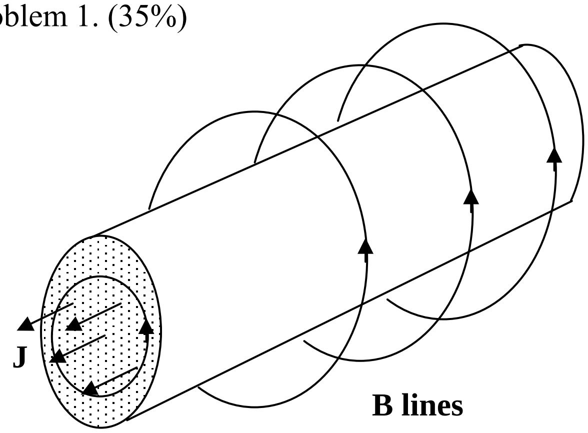
Massachusetts Institute of Technology

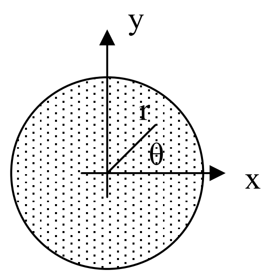
Subject 6.651J/8.613J/22.611J — R. Parker
25 October 2005 — 1.5 Hours

Mid-Term Quiz

Note: Closed book, one 8.5”x11” sheet of notes is permitted.

Problem 1. (35%)





The figure depicts a long z-pinch with current density $\vec{J} = \hat{z}J(r)$, magnetic field $\vec{B} = \hat{\theta}B(r)$, and pressure $p = p(r)$. There is no flow and the resistivity is assumed to be zero. Also $n_e = n_i$, $p_e = p_i$ and $Z_i = 1$.

a) What is the relation between p, B and J required to assure MHD equilibrium?

b) Calculate the current density $\vec{J}_D$ associated with the electron and ion drifts.

c) The relation between J and $\vec{J}_D$ in parts a) and b) is not at all obvious and in fact they are not the same. The missing link is the so-called diamagnetic current which is given by

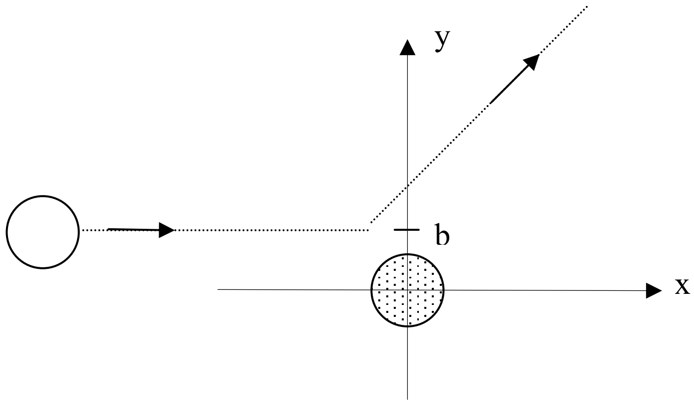
$$\vec{J}_M = -\nabla \times \vec{M}$$

where the magnetization is $\vec{M} = \frac{nw_\perp}{B}\hat{b} = \frac{p}{B}\hat{b}$. Compute $\vec{J}_M + \vec{J}_D$ and compare with $\vec{J}$ as determined in part a).

Note: In cylindrical coordinates: $\nabla \times \vec{A} = \hat{r}\left(\frac{\partial A_z}{r\partial\theta} - \frac{\partial A_\theta}{\partial z}\right) + \hat{\theta}\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \hat{z}\left(\frac{\partial A_\theta}{\partial r} + \frac{A_\theta}{r} - \frac{\partial A_r}{r\partial\theta}\right)$

Problem 2. (35%)

The figure below illustrates a collision between two hard spheres of the same radius R. In the case



shown, the shaded sphere has infinite mass and is therefore stationary during the collision. The final velocity of the moving sphere is then given by

$$\vec{V}_f = \hat{x}V_0\left(\frac{b^2}{2R^2}-1\right)+\hat{y}V_0\frac{b}{R}\sqrt{1-\frac{b^2}{4R^2}} \qquad \text{for } b<2R$$

$$\vec{V}_f = \hat{x}V_0 \qquad \text{for } b>2R$$

where b is the impact parameter and V_0 is the moving sphere's initial velocity.

a) Assume now that in the collision the incident sphere has mass m_1 while the initially stationary sphere has mass m_2. What will be the final velocity of the incident sphere?

b) A beam of such spheres with mass m_1 and initial velocity $\vec{v}_0$ is passing through a "sea" of spheres with mass m_2 and density n(m^{-3}). Calculate the frequency for slowing down ν_{sd} defined by

$$\frac{d\vec{v}_0}{dt} = -\nu_{sd}\vec{v}_0\,.$$

Problem 3. (30%)

A cold homogeneous plasma supports oscillations at the plasma frequency $\omega = \omega_p$. Perhaps surprisingly, the oscillation occurs at the same frequency regardless of the wavenumber $k = 2\pi / \lambda$. However, when pressure is included in the equations describing a plasma oscillation, the situation changes and the frequency of oscillation depends on *k*. In this problem you are asked to find the $\omega - k$ relationship for a plasma in which pressure plays a role.

a) The electron fluid equations are:

$$\frac{\partial n}{\partial t} + \nabla \bullet n\vec{v} = 0$$

$$m_e n \frac{d\vec{v}}{dt} = -en\vec{E} - \nabla p$$

$$pn^{-\gamma} = p_0 n_0^{-\gamma}$$

$$\nabla \bullet \varepsilon_0 \vec{E} = -e(n - n_{0)}$$

Let $n = n_0 + n_1$, $\vec{v} = \vec{v}_1$, $p = p_0 + p_1$ and $\vec{E} = \vec{E}_1$ where quantities with subscript 0 refer to the spatially homogeneous equilibrium and those with subscript 1 indicate small perturbations. Develop a set of linear equations sufficient to solve for the perturbed variables.

b) Assume that all variables have time-space dependence proportional to $\exp(-i\omega t + ikx)$. Determine the relation between ω and *k* that permits a nontrivial solution to the equations that you found in part a).