Collisions
and
Transport Theory
(22.616: Class Notes)

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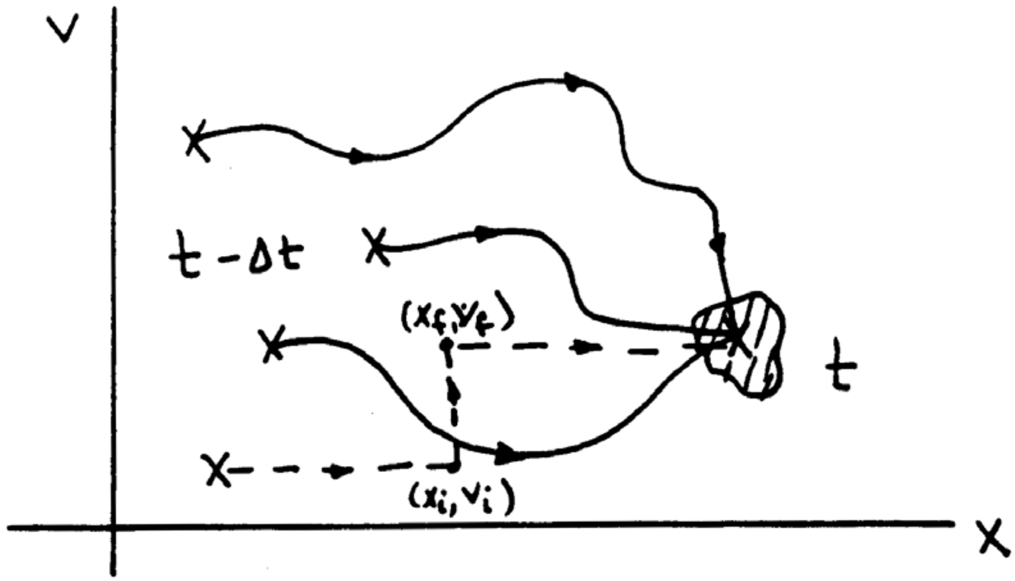
1 Probabilistic Treatment of Scattering: The Fokker-Planck Equation

In our previous study of plasmas in the Vlasov limit (particularly the heuristic derivation in Chap. 6, Sec. 2), it was convenient to interpret the distribution function, $f(\mathbf{x}, \mathbf{v}, t)$, as a phase space density. The Vlasov equation describes the evolution of the distribution function resulting from particles moving along the orbits determined by the self-consistent averaged fields. In this picture, there is a mapping of phase space through time that is one to one. For each phase space point $(\mathbf{x}, \mathbf{v})$ and time, t, there corresponds a unique point $(\mathbf{x}', \mathbf{v}')$ at time t' from which that particle originated. One has deterministic, reversible dynamics, albeit complicated in general.

Often we need to treat processes which are statistical in nature, either because we lack precise knowledge, or because we do not want to describe the'microscopic level on which the process is deterministic. Collisions are an example of the later situation where the interaction potential is known and one has, microscopically, just a complicated many body problem that is solvable exactly, in principle. In practise when treating macroscopic systems it is often preferable to describe the system on a larger scale where the microscopic binary encounters are not followed in detail but give rise to a statistical process - collisions. In fact the term "collision" implies a statistical description of this situation.

In any event on this level on description collisions and other scattering processes introduce a random, or probabilistic, nature to the particle orbits. A given phase space point does not have a unique mapping through time but rather a *probability* of arriving in a certain phase space volume at some later time. Here the probabilistic interpretation of the distribution, f, as an expected number of particles per unit phase space volume, is essential. This more general probabilistic situation is illustrated in Fig. (14.1.1) to be contrasted with Fig. (6.2.1) for the Vlasov plasma.



Evolution of phase space with scattering present. Arrows depict possible paths. The dotted path suggests the route followed by a particle undergoing a collision between $t - \Delta t$ and t.

In fact the actual case with collisions is very complicated because collisions, at this level of description, are virtually instantaneous jumps in velocity, $\mathbf{v}$, at a fixed spatial position, $\mathbf{x}$, which can not really be diagrammed as paths. The dotted path in Fig. (14.1.1) shows a trajectory suggesting an intervening collision intersecting another trajectory even though no interaction between the paths has occurred. Actually the colliding particle has jumped instantaneously from (x_i, v_i) to (x_j, v_j), thereby avoiding the intersecting path.

