langle \vec{v}\cdot\vec{v} \rangle = \langle \vec{V}\cdot\vec{V} + 2\vec{w}\cdot\vec{V} + \vec{w}\cdot\vec{w} \rangle$$

$$= \langle V^2 + \underset{\substack{\| \\ 0 \text{ (since } \vec{w} \text{ is "random")}}}{2\vec{w}\cdot\vec{V}} + w^2 \rangle = V^2 + \langle w^2 \rangle = \boxed{V^2 + \frac{3P}{nm}},$$

Assuming $P = nT [=] \frac{J}{m^3} \doteq \left(\underbrace{\frac{1}{2} nm \langle w^2 \rangle}\right)\frac{2}{3}$
(since $E = \frac{3}{2} nT$ for a max.)

P is the scalar pressure!

$$\langle v^2 \vec{v} \rangle = \langle (\vec{v}\cdot\vec{v})\vec{v} \rangle = \langle (\vec{v}\cdot\vec{v})(\vec{V} + \vec{w}) \rangle$$

$$= \langle (V^2 + \underset{\substack{\| \\ 0}}{2\vec{w}\cdot\vec{V}} + w^2)\vec{V} + \vec{w}(V^2 + 2\vec{w}\cdot\vec{V} + w^2) \rangle$$

$$= \langle (V^2 + w^2)\vec{V} + \vec{w}(V^2 + w^2 + 2\vec{w}\cdot\vec{V})$$

$$= \vec{V}V^2 + \vec{V}\langle w^2 \rangle + \underset{\substack{\| \\ 0}}{\langle \vec{w} \rangle} V^2 + \langle \vec{w} w^2 \rangle + 2\langle \vec{w}(\vec{w}\cdot\vec{V}) \rangle$$

$$= \boxed{\vec{V}V^2 + \frac{3P\vec{V}}{nm} + \underbrace{2\langle \vec{w}(\vec{w}\cdot\vec{V}) \rangle + \langle \vec{w} w^2 \rangle}_{\text{Let's look at these terms in more detail:}}}$$

for $\langle \vec{w} w^2 \rangle$, this really is just

$\langle \vec{w} w^2 \rangle = \int \underset{\substack{\uparrow \\ \text{odd}}}{\vec{w}}\, \underset{\substack{\uparrow \\ \text{even}}}{w^2}\, \underset{\substack{\uparrow \\ \text{odd or even?}}}{f}\, d^3w$, which is equal to zero if f is an even function (maxwellian).

$\boxed{\langle \vec{w} w^2 \rangle = 0 \text{ if } f \text{ is even}}$ → this is the "random heat flux" term.
→ for a maxwellian in equilibrium, it's physically intuitive that it's zero.

What about

$2\langle \vec{w}(\vec{w}\cdot\vec{V})\rangle$?

Let us define

$$\langle mn\vec{W}\vec{W}\rangle = \underset{\substack{\uparrow\\ \text{total pressure tensor}}}{\overline{\overline{P}}} = \underset{\substack{\uparrow\\ \text{scalar pressure}}}{P}\overline{\overline{I}} + \underset{\substack{\uparrow\\ \text{anisotropic pressure tensor (viscosity, etc.)}}}{\overline{\overline{\pi}}}$$

where $P = \frac{1}{3}nm\langle w^2\rangle$

$$\overline{\overline{\pi}} = (nm\langle \vec{w}\vec{w} - \frac{1}{3}w^2\overline{\overline{I}}\rangle)$$

but $\overline{\overline{\pi}}$ is zero for f = Maxwellian, so we're left w/ $\overline{\overline{P}} = P\overline{\overline{I}} \Rightarrow$ scalar pressure

then,

$$2\langle \vec{w}(\vec{w}\cdot\vec{V})\rangle = 2\langle \vec{V}\cdot(\vec{w}\vec{w})\rangle = \underbrace{\left(\frac{2}{3}\frac{\langle w^2\rangle}{1}\right)\vec{V}}_{\text{from } \overline{\overline{\pi}}=0 \Rightarrow \langle\vec{w}\vec{w}\rangle = \langle\frac{1}{3}w^2\overline{\overline{I}}\rangle}$$

