Fall Term 2002
Introduction to Plasma Physics I

22.611J, 6.651J, 8.613J

Problems for study

1. *Collision Operator Properties.* The Coulomb collision operator for like-particle collisions in Landau form is given by,

$$\mathcal{C}(f,f) = \Gamma \frac{\partial}{\partial \mathbf{v}} \cdot \int d^3v' \mathbf{U}(\mathbf{v}-\mathbf{v}') \cdot \left(\frac{\partial}{\partial \mathbf{v}} - \frac{\partial}{\partial \mathbf{v}'} \right) f(\mathbf{v}) f(\mathbf{v}')$$

with,

$$\begin{aligned} \mathbf{U}(\mathbf{v}-\mathbf{v}') &= \frac{1}{|\mathbf{v}-\mathbf{v}'|} \left(\mathbf{I} - \frac{(\mathbf{v}-\mathbf{v}')(\mathbf{v}-\mathbf{v}')}{|\mathbf{v}-\mathbf{v}'|^2} \right) \\ \Gamma &= \frac{2\pi q^4 \ln \Lambda}{m^2} \end{aligned}$$

Prove the conservation laws for the collision operator:

Particle Conservation	$0 = \int d^3v \mathcal{C}(f,f)$
Momentum Conservation	$0 = \int d^3v m \mathbf{v} \mathcal{C}(f,f)$
Energy Conservation	$0 = \int d^3v \frac{1}{2} m v^2 \mathcal{C}(f,f)$

2. *Two Species Collisions*: Show that the conservation laws also hold for collision between two species, but that the sum over species is now required:

Particle Conservation	$0 = \int d^3v \mathcal{C}_{\alpha\beta}(f_\alpha, f_\beta)$
Momentum Conservation	$0 = \int d^3v \left[m_\alpha \mathbf{v} \mathcal{C}_{\alpha\beta}(f_\alpha, f_\beta) + m_\beta \mathbf{v} \mathcal{C}_{\beta\alpha}(f_\beta, f_\alpha) \right]$ $0 = \int d^3v m_\alpha \mathbf{v} \mathcal{C}_{\alpha\alpha}(f_\alpha, f_\alpha)$
Energy Conservation	$0 = \int d^3v \left[\frac{1}{2} m_\alpha v^2 \mathcal{C}_{\alpha\beta}(f_\alpha, f_\beta) + \frac{1}{2} m_\beta v^2 \mathcal{C}_{\beta\alpha}(f_\beta, f_\alpha) \right]$ $0 = \int d^3v \frac{1}{2} m_\alpha v^2 \mathcal{C}_{\alpha\alpha}(f_\alpha, f_\alpha)$

Note that particle's are conserved by *each* collision operator, while momentum and energy is only conserved as a species sum. Make a statement of the physical meaning of the two separate conservation laws for each of momentum and energy. Recall that the collision operator between species, α, and, β, is given by,

$$\begin{aligned} \mathcal{C}_{\alpha\beta}(f_\alpha, f_\beta) &= \Gamma_{\alpha\beta} \frac{\partial}{\partial \mathbf{v}} \cdot \int d^3v' \mathbf{U}(\mathbf{v}-\mathbf{v}') \cdot \left(\frac{m_\beta}{m_\alpha} \frac{\partial}{\partial \mathbf{v}} - \frac{\partial}{\partial \mathbf{v}'} \right) f_\alpha(\mathbf{v}) f_\beta(\mathbf{v}') \\ \Gamma_{\alpha\beta} &= \frac{2\pi Z_\alpha^2 Z_\beta^2 e^4 \ln \Lambda}{m_\alpha m_\beta} \end{aligned}$$

and the tensor, $\mathbf{U}$, defined in terms of particle velocities, has the same definition as above.

3. *Temperature Equilibration*: Show that temperature equilibration proceeds according to a term of the form,

$$\frac{\partial}{\partial t}\frac{3}{2}nT_e = -\nu_{ei}\frac{m_e}{m_i}\sqrt{\frac{2}{\pi}}n\left(T_e - T_i\right)$$

by taking the energy moment,

$$\int d^3v\frac{1}{2}m_e v^2\mathcal{C}_{ei}\left(f_e^{\max}, f_i^{\max}\right)$$

You may use the expanded form of, $\mathcal{C}_{ei}$, obtained by assuming a Maxwellian distribution for the ions,

$$\mathcal{C}_{ei} \simeq \mathcal{C}_{ei}^L\left(f_e\right) + \mathcal{C}_{ei}^E\left(f_e\right)$$

with the Lorentz collision operator,

$$\begin{aligned}\mathcal{C}_{ei}^L\left(f_e\right) &\equiv \nu_{ei}\frac{v_e^3}{v^3}\mathcal{L}\left(f_e\right)\\ \mathcal{L} &\equiv \frac{1}{2}\frac{\partial}{\partial\mu}\left(1-\mu^2\right)\frac{\partial}{\partial\mu}\end{aligned}$$

and the energy exchange operator,

$$\mathcal{C}_{ei}^E\left(f_e\right) = \nu_{ei}\frac{m_e}{2m_i}v_e^3\left[\frac{1}{v^2}\frac{\partial}{\partial v} + \frac{T_i}{m_e v^2}\frac{\partial}{\partial v}\frac{1}{v}\frac{\partial}{\partial v}\right]f_e$$

The electron-ion collision frequency defined as,

$$\nu_{ei} \equiv \frac{4\pi ne^4\ln\Lambda}{m_e v_e^3}$$

and we have used the convention, $v_e = \sqrt{T_e/m_e}$, for the thermal velocity. *n.b. different from convention in Landau problem to make expressions cleaner. . .*

What is the relative rate of angle scattering vs. thermalization for electrons scattering off ions?