In any event, for this general probabilistic evolution the basic conservation law determining the distribution, f, is the *Chapman-Kolmogorov Equation*,

$$f(\mathbf{x}, \mathbf{v}, t + \Delta t) = \int d^3\Delta\mathbf{x} d^3\Delta\mathbf{v} P_{\Delta t}(\Delta\mathbf{x}, \mathbf{x}{-}\Delta\mathbf{x}; \Delta\mathbf{v}, \mathbf{v}{-}\Delta\mathbf{v}) f(\mathbf{x}{-}\Delta\mathbf{x}, \mathbf{v}{-}\Delta\mathbf{v}, t) \tag{1}$$

where,

$$1 = \int d^3\Delta\mathbf{x} d^3\Delta\mathbf{v} P_{\Delta t}(\Delta\mathbf{x}, \mathbf{x}; \Delta\mathbf{v}, \mathbf{v}) \tag{2}$$

Here P is a transition (or conditional) probability that a particle is at point $(\mathbf{x}, \mathbf{v})$ at time $t + \Delta t$, given that it was at point $(\mathbf{x}{-}\Delta\mathbf{x}; \mathbf{v}{-}\Delta\mathbf{v})$at time t. Equation (1) should, in a sense, be self-evident since it is equivalent to the well known rule for the conditional probability. In the conventional notation of probability theory, the conditional probability would be denoted $P_{\Delta t}(\mathbf{x}, \mathbf{v} \mid \mathbf{x}{-}\Delta\mathbf{x}, \mathbf{v}{-}\Delta\mathbf{v})$ and the sum, or integral would run over initial states $\mathbf{x}{-}\Delta\mathbf{x}, \mathbf{v}{-}\Delta\mathbf{v}$. The notation of eq. (1), for reasons that will be clear shortly, is chosen to give explicit functional dependence on the magnitude of the jump $\Delta\mathbf{x}$,$\Delta\mathbf{v}$. Equation (2) states that all particles end up somewhere (with probability 1), irrespective of the initial phase space position $(\mathbf{x},\mathbf{v})$, so that the scattering process conserves particles. It is written such that the starting point, $(\mathbf{x},\mathbf{v})$, is fixed and the sum runs over all possible jumps, $\Delta\mathbf{x}$,$\Delta\mathbf{v}$.

Note that the Chapman-Kolmogorov equation does not, in general, give a complete description of the system, since P is unspecified and may depend on auxiliary functions in addition to the distribution function, f. Thus eq. (1) must be augmented by equations or conditions determining the transition probability.

Note also that the Chapman-Kolmogorov equation reduces to the Vlasov equation in a singular limit where the conditional probability is taken to be a delta function along the self-consistent Vlasov trajectory, or characteristic,

$$P \to \delta\left(\mathbf{x}-\Delta\mathbf{x}-\mathbf{x}'\left(\mathbf{x},\mathbf{v},t+\Delta t;t\right)\right)\delta\left(\mathbf{v}-\Delta\mathbf{v}-\mathbf{v}'\left(\mathbf{x},\mathbf{v},t+\Delta t;t\right)\right) \tag{3}$$

Using eq.(3) in (1) gives $f\left(\mathbf{x},\mathbf{v},t+\Delta t\right)=f\left(\mathbf{x}',\mathbf{v}',t\right)$, which is essentially eq. (6.2.4), since the volume elements $d^3\mathbf{x}d^3\mathbf{v}$ and $d^3\mathbf{x}'d^3\mathbf{v}'$ are equal. This is the integral form of the Vlasov equation, expressing the constancy of f along particle orbits. The differential form is recovered by taking $\Delta t \to 0$ as outlined in chapter 6.2. This shows that the Chapman-Kolmogorov equation doesn't necessarily imply irreversible dynamics since it contains the Vlasov equation as a limiting case. In general, however, for smooth, non-singular conditional probabilities, irreversibility will be implied by eq. (1).

We now specialize to a homogeneous plasma and drop the spatial variable from eqs. (1) and (2). The basic equation is then

$$f\left(\mathbf{v},t+\Delta t\right)=\int d^3\Delta\mathbf{v}P_{\Delta t}\left(\Delta\mathbf{v},\mathbf{v}-\Delta\mathbf{v}\right)f\left(\mathbf{v}-\Delta\mathbf{v},t\right) \tag{4}$$

with

$$1=\int d^3\Delta\mathbf{v}P_{\Delta t}\left(\Delta\mathbf{v},\mathbf{v}\right) \tag{5}$$

It should be clear that the results can be generalized to include spatial scattering, as we will consider later in this chapter. For the present we have in mind velocity scattering due to Coulomb collisions and wish to reduce eq. (4) to a simpler form for such a system. In taking eq. (4) with a smooth probability, P, we are implicitly taking the coarse grained or macroscopic description of collisions as a stochastic process. We do not look on the microscopic (Klimontovich) phase-space and time scales where collisions are a reversible part of the many body trajectory resulting from two particles coming close together in their mutual Coulomb fields. By removing this fine scale information from the system, one gets a relatively simple description of collisions which can exhibit irreversible behavior.

