Massachusetts Institute of Technology

Subject 6.651J/8.613J/22.611J — R. Parker
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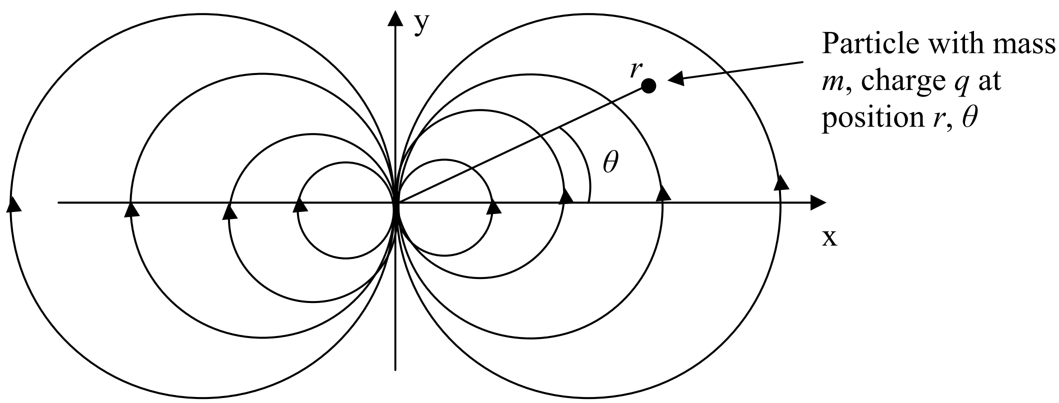
Mid-Term Quiz

Note: Closed book, one 8.5"x11" sheet of notes is permitted.

Problem 1. (35%)

The figure below shows a charged particle moving in a 2-dimensional dipole magnetic field. The particle's mass is m and its charge is q. The field is given by

$$\vec{B} = B_0 \frac{R^2}{r^2}(-\hat{r}\sin\theta + \hat{\theta}\cos\theta)$$



where R and B_0 are constants and r, θ are the usual cylindrical coordinates. The field is independent of the z-coordinate. **Hint: The field lines for this 2-D dipole are circular and are given by $r \propto \cos\theta$.**

a) At $\theta = 0$, the particle is located at r_0, its perpendicular energy is $W_{\perp 0}$ and its total energy is W_0. Determine the particle's maximum angular displacement θ_{max} (measured from the x-axis.)

b) Determine the particle's drift velocity as a function ***only*** of its θ-coordinate and other fixed parameters.

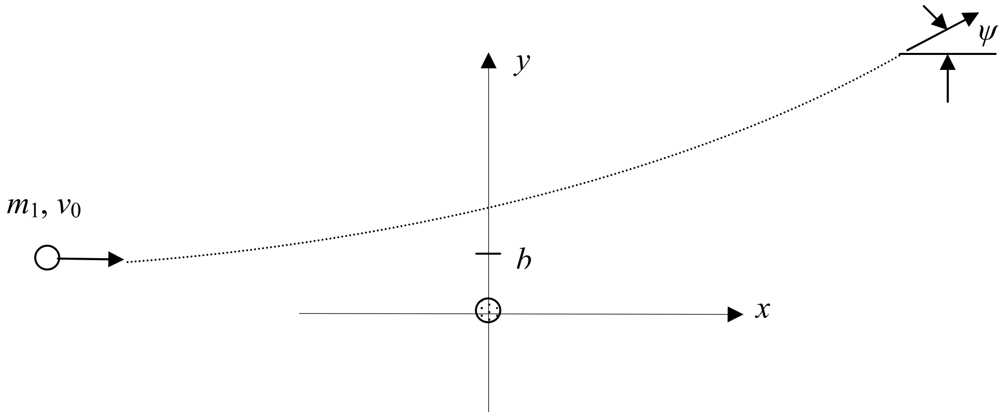
c) The particle's equation of motion parallel to the field has the form $\frac{d^2\theta}{dt^2} = f(\theta)$ where $f(\theta)$ is a function of θ and the other fixed parameters of the problem, but not of r. Determine $f(\theta)$.

Note: In cylindrical coordinates:

$$\nabla \times \vec{A} = \hat{r}\left(\frac{\partial A_z}{r\partial\theta} - \frac{\partial A_\theta}{\partial z}\right) + \hat{\theta}\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \hat{z}\left(\frac{\partial A_\theta}{\partial r} + \frac{A_\theta}{r} - \frac{\partial A_r}{r\partial\theta}\right), \quad \nabla\phi = \hat{r}\frac{\partial\phi}{\partial r} + \hat{\theta}\frac{\partial\phi}{r\partial\theta} + \hat{z}\frac{\partial\phi}{\partial z}$$

Problem 2. (35%)

The figure below illustrates a collision between two particles. In the case shown, the shaded particle



has infinite mass and is therefore stationary during the collision. The moving particle has mass m_1, speed v_0 and the impact parameter for the collision is b.

The force between the two particles is $\boldsymbol{k/r^3}$ where k is a constant and r is the distance of separation. The force is repulsive and acts along a line connecting the particles.

a) Determine the distance of closest approach, $r_{\min}$, i.e., the minimum distance between the particles during the collision event.

b) Calculate the angular deflection ψ of the incident particle and the impact parameter, b_{90}, for a 90º collision. One of the integrals on the next page should be useful to you in answering this part.