$$= \underbrace{\frac{2}{\cancel{3}}\frac{\cancel{3}P}{nm}}_{\text{from } \langle w^2\rangle = \frac{3P}{nm}}$$

$$\boxed{2\langle \vec{w}(\vec{w}\cdot\vec{V})\rangle = \frac{2P}{nm}\vec{V}}$$

Now, collecting Terms, we've:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}mn\langle v^2\rangle\right) = \frac{\partial}{\partial t}\left(\frac{1}{2}mn\left(V^2 + \frac{3P}{nm}\right)\right) = \frac{\partial}{\partial t}\left(\frac{1}{2}mnV^2 + \frac{3}{2}nT\right)$$

$$\nabla\cdot\left(\frac{1}{2}mn\langle v^2\vec{v}\rangle\right) = \nabla\cdot\left(\frac{1}{2}mn\left(\vec{V}V^2 + \frac{3P\vec{V}}{nm} + \frac{2P\vec{V}}{nm}\right)\right)$$

$$= \nabla\cdot\left(\frac{1}{2}mn\vec{V}V^2 + \frac{3}{2}nT\vec{V} + P\vec{V}\right)$$

Thus, we've

$$\underbrace{\frac{\partial}{\partial t}\left(\frac{1}{2}mnV^2+\frac{3}{2}nT\right)}_{\text{change in tot. E vs. time}} + \nabla\cdot\left(\overbrace{\frac{1}{2}mn\vec{V}V^2}^{\text{hydro term}} + \underbrace{\frac{3}{2}nT}_{\text{thermal energy}}\vec{V} + \overbrace{P\vec{V}}^{P\Delta V\text{ work}}\right)$$

$$- \underbrace{qn\vec{V}\cdot\vec{E}}_{\text{E\&M work}} = 0$$

The momentum equation is:

$$mn\frac{d\vec{V}}{dt} + \nabla P - qn(\vec{E}+\vec{V}\times\vec{B}) = 0,$$

where $\frac{d}{dt} = \frac{\partial}{\partial t} + (\vec{V}\cdot\nabla\ )$, the convective derivative.

Now, take $\vec{V}\cdot$ (momentum equation):

$$\Rightarrow \frac{mn}{2}\frac{dV^2}{dt} + \vec{V}\cdot\nabla P - qn\vec{E}\cdot\vec{V} = 0$$

$$\Rightarrow \boxed{\frac{d\frac{mnV^2}{2}}{dt} + \vec{V}\cdot\nabla P - qn\vec{E}\cdot\vec{V} - V^2\frac{d\,mn/2}{dt} = 0}$$

$\Rightarrow$ Now, Rewrite energy equation in terms of the full / convective derivative:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}mnV^2+\frac{3}{2}nT\right) + \vec{V}\cdot\left(\nabla\left(\frac{1}{2}mnV^2+\frac{3}{2}nT+P\right)\right)$$
$$+ \left(\frac{1}{2}mnV^2+\frac{3}{2}nT+P\right)\nabla\cdot\vec{V} - qn\vec{V}\cdot\vec{E} = 0$$

$$\Rightarrow \frac{d}{dt}\left(\frac{mnV^2}{2}+\frac{3}{2}nT\right) + \vec{V}\cdot(\nabla P) + \left(\frac{1}{2}mnV^2+\frac{3}{2}nT+P\right)\nabla\cdot\vec{V}$$
$$- qn\vec{V}\cdot\vec{E} = 0$$

Now subtract $\vec{V}\cdot$(mom. eq) from above

$$\Rightarrow \boxed{\frac{d}{dt}\left(\frac{3}{2}nT\right) + \left(\frac{1}{2}mnV^2 + \frac{3}{2}nT + P\right)\nabla\cdot\vec{V} + \frac{V^2 m}{2}\frac{dn}{dt} = 0}$$