Coulomb interactions are characterized by an infinite range of interaction, so that small angle (or small $\Delta\mathbf{v}$) scatterings dominate the scattering process. These lead to the point we discussed heuristically at the very beginning of this text, namely the divergence of the cross section and associated required cutoffs giving the so-called Coulomb Logarithm. This property of Coulomb scattering implies that the transition probability $P_{\Delta t}\left(\Delta\mathbf{v},\mathbf{v}-\Delta\mathbf{v}\right)$, will be strongly peaked as a function of the first argument about small $\mathbf{\Delta v}$. In particular, if the distribution function, f, is relatively smooth with a characteristic scale on the order of the thermal velocity, $v_T = \sqrt{2T/m}$, we expect P to be concentrated with respect to its first argument in a region, $\mathbf{\Delta v} \ll \mathbf{v}_T$. The second argument of P, namely, $\mathbf{v}-\mathbf{\Delta v}$, the particle velocity before the collision, is expected to have smooth scales, of order v_T as we will verify shortly. These properties suggest that the distribution, f and the second argument of the probability, P in the integrand of eq. (4), be Taylor expanded in powers of $\mathbf{\Delta v}$. If we also assume f has a characteristic time scale longer than Δt and expand accordingly (a more precise identification of the time scale condition will be determined below), there results,

$$\begin{aligned} f(\mathbf{v},t) + \Delta t \frac{\partial f}{\partial t}(\mathbf{v},t) &= \int d^3\mathbf{\Delta v} \{ P_{\Delta t}(\mathbf{\Delta v},\mathbf{v}) f(\mathbf{v},t) - \mathbf{\Delta v} \cdot \frac{\partial}{\partial \mathbf{v}} \left(P_{\Delta t}(\mathbf{\Delta v},\mathbf{v}) f(\mathbf{v},t) \right) \\ + \frac{1}{2} \mathbf{\Delta v \Delta v} &: \frac{\partial^2}{\partial \mathbf{v} \partial \mathbf{v}} \left(P_{\Delta t}(\mathbf{\Delta v},\mathbf{v}) f(\mathbf{v},t) \right) + \ldots \} \end{aligned} \tag{6}$$

The first terms on both sides cancel by virtue of eq. (5). The $\mathbf{\Delta v}$ integrals can then be brought inside the $\mathbf{v}$ derivatives, so that (6) becomes,

$$\begin{aligned} \frac{\partial f}{\partial t}(\mathbf{v},t) &= -\frac{\partial}{\partial \mathbf{v}} \cdot \left\{ f(\mathbf{v},t) \int d^3\mathbf{\Delta v} \frac{\mathbf{\Delta v} P_{\Delta t}(\mathbf{\Delta v},\mathbf{v})}{\Delta t} \right\} \\ + \frac{1}{2} \frac{\partial^2}{\partial \mathbf{v} \partial \mathbf{v}} &: \left\{ f(\mathbf{v},t) \int d^3\mathbf{\Delta v} \frac{\mathbf{\Delta v \Delta v} P_{\Delta t}(\mathbf{\Delta v},\mathbf{v})}{\Delta t} \right\} \end{aligned} \tag{7}$$

It is often convenient to denote the integrals over P by angular brackets and to define the coefficients of *friction*,

$$\mathbf{A} \equiv \frac{1}{\Delta t} \int d^3\mathbf{\Delta v} P_{\Delta t}(\mathbf{\Delta v},\mathbf{v}) \mathbf{\Delta v} \equiv \left\langle \frac{\mathbf{\Delta v}}{\Delta t} \right\rangle \tag{8}$$

which is a vector and *diffusion*,

$$\mathbf{D} \equiv \frac{1}{2} \frac{1}{\Delta t} \int d^3\mathbf{\Delta v} P_{\Delta t}(\mathbf{\Delta v},\mathbf{v}) \mathbf{\Delta v \Delta v} \equiv \frac{1}{2} \left\langle \frac{\mathbf{\Delta v \Delta v}}{\Delta t} \right\rangle \tag{9}$$

which is a tensor.

With these definitions, equation (7)can be written,

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial \mathbf{v}} \cdot (\mathbf{A} f) + \frac{\partial^2}{\partial \mathbf{v} \partial \mathbf{v}} : (\mathbf{D} f) \tag{10}$$

which is the standard form of the Fokker-Planck equation.

Do we have an irreversible equation at this point?

