

Fall Term 2002
Introduction to Plasma Physics I

22.611J, 6.651J, 8.613J

Problem Set #5

1. *Polarization Drift “Conductivity”:*

In class we derived the dispersion relations for the Alfvén wave from ideal MHD. Show that the same wave results from a simple model of plasma response where the perpendicular current results from the ion polarization drift (including electrons doesn't change the answer to leading order in the mass ratio). To be more mathematical, show that the conductivity resulting from the ion polarization drift response, for appropriately oriented magnetic and electric wave perturbations, is,

$$\boldsymbol{\sigma} = -i \frac{\omega \rho c^2}{B_0^2} \mathbf{I}$$

where, \mathbf{I} , is the identity tensor. This result seems to imply that MHD *contains* polarization drift effects. How can you reconcile this with the ideal MHD Ohm's law which implies that the fluid velocity is equal to the, $\mathbf{E} \times \mathbf{B}$, drift (and nothing else!)?

2. *Wave Packets and Group Velocity:*

Show that a Gaussian envelop wave packet,

$$A(x) = \text{Re} \left[\exp(ik_0x) \exp(-x^2/2\sigma^2) \right]$$

can be written as a Gaussian superposition of plane waves,

$$A(x) = \text{Re} \left[\frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk \exp(ikx) \exp\left(-\frac{\sigma^2(k - k_0)^2}{2}\right) \right]$$

You can do this by completing the square in the exponent to yield a simple integral over a Gaussian, which, for fixed, x , can be evaluated using,

$$\frac{\sqrt{\pi}}{a} = \int_{-\infty}^{+\infty} dx \exp(-a^2x^2)$$

If now the individual waves, $\sim \exp(ikx)$ evolve in time according to, $\sim \exp(-i\omega(k)t)$, one can write the complete space-time wave packet as,

$$A(x, t) = \text{Re} \left[\frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk \exp(ikx - i\omega(k)t) \exp\left(-\frac{\sigma^2(k - k_0)^2}{2}\right) \right]$$

Make a drawing of the wavepacket, assuming, $k_0 \gg 1/\sigma$. For a narrow wave packet, i.e. such that, $\sigma d \ln \omega(k)/dk \ll 1$, the frequency can be approximated by,

$$\omega(k) \simeq \omega(k_0) + \frac{\partial \omega}{\partial k} \Big|_{k=k_0} (k - k_0) + \dots$$

Use this approximation for the frequency to show that that wave packet envelope moves without distortion at the group velocity,

$$v_g = \left. \frac{\partial \omega}{\partial k} \right|_{k=k_0}$$

Extra Credit: Show that higher order terms,

$$\sim \left. \frac{\partial^2 \omega}{\partial k^2} \right|_{k=k_0} (k - k_0)^2$$

contribute to dispersion or spreading of the wave packet and that therefor if,

$$\left. \frac{\partial^2 \omega}{\partial k^2} \right|_{k=k_0} = 0$$

as is the case for light waves in vacuum, for example, then wave packets propagate without dispersion.

3. *Ion Acoustic Waves:*

Derive the dispersion relation for low frequency waves in an unmagnetized plasma using a simple model of the plasma response wherein:

- (a) The electrons behave adiabatically,

$$\tilde{n}_e \simeq n \frac{e\tilde{\phi}}{T_e}$$

- (b) The ions behave like free particles responding to the electric field
 (c) The overall response is quasi-neutral, i.e.

$$\tilde{n}_e \simeq \tilde{n}_i$$

Show that such waves obey a dispersion relation,

$$\omega^2 = k^2 c_{IA}^2$$

where, $c_{IA} \equiv \sqrt{T_e/m_i}$, is the ion acoustic speed (sometimes called ion sound speed).