Using

$$\frac{dn}{dt} = \frac{\partial n}{\partial t} + \vec{V}\cdot\nabla n$$

$$\frac{\partial n}{\partial t} + \vec{V}\cdot\nabla n + n\,\nabla\cdot\vec{V} = 0$$

$\Rightarrow$ Let's examine $\frac{1}{2}mnV^2\,\nabla\cdot\vec{V} + \frac{V^2 m}{2}\frac{dn}{dt}$ terms:

$$\rightarrow \frac{1}{2}mnV^2\,\nabla\cdot\vec{V} + \frac{V^2 m}{2}\Big(\underbrace{\frac{\partial n}{\partial t} + \vec{V}\cdot\nabla n}_{= -n\,\nabla\cdot\vec{V}}\Big)$$

$$\Rightarrow \frac{1}{2}mnV^2\,\nabla\cdot\vec{V} - \frac{V^2 m}{2}n\,\nabla\cdot\vec{V} = 0$$ they cancel!

then energy equation is:

$$\frac{d}{dt}\left(\frac{3}{2}nT\right) + \left(\frac{3}{2}nT + P\right)\nabla\cdot\vec{V} = 0$$

$$\frac{d}{dt}\left(\frac{3}{2}P\right) + \left(\frac{3}{2}P + P\right)\nabla\cdot\vec{V} = 0$$

$$-\nabla\cdot\vec{V} = \frac{1}{n}\frac{\partial n}{\partial t} + \frac{\vec{V}}{n}\cdot\nabla n$$

OR

$$\nabla\cdot\vec{V} = -\frac{1}{n}\frac{dn}{dt}$$

$$\frac{d}{dt}(3P) + (3P + 2P)\left(-\frac{1}{n}\frac{dn}{dt}\right) = 0$$

$$\frac{d}{dt}3P + \frac{5P}{n}\frac{dn}{dt} = 0$$

$$\frac{1}{P}\frac{dP}{dt} + \frac{5}{3n}\frac{dn}{dt} = 0$$

$$\frac{d\ln P}{dt} - \frac{d\ln n^{5/3}}{dt} = 0$$

$$\frac{d\ln\left(\frac{P}{n^{5/3}}\right)}{dt} = 0$$

$$\boxed{\frac{D}{Dt}\left(\frac{P}{n^{5/3}}\right) = 0}$$ the adibatic law!

Note: the type of density used doesn't matter, since only a factor is involved.

2a) $\eta \vec{J}$, $\frac{m_i}{\rho q_i}(\vec{J}\times\vec{B} - \nabla P_e)$, & $-\frac{m_i m_e}{q_e q_i}\frac{\partial}{\partial t}\left(\frac{\vec{J}}{\rho}\right)$

are neglected in ideal MHD

b) To justify omitting these terms, we've to show they're small compared w/ $\vec{v}\times\vec{B}$, since in ideal MHD,

$\vec{E} + \vec{v}\times\vec{B} = 0$. Take L & τ as characteristic MHD length and time.

→ Let's start w/ $\eta \vec{J}$:

- ηJ is the resistive term. For very hot plasmas, $\eta \to 0$ (superconducting)
- $\eta \sim$ spitzer resistivity $\propto \frac{1}{T^{3/2}}$, as calculated on the last homework
- typically, in a tokamak,

$$\eta \sim 10^{-8}\,\Omega\text{-m} \quad (2\,\text{keV Plasma})$$

- then, $\eta \vec{J} \sim 10^{-8}\,\Omega\text{m}\; 10^5 A \sim 10^{-3}\frac{V}{m}$
- $\frac{|\eta \vec{J}|}{|\vec{v}\times\vec{B}|} \sim \frac{10^{-3}}{|v_{MHD}B|} \sim \frac{10^{-3}}{|v_{th_i} B|} \sim \frac{10^{-3}}{|10^5 \cdot 5|} \ll 1$

(for a typical tokamak V_{MHD} instability)

- In general, ideal MHD is good for times shorter then resistive times.