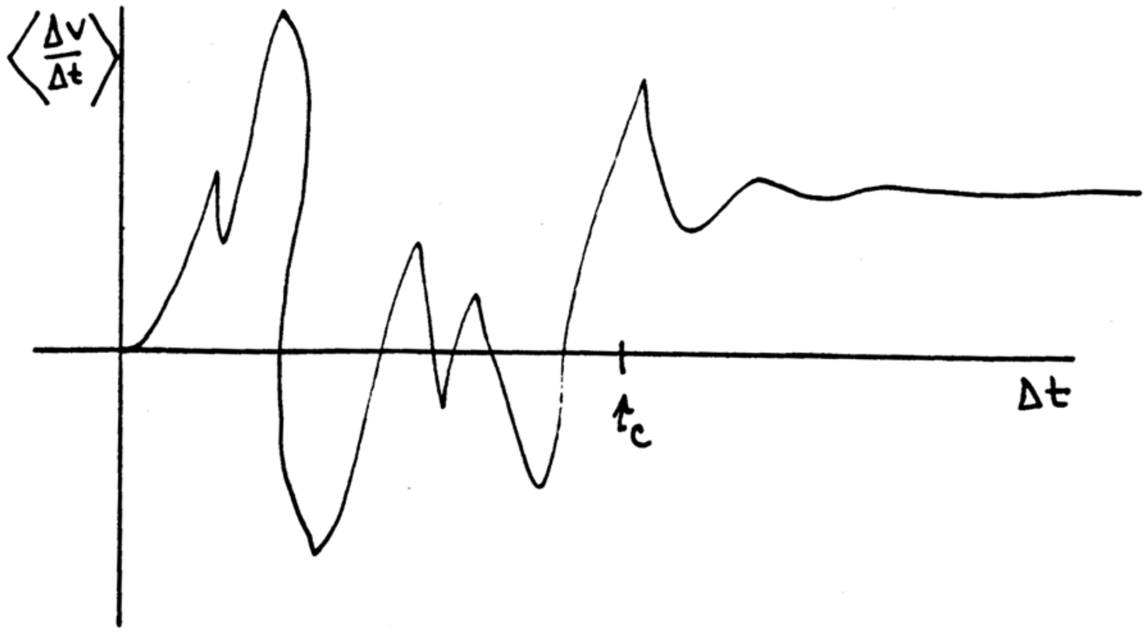
This is a very important feature to understand as we develop statistical theories. Certainly collisions conjure up notions of irreversibility. We have discussed this issue at length already and the reader may have assumed that such a statistical theory is irreversible. But statistics do not imply irreversibility as we know from the Vlasov equation. And then if equation (10) is irreversible where in the preceding developements was it put in?

The first question has been answered already really, since we know equation (10) contains the Vlasov equation as a special case. One should recognize that the Fokker-Planck equation, per se, needn't give irreversible behavior. If we had used the singular probability of eq. (3) to compute the coefficients, the diffusion tensor, $\mathbf{D}$, would have been found to be zero and $\mathbf{A} = -\frac{q}{m}\left(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}\right)$, so that eq. (10) would have reduced to the homogeneous Vlasov equation. In this case the terminology friction would not really be appropriate.

Of course, in the present chapter, we have in mind friction and diffusion coefficients that are due to collisions and the equation is irreversible. The friction term represents a slowing down, or drag, on particles due to collisions, and the diffusion term produces a spreading of the distribution function. But we still want to know where the irreversibility was put in. The interested student might attempt to answer this question for himself at this point. We will return to it in the text shortly.

Now one might ask why the expansion in $\Delta v/v_T$ in eq. (6) was stopped after second order or why the second order term was retained at all since the first order term is presumably the biggest. It turns out, as we will show in the next section, that the retained terms are of the same order - actually divergent and requiring the introduction of cutoffs to give terms that are large in the Coulomb logarithm. The third and higher order terms are not divergent and therefore smaller by a factor of order the Coulomb logarithm than the retained terms. The Fokker-Planck equation consists of the "dominant" terms in the relaxation of a quiescent plasma when $\ln\Lambda \equiv \ln\left(n\lambda_D^3\right) \gg 1$.

The expansion in $\frac{\Delta t}{f}\frac{\partial f}{\partial t} \ll 1$ is made possible by the very short time that characterizes the retention of correlation by particles undergoing a collision. The correlation will persist for time on the order of the interaction time, or $\tau_c \simeq \lambda_D/v_T$, where λ_D, the Debye length, is the approximate range of interaction. Thus the transition probability, $P_{\Delta t}(\Delta\mathbf{v}, \mathbf{v})$, will be meaningful only for time intervals $\Delta t \gtrsim \tau_c \simeq 1/\omega_p$. For intervals less than this, P and the moments $\left\langle\frac{\Delta\mathbf{v}}{\Delta t}\right\rangle$, $\left\langle\frac{\Delta\mathbf{v}\Delta\mathbf{v}}{\Delta t}\right\rangle$ would exhibit fluctuations corresponding to the individual collisions and signifying the failure of a probabilistic treatment of collisions on so short a time scale.



