Fall Term 2002
Introduction to Plasma Physics I

22.611J, 6.651J, 8.613J

Problem Set #6

1. Prove the following identities, both by direct integration and use of Poisson's equation:

$$\begin{aligned}\int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}} \frac{1}{|\mathbf{x}|} &= \frac{4\pi}{k^2} \\ \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}} \frac{\partial}{\partial \mathbf{x}} \frac{1}{|\mathbf{x}|} &= \frac{4\pi i}{k^2}\mathbf{k}\end{aligned}$$

2. Evaluate,

$$f(\mathbf{x}) = \frac{1}{(2\pi)^3} \int d^3k e^{-i\mathbf{k}\cdot\mathbf{x}} \frac{4\pi}{k^2 + k_0^2}$$

3. *Maxwellian Distribution from Entropy Maximization.* Show that the Maxwellian is what results from achieving maximum entropy using the non-equilibrium formula,

$$S = -\int d^3v f \ln f$$

Note that we are assuming a homogeneous system, so that you only need determine the velocity dependence of the distribution, f. You need to perform a functional variation of, S, with respect to variations in the distribution, $f \to \overline{f} + \delta f$, (i.e. $\delta S = \delta f \cdot \partial S / \partial f$), while maintaining the constraints of constant particle number and constant energy,

$$\begin{aligned}N &= \int d^3v f \\ E &= \int d^3v \frac{1}{2} m v^2 f\end{aligned}$$

You then compute the distribution, $\overline{f}$, such that, $\delta S = 0$, for all, δf. This can be done by the method of Lagrange multipliers, using the variational form,

$$G = S + \alpha N + \beta E$$

and calculating the multipliers, α, and, β, from the constraints once the form of the distribution, f, is known.

4. *Vlasov Mush Transformation*: Discreteness can be eliminated by sub-dividing particles indefinately while preserving the charge and mass. This sub-division can be described by the tranformation,

$$\begin{aligned}q &\to \frac{q}{N} \\ m &\to \frac{m}{N} \\ n &\to Nn\end{aligned}$$

where, N, is the number of subdivisions, and, q, m, and, n, are particle charge, mass and density. Show that the Vlasov equation is invariant under this subdivision, while the discreteness parameter,

$$\frac{1}{n\lambda_D^3}$$

is not. Furthermore show that,

$$\lim_{N\to\infty}\frac{1}{n\lambda_D^3}=0$$

Thereby proving that the Vlasov equation becomes exact in the limit of zero discreteness.

5. Prove the *Final Value Theorem* for *Laplace Transforms*. Recall our convention for the transform and its inverse,

$$\begin{aligned} g_\omega &= \int_0^\infty dt e^{i\omega t} g(t) \\ g(t) &= \frac{1}{2\pi}\int_L d\omega e^{-i\omega t} g_\omega \end{aligned}$$

where, L, denotes the Laplace inversion contour residing above all singularities of, g_ω. (In other words, g_ω, is analytic for, $\operatorname{Im}\omega > L$). The Final Value Theorem states,

$$\lim_{t\to\infty} g(t) = \lim_{\omega\to 0}\left(-i\omega g_\omega\right)$$

This theorem only applies to *stable* systems, of course. Elaborate on the meaning of “stable” as applied to this system, and indicate any changes in the theorem needed to account for systems that oscillate indefinitely without decay.