→ Now consider the $\frac{m_i}{\rho q_i}(\vec{J}\times\vec{B} - \nabla P_e)$ terms

$$\frac{\left|\frac{\nabla P_e}{en}\right|}{|\vec{v}\times\vec{B}|} \sim \frac{|T_e \nabla n|}{|env B|} \sim \frac{T_0}{LevB}$$

using $eB \sim \Omega_i m_i$

2b) Con't

$$\frac{T_0}{LevB} \sim \frac{T}{L v m_i \Omega_i} \sim \frac{v_{thi}^2}{L v \Omega} = \frac{r_{Li}}{L}\frac{v_{thi}}{v}$$

so, for $v \simeq v_{thi}$ (violent MHD instability) (i.e. $v_{E\times B} \gtrsim v_{thi}$)

$$\boxed{\frac{r_{Li}}{L}\cancel{\frac{v_{thi}}{v_{thi}}} \ll 1}$$

, since $L_{MHD} \sim 1m$,

$r_{Li} \simeq \frac{m v_{thi}}{qB} \sim \frac{10^{-27}10^{5}}{10^{-19}5} \sim 10^{-}$ for fusion plasmas

however, if $v \ll v_{thi}$,
then this approximation is not valid!

→ What about $\vec{J}\times\vec{B}$?

Well, from eq. of momentum, $|\nabla p| \sim |\bar{J}\times\bar{B}|$,

so $m_i \frac{|\bar{J}\times\bar{B}|}{\rho q_i} = \frac{\frac{\bar{J}\times\bar{B}}{en}}{v\times B} \ll 1$ also, as long as the conditions for $\frac{r_{Li}}{L}\frac{v_{thi}}{v} \ll 1$ are satisfied.

→ Finally, what about $-\frac{m_i m_e}{q_e q_i}\frac{\partial}{\partial t}\left(\frac{\bar{J}}{\rho_m}\right)$?

let's rewrite it as $\sim \frac{m_i m_e}{q_e q_i m_i n}\frac{\partial}{\partial t}\left(\frac{\bar{J}}{1}\right)$

then $\sim \frac{m_e}{q_e q_i n}\frac{\partial}{\partial t}\bar{J}$ ∴ In MHD, we neglect m_e inertia by letting $m_e \to 0$... So this term vanishes.

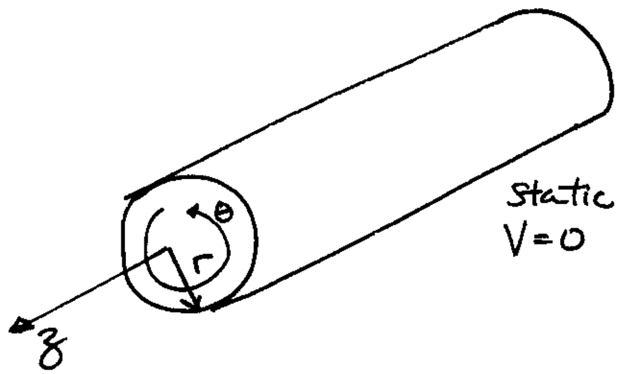
Let's see the size

$$\frac{\left|\frac{m_e}{e^2 n}\frac{\partial}{\partial t}\bar{J}\right|}{v\times B} \sim \frac{m_e \cancel{e} \cancel{n_e} v_e}{e^2 \cancel{n}\, \tau (v \cdot B)} \sim \frac{m_e v_e}{\tau v\, eB} \rightarrow \text{but } (\tau v)_{MHD} \sim L_{MHD} \sim 1m$$

$$\rightarrow \frac{m_e v_e}{eB} \sim \frac{10^{-31}\cdot 10^{7}}{10^{-19}\cdot 5} \ll 1$$