Behavior of the friction coefficient showing fluctuations prior to the correlation time, τ_c. When the limit of small Δt is taken in eq. (6) then, we are restricted by $\Delta t \gtrsim \tau_c$ to insure validity of the probabilistic description. This is a limit to the accuracy imposed by the statistical description of collisions. It prohibits going to the limit, $\Delta t \to 0$, as one does in deriving the Vlasov equation. Consequently the neglected terms in equation (6) are finite and there is an unavoidable error. It is here that irreversibility is introduced. By not allowing Δt to go to the very small values where the complex reversibility occurs, and at the same time freezing the evolution of the distribution function, we remove the linkage between the distribution function and the microscopic dynamics - giving irreversibility.

Since the time scale of variation of the distribution, f, is characterized by ν^{-1}, the collision time, the order of approximation involved here is $\nu\tau_c \simeq \nu/\omega_p$. For electrons this parameter is,

$$\frac{\nu_{ei}}{\omega_{pe}} = \frac{4\pi n e^4 \ln\Lambda}{m_e^2 v_{Te}^3}\left(\frac{m_e}{4\pi n e^2}\right)^{1/2} = \frac{\ln\Lambda}{4\pi n \lambda_D^3} \ll 1$$

For the ions it is comparable. For typical fusion plasmas, interstellar gas and magnetospheric plasmas this is on the order of 10^{-6} or less. The expansion leading to a first order time derivative is thus extremely well justified. In denser plasma conditions like the center of the sun, however, $n\lambda_D^3$, can be of order unity and this description is questionable.

Finally, although we have emphasized the specific properties of Coulomb collisions for the derivation in this section, the Fokker-Planck equation has a much wider applicability. The essential points are that the correlation time for the stochastic process be small compared to the time scales of interest and the jumps in velocity, $\mathbf{\Delta v}$, (or some generalized coordinate) be small compared to the velocity (or other) scales characterizing the distribution function, f.

2 Fokker-Planck Coefficients for Coulomb Collisions

The Fokker-Planck coefficients due to collisions will have additive contributions from each species in the plasma. To generalize eq. (10) for a multispecies plasma we define friction and diffusion coefficients,

$$\mathbf{A}^{ab} \equiv \text{Friction on species } a \text{ due to collisions with species } b$$
$$\mathbf{D}^{ab} \equiv \text{Diffusion of species } a \text{ due to collisions with species } b$$

Particle a, the scatteree, is often termed the *test particle*, while particle b, the scatter, is called a *field particle*. A distribution of test particles will evolve in time due to collisions with all possible field particles, including particles of the same species acting as scatterers. Thus we define a *collision operator*, $\mathcal{C}_{ab}$, for collisions of test particles a off field particles b,

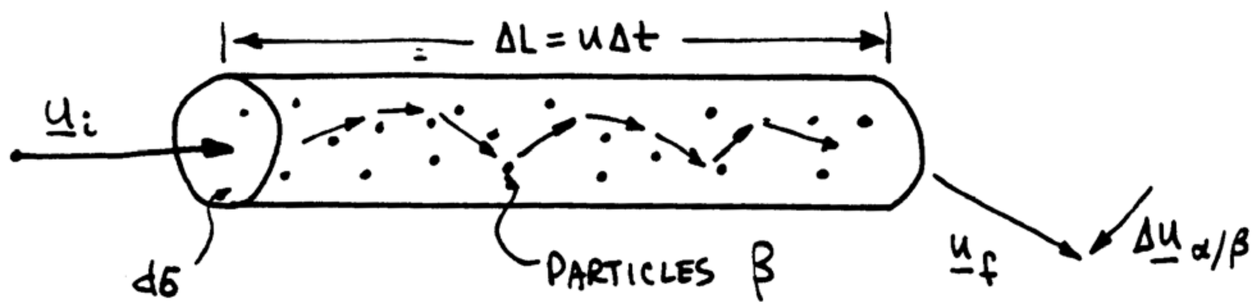
$$\mathcal{C}_{ab}\left(f_a, f_b\right) = -\frac{\partial}{\partial \mathbf{v}} \cdot \left(\mathbf{A}^{ab} f_a\right) + \frac{\partial^2}{\partial \mathbf{v} \partial \mathbf{v}} : \left(\mathbf{D}^{ab} f_a\right) \tag{11}$$

The evolution of the distribution function of a given species due to all collisions is then,

$$\frac{\partial f_a}{\partial t} = \mathcal{C}_a\left(f_a\right) = \sum_b \mathcal{C}_{ab}\left(f_a, f_b\right)$$

where the full operator is denoted by a single subscript.