Assume now that the particle at the origin has finite mass m_2 and is stationary before the collision.

c) Calculate, in terms of ψ, the loss in x-directed momentum suffered by particle 1 as a result of the collision.

d) Consider now a beam of particles with mass m_1 and velocity $\hat{x}v_x$ injected into a "sea" of initially stationary particles with mass m_2. (The force of interaction continues to be k/r^3.) The initial rate of momentum loss of the beam particles is $\upsilon_p m_1 v_x$ where

$$\upsilon_p = n_2 \sigma_p v_x .$$

and n_2 is the density of particles with mass m_2.. The cross-section σ_p is determined by an integral of the form

$$\sigma_p = \int_0^\infty f(b)db$$

Determine $f(b)$.

Integrals:

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln\left(x + \sqrt{x^2 - a^2}\right)$$

$$\int \frac{xdx}{\sqrt{x^2 - a^2}} = \sqrt{x^2 - a^2}$$

$$\int \frac{x^2dx}{\sqrt{x^2 - a^2}} = \frac{x\sqrt{x^2 - a^2}}{2} + \frac{a^2}{2}\ln(x + \sqrt{x^2 - a^2})$$

$$\int \frac{x^3dx}{\sqrt{x^2 - a^2}} = \frac{(x^2 - a^2)^{3/2}}{3} + a^2\sqrt{x^2 - a^2}$$

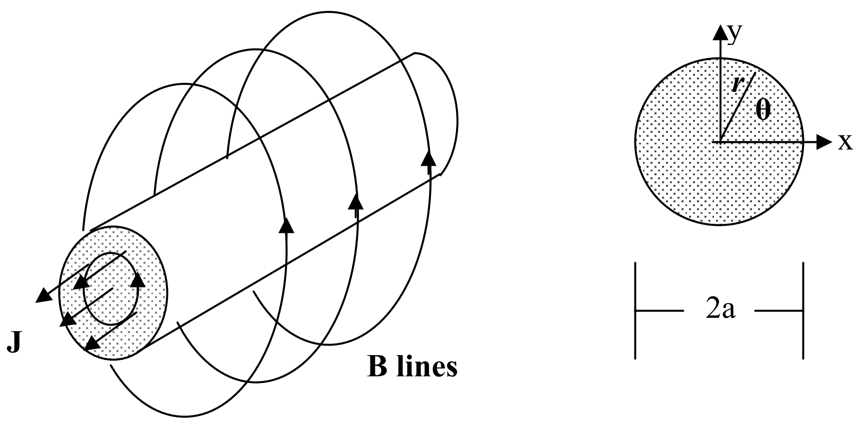
$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\cos^{-1}\left|\frac{a}{x}\right|$$

$$\int \frac{dx}{x^2\sqrt{x^2 - a^2}} = \frac{\sqrt{x^2 - a^2}}{a^2x}$$

$$\int \frac{dx}{x^3\sqrt{x^2 - a^2}} = \frac{\sqrt{x^2 - a^2}}{2a^2x^2} + \frac{1}{2a^3}\sec^{-1}\left|\frac{x}{a}\right|$$

Problem 3. (30%)

The figure below depicts a long z-pinch with magnetic field $\vec{B} = \hat{\theta}B(r)$ and pressure $p(r)$. The pinch has a circular cross-section radius and the pressure vanishes at the radius ***a***. The mass density ρ is constant, $\rho = \rho_0$. **Assume in this problem that the usual MHD equation of state $p\rho^{-\gamma}$=const is not valid for the equilibrium, and that the pressure can be determined independent of the density.**



a) The plasma is undergoing rigid-body rotation in the θ-direction, i.e., the plasma velocity is $\vec{V} = \hat{\theta}r\Omega$ where Ω is a constant. What is the relationship among $B(r)$, $p(r)$ and Ω necessary to assure MHD equilibrium?

b) The current density in the z-pinch is constant, i.e., $\vec{J} = \hat{z}J_0$ where J_0 is a constant. Solve the equation found in part a) to determine $p(r)$ assuming that $p(a) = 0$ where ***a*** is the outer radius of the z-pinch.

c) Determine the β of the plasma define as $\beta = \dfrac{(\frac{2}{a^2}\int_0^a p(r)rdr)}{\frac{B_\theta^2(a)}{2\mu_0}}$.

Possibly useful formula:

$$\vec{a}\bullet\nabla\vec{b} = \hat{r}\left(a_r\frac{\partial b_r}{\partial r} + \frac{a_\theta}{r}\frac{\partial b_r}{\partial\theta} + a_z\frac{\partial b_r}{\partial z} - \frac{a_\theta b_\theta}{r}\right) + \hat{\theta}\left(a_r\frac{\partial b_\theta}{\partial r} + \frac{a_\theta}{r}\frac{\partial b_\theta}{\partial\theta} + a_z\frac{\partial b_\theta}{\partial z} + \frac{a_\theta b_r}{r}\right)$$
$$+\hat{z}\left(a_r\frac{\partial b_z}{\partial r} + \frac{a_\theta}{r}\frac{\partial b_z}{\partial\theta} + a_z\frac{\partial b_z}{\partial z}\right)$$