3. a)

$$J_z = \frac{c_1 r^2/a^2}{(1+r^2/a^2)^3}$$



Static:
$V=0$

$$\overline{\nabla} \times \overline{B} = \mu_0 J$$

$$(\nabla \times B_\theta)_z = \mu_0 \vec{J}_z$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) = \mu_0 \frac{c_1 r^2/a^2}{(1+r^2/a^2)^3}$$

$$\int_0^{rB_\theta} \partial(rB_\theta) = \int_0^r \mu_0 r \frac{c_1 r^2/a^2}{(1+r^2/a^2)^3}\,\partial r$$

$$rB_\theta = \frac{1}{2\pi}\mu_0 \underbrace{\int_0^r 2\pi r \frac{c_1 r^2/a^2}{(1+r^2/a^2)^3}\,\partial r}_{=I(r)} \qquad \left(\text{hence } B_\theta = \frac{\mu_0}{2\pi r} I(r)\right)$$

make substitution $x = r^2/a^2$, $\quad xa^2 = r^2$

$$\frac{dx}{dr} = \frac{2r}{a^2} \qquad r = \pm (xa^2)^{\frac{1}{2}}$$

$$I(r) = \int_0^{r=(xa^2)^{\frac{1}{2}}} a^2 \pi \frac{c_1 x}{(1+x)^3}\,dx$$

$$= a^2 \pi c_1 \int_0^r \frac{x}{(1+x)^3}\,dx = a^2\pi c_1 \left[\frac{1}{2(1+x)^2} + -\frac{1}{1+x}\right]_0^r$$

$$I(r) = a^2\pi C_1 \left[\frac{1}{2(1+\frac{r^2}{a^2})^2} - \frac{1}{1+\frac{r^2}{a^2}} \right]_0^r$$

$$= a^2\pi C_1 \left[\frac{1}{2(1+\frac{r^2}{a^2})^2} - \frac{1}{1+\frac{r^2}{a^2}} - \frac{1}{2} + 1 \right]$$

$$\boxed{I(r) = a^2\pi C_1 \left[\frac{1}{2(1+\frac{r^2}{a^2})^2} - \frac{1}{1+\frac{r^2}{a^2}} + \frac{1}{2} \right] \qquad B_\theta(r) = \frac{\mu_0}{2\pi r} I(r) \quad (\text{in SI})}$$

$I_{tot} = \frac{a^2\pi C_1}{8}$ (for $r=a$)

Now, solve for $P(r)$

$$\vec{J} \times \vec{B} = \vec{\nabla} P \qquad \begin{vmatrix} r & \theta & z \\ 0 & 0 & J_z \\ 0 & B_\theta & 0 \end{vmatrix} = \hat{r}(-J_z B_\theta)$$

$$-J_z B_\theta = \frac{\partial P}{\partial r} \quad \rightarrow \text{take } P(a) = 0$$

$$\int_r^a -\frac{C_1 r^2/a^2}{(1+r^2/a^2)^3} \frac{\mu_0}{2\pi r} I(r)\, \partial r = \int_P^0 \partial P$$

$$a^2\pi C_1 \cdot \frac{C_1 \mu_0}{2\pi} \int_r^a \frac{r/a^2}{(1+r^2/a^2)^3} \left[\frac{1}{2(1+\frac{r^2}{a^2})^2} - \frac{1}{1+\frac{r^2}{a^2}} + \frac{1}{2} \right] \partial r = P$$

let $x = r/a \qquad dx = \dfrac{dr}{a}$

3 a) Con't

$$\frac{a^2 C_1^2 \mu_0}{2} \int_r^a \frac{x}{(1+x^2)^3} \Bigg(\underbrace{\frac{1}{2(1+x^2)^2}}_{A} - \underbrace{\frac{1}{1+x^2}}_{B} + \underbrace{\frac{1}{2}}_{C} \Bigg) dx = P$$