We now evaluate the friction coefficient $\mathbf{A}^{ab}$ by considering a given labeled particle, a, the test particle, colliding with a background distribution of field particles, b, as depicted in the figure.



Schematic of test particle a scattering from a distribution of field particles, b.

Take a large number of field particles, all having the same velocity $\mathbf{v}'$, but distributed in space. In a frame moving at velocity $\mathbf{v}'$, one has the situation in the figure, where, $\mathbf{u}_i = \mathbf{v} - \mathbf{v}'$, is the relative velocity between the test particle and all the field particles before any collision has occurred. The field particles present an effective area, $d\sigma = \sigma\left(\theta\right) \sin\theta d\theta d\phi$, where $\sigma\left(\theta\right)$ is the differential cross section, for scattering of the test particles, so that in a time Δt the number of scatters will be,

$$dN_b = d\sigma u \Delta t f_b\left(\mathbf{v}'\right) d^3\mathbf{v}'$$

When the interval Δt is large compared to the interaction time, $\tau_c \simeq 1/\omega_{pe}$, we expect the statistical average over the conditional probability, $P_{\Delta t}$, defining the friction coefficient in eq. (8), to be equivalent to a single arithmetic sum over the scatters. Thus,

$$\begin{aligned} \mathbf{A}^{ab} &\equiv \left\langle \frac{\Delta \mathbf{v}_{ab}}{\Delta t} \right\rangle \simeq \int dN_b \frac{\Delta \mathbf{v}_{ab}}{\Delta t} \\ &= \int d^3\mathbf{v}' f_b\left(\mathbf{v}'\right) \int d\sigma u \Delta \mathbf{v}_{ab} \end{aligned} \tag{12}$$

and, similarly,

$$\mathbf{D}^{ab} = \frac{1}{2} \int d^3\mathbf{v}' f_b\left(\mathbf{v}'\right) \int d\sigma u \Delta \mathbf{v}_{ab} \Delta \mathbf{v}_{ab} \tag{13}$$

Expressions (12) and (13) assume that the collisions are statistically independent events, or that the test and field particles become uncorrelated between individual collisions. This is to be expected when the interaction time, $\tau_c \simeq 1/\omega_{pe}$, is short compared to the time between collisions, $1/\nu$, a condition that is satisfied when the plasma parameter, $n\lambda_D^3$, is large. In fact one expects τ_c to be of the order of the time for decay of correlation between field and test particle, and accordingly refer to τ_c alternatively, as a correlation time.

This independence assumption is what introduces irreversibility into the problem for the stochastic process of particle scattering. As we saw first in the Landau problem and again in the formal derivation of the collision operator in Chapter ?, the development of irreversible behavior from reversible underlying dynamics is a subtle problem, requiring, in the final analysis, some kind of *truly random* interference by an external agency. The implications and profundity of this step were well understood by Boltzmann in his first derivation of the general collision operator in 1876. He emphasized it as the "STOSSAHLANSATZ", which in German, means stochastic assumption.

It remains to use the kinematics of Coulomb collisions to evaluate the cross section and velocity increment. This is most easily done in the center of mass frame for fixed test and field particle velocities before collision, $\mathbf{v}$ and $\mathbf{v}'$, respectively. The center of mass dynamics are easily found from the individual equations of motion,

$$\begin{aligned} m_a \frac{d^2}{dt^2}\mathbf{x}_a &= e^2 Z_a Z_b \frac{\mathbf{x}_a - \mathbf{x}_b}{\left|\mathbf{x}_a - \mathbf{x}_b\right|^3} \rightarrow \left[\frac{e^2 Z_a Z_b}{4\pi\epsilon_0} \frac{\mathbf{x}_a - \mathbf{x}_b}{\left|\mathbf{x}_a - \mathbf{x}_b\right|^3}\right]_{MKS} \\ m_b \frac{d^2}{dt^2}\mathbf{x}_b &= e^2 Z_a Z_b \frac{\mathbf{x}_b - \mathbf{x}_a}{\left|\mathbf{x}_a - \mathbf{x}_b\right|^3} \rightarrow \left[\frac{e^2 Z_a Z_b}{4\pi\epsilon_0} \frac{\mathbf{x}_b - \mathbf{x}_a}{\left|\mathbf{x}_a - \mathbf{x}_b\right|^3}\right]_{MKS} \end{aligned}$$

The arrow denotes the same equation written in MKS units. Adding these equations gives,

$$0 = \frac{d^2}{dt^2}\left(m_a \mathbf{x}_a + m_b \mathbf{x}_b\right) = \left(m_a + m_b\right) \frac{d^2}{dt^2}\mathbf{X}_{CM} \tag{14}$$

where $\mathbf{X}_{CM} \equiv \left(m_a \mathbf{x}_a + m_b \mathbf{x}_b\right) / \left(m_a + m_b\right)$ is the center of mass coordinate and eq. (14) shows that the center of mass velocity, $\mathbf{V}_{CM} \equiv \left(m_a \mathbf{v}_a + m_b \mathbf{v}_b\right) / \left(m_a + m_b\right)$, is conserved. Note also that the individual particle coordinate is related to the center of mass coordinate, $\mathbf{X}_{CM}$, by,

$$\mathbf{x}_a = \mathbf{X}_{CM} + \frac{\mu}{m_a}\mathbf{x}$$

where, $\mathbf{x} = \mathbf{x}_a - \mathbf{x}_b$, is the relative coordinate. By subtracting the equations of motion one obtains the equation of motion for the relative coordinate,

$$\frac{d^2}{dt^2}\mathbf{x} = \frac{Z_a Z_b e^2}{\mu}\frac{\mathbf{x}}{|\mathbf{x}|^3} \rightarrow \left[\frac{Z_a Z_b e^2}{4\pi\epsilon_0 \mu}\frac{\mathbf{x}}{|\mathbf{x}|^3}\right]_{MKS}$$

where $\mu \equiv m_a m_b / (m_a + m_b)$ is the reduced mass. Thus we have a one-body problem for a particle of mass μ, scattering from a fixed scattering center. Also,

$$\mathbf{v}_a = \mathbf{V}_{CM} + \frac{\mu}{m_a}\mathbf{u}$$

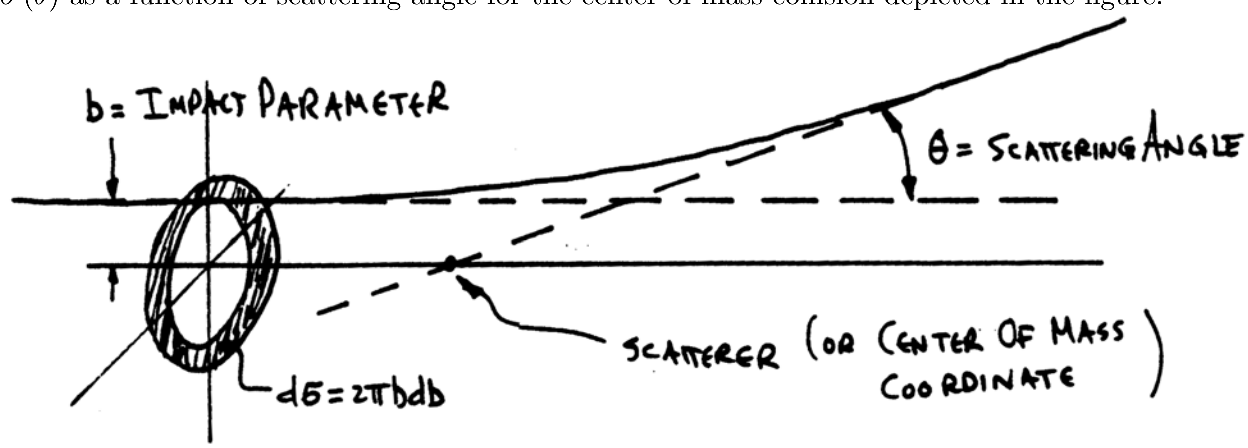
where $\mathbf{u} = \mathbf{v}_a - \mathbf{v}_b$ is the relative velocity. Since $\mathbf{V}_{CM}$ is conserved in the collision,

$$\Delta\mathbf{v}_a = \frac{\mu}{m_a}\Delta\mathbf{u}$$

and the friction coefficient can be expressed in terms of center of mass quantities,

$$\mathbf{A}^{ab} = \int d^3\mathbf{v}' \frac{\mu}{m_a} f_b(\mathbf{v}') \int d\sigma u \Delta\mathbf{u} \tag{15}$$

To evaluate eq. (15), we need to know the velocity increment, $\Delta\mathbf{u}$, and differential cross section $\sigma(\theta)$ as a function of scattering angle for the center of mass collision depicted in the figure.



Collision in center of mass frame showing trajectory of relative coordinate.