Let's do this term by term:

$$\frac{1}{2} \int_{ax}^{a} \frac{x}{(1+x^2)^5} = \frac{1}{2} \Bigg[-\frac{1}{8(1+x^2)^4} \Bigg]_{ax}^{a} = A$$

$$-\int_{ax}^{a} \frac{x}{(1+x^2)^4} = + \Bigg[\frac{1}{6(1+x^2)^3} \Bigg]_{ax}^{a} = B$$

$$\frac{1}{2} \int_{ax}^{a} \frac{x}{(1+x^2)^3} = \frac{1}{2} \Bigg[-\frac{1}{4(1+x^2)^2} \Bigg]_{ax}^{a} = C$$

$$A = -\frac{1}{16} \Bigg[\frac{1}{(1+r^2/a^2)^4} \Bigg]_r^a = -\frac{1}{16} \Bigg[\frac{1}{16} - \frac{1}{(1+r^2/a^2)^4} \Bigg]$$

$$B = \Bigg[\frac{1}{6(1+\frac{r^2}{a^2})^3} \Bigg]_r^a = \frac{1}{6} \Bigg[\frac{1}{8} - \frac{1}{(1+\frac{r^2}{a^2})^3} \Bigg]$$

$$C = -\frac{1}{8} \Bigg[\frac{1}{(1+\frac{r^2}{a^2})^2} \Bigg]_r^a = -\frac{1}{8} \Bigg[\frac{1}{4} - \frac{1}{(1+\frac{r^2}{a^2})^2} \Bigg]$$

$$P = \frac{a^2 C_1^2 \mu_0}{2}\left[A + B + C \right]$$

$$= \frac{a^2 C_1^2 \mu_0}{2}\left[-\frac{1}{16^2} + \frac{1}{16(1+r^2/a^2)^4} + \frac{1}{48} - \frac{1}{6(1+\frac{r^2}{a^2})^3} - \frac{1}{32} + \frac{1}{8(1+\frac{r^2}{a^2})^2} \right]$$

$$\boxed{P(r) = \frac{a^2 C_1^2 \mu_0}{2}\left[-1.43\times10^{-2} + \frac{1}{16(1+r^2/a^2)^4} - \frac{1}{6(1+\frac{r^2}{a^2})^3} + \frac{1}{8(1+\frac{r^2}{a^2})^2} \right]}$$

for $P=0$ at $r=a$ (otherwise add a $P(a)$)

In SI units

Rewriting our answers, we've

$$I(r) = a^2 \pi C_1 \left[\frac{\left((1+\frac{r^2}{a^2}) - 1\right)^2}{2(1+r^2/a^2)^2} \right], \quad \text{w/ } I_{total} = \frac{a^2 \pi C_1}{8}$$

$$B_\theta(r) = \frac{\mu_0}{2\pi r} I(r)$$

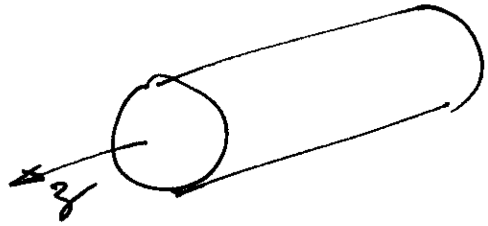
$$P(r) = \frac{C_1 I_{tot} \mu_0}{\pi \; 4}\left[\frac{2(1+\frac{r^2}{a^2})^2 - \frac{8}{3}(1+\frac{r^2}{a^2}) - 0.2288(1+\frac{r^2}{a^2})^4 + 1}{(1+r^2/a^2)^4} \right]$$

Notes:

→ A easier way to do these integrals would be to make a $x = (1 + r^2/a^2)$ substitution!

→ See next page for numerics & graphs →

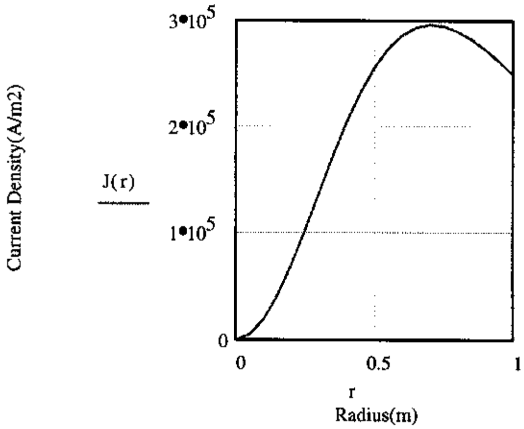
problem 3a)



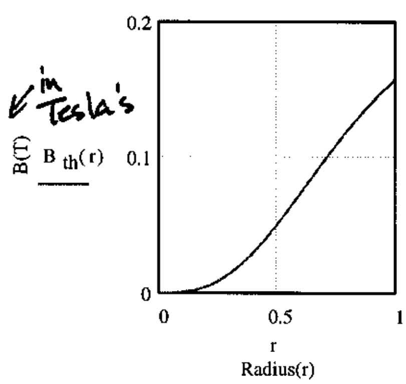
$a := 1\ \mathrm{m} \qquad c := 2 \cdot 10^{6}\ \dfrac{\mathrm{A}}{\mathrm{m}^2} \qquad r := 0, .05 .. 1\ \mathrm{m} \qquad J(r) := c \cdot r^2 \cdot \dfrac{a^4}{\left(a^2 + r^2\right)^3}$

$$J(r) := c \cdot a^4 \cdot \frac{r^2}{\left(a^2 + r^2\right)^3}$$

We take the case a = 1m = edge of plasma

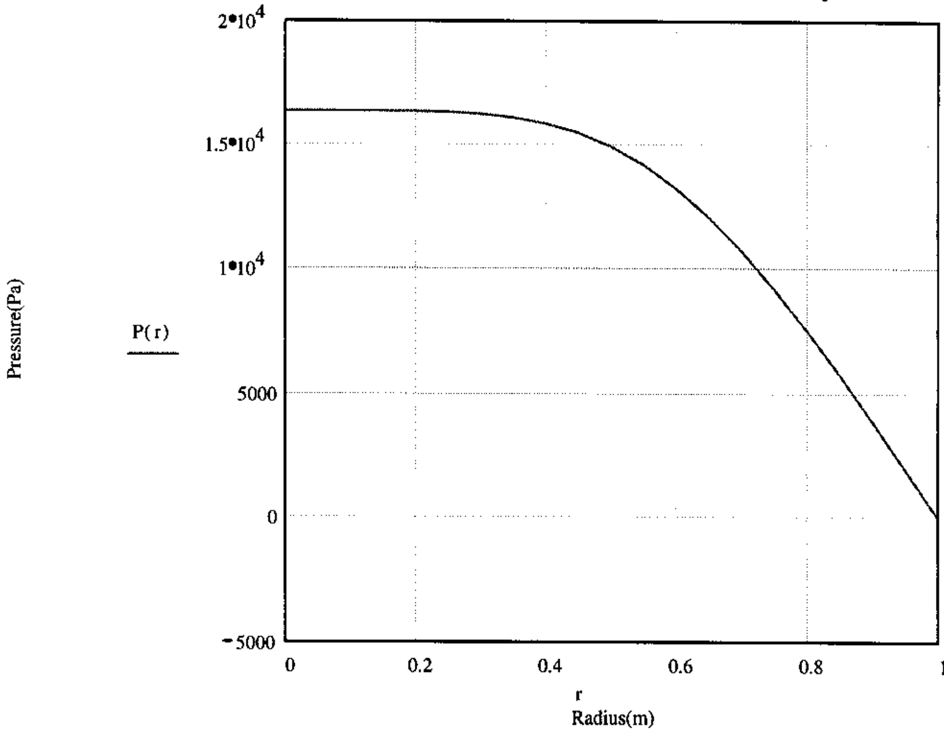


$pi := 3.1456 \qquad \mu := 4 \cdot pi \cdot 10^{-7}\ \dfrac{\mathrm{H}}{\mathrm{m}} \qquad B_{th}(r) := \dfrac{\mu \cdot \int_0^r J(r) \cdot 2 \cdot pi \cdot r\, dr}{2\ pi \cdot r}$



$$P(r) := \int_r^a J(r) \cdot B_{th}(r)\,dr$$

(P=0 at r=a)
(Just a constant/factor otherwise)



3b) Extra credit:

→ One explaination:

$J(r)$ only vanishes at $r \to \infty$.

- Hence, this is not really confinement, since it indicates a physical plasma in <u>all space</u>!