The trajectory in the figure is found by computing the unbounded orbit in a Coulomb field and is well known from elementary mechanics [2],

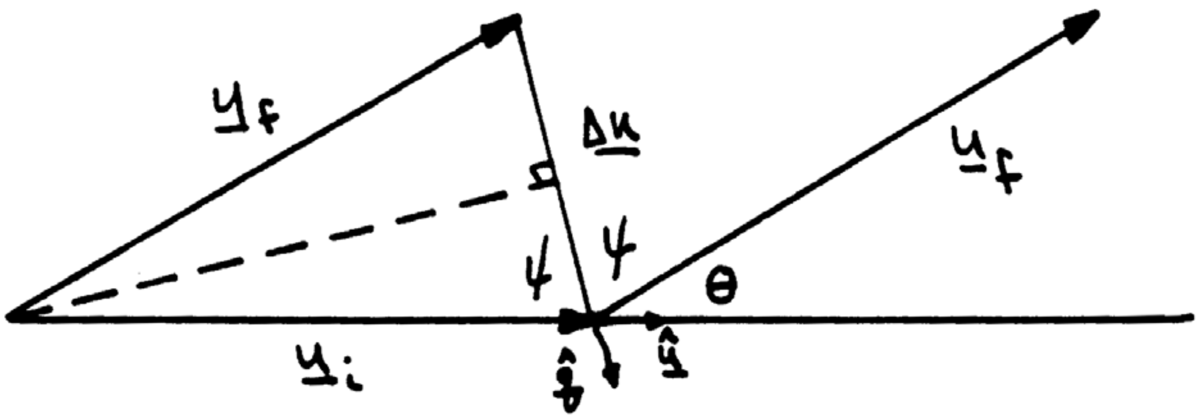
$$\tan\frac{\theta}{2} = \frac{Z_a Z_b e^2}{\mu u^2 b} \rightarrow \left[\frac{Z_a Z_b e^2}{4\pi\epsilon_0 \mu u^2 b}\right]_{MKS} \tag{16}$$

Using the relation between impact parameter and cross section, $d\sigma = 2\pi b db$, and the definition of the differential cross section, $2\pi b db = 2\pi\sigma(\theta)\sin\theta d\theta$, one obtains,

$$\sigma(\theta) = \frac{Z_a^2 Z_b^2 e^4}{4\mu^2 u^4 \sin^4\frac{1}{4}\theta} \rightarrow \left[\frac{Z_a^2 Z_b^2 e^4}{64\pi\epsilon_0 \mu^2 u^4 \sin^4\frac{1}{4}\theta}\right]_{MKS} \tag{17}$$

which is the *Rutherford cross section* for Coulomb scattering. This exhibits a strong peak (in fact singularity) at zero degrees, showing the dominance of small angle scatterings that was the basis of the Fokker-Planck treatment.

The velocity increment $\Delta\mathbf{u}$, can be simply determined from the geometry of the scatter shown in the figure.



Geometry of the scattering event showing relative velocities before, $\mathbf{u}_i$, and after, $\mathbf{u}_f$, the collision. The unit vectors, $\hat{\mathbf{u}}$ and $\hat{\mathbf{q}}$, point in the directions, $\mathbf{u}_i$, and $\Delta\mathbf{u}$, respectively

Since energy is conserved in the center of mass frame, $|\mathbf{u}_i| =,|\mathbf{u}_f|$ the vector, $\Delta\mathbf{u}$, bisects the angle, $\pi - \theta$, between $\mathbf{u}_i$ and $\mathbf{u}_f$. We thus have the relations,

$$\begin{aligned}\psi &= \frac{1}{2}(\pi - \theta)\\ \Delta\mathbf{u} &= -\mathbf{2}\hat{\mathbf{q}}\hat{\mathbf{q}}\cdot\mathbf{u}_i\end{aligned}$$

Writing $\hat{\mathbf{q}}$ in polar coordinates oriented with the z-axis along $\mathbf{u}$, gives,

$$\begin{aligned}\hat{\mathbf{q}} &= (\cos\psi, \sin\psi\cos\phi, \sin\psi\sin\phi)\\ &= \left(\sin\frac{\theta}{2}, \cos\frac{\theta}{2}\cos\phi, \cos\frac{\theta}{2}\sin\phi\right)\end{aligned}$$

The friction coefficient is then expressed as,

$$\begin{aligned}\mathbf{A}^{ab} &= -\int d^3\mathbf{v}'\frac{\mu}{m_a}f_b(\mathbf{v}')\int d\phi d\theta\sigma(\theta)\,2u^2\hat{\mathbf{q}}\hat{\mathbf{q}}\cdot\hat{\mathbf{u}} \quad (18)\\ &= -\int d^3\mathbf{v}'\frac{\mu}{m_a}f_b(\mathbf{v}')\int d\phi b(\theta)\,db(\theta)\,2u^2\sin\frac{\theta}{2}\left(\sin\frac{\theta}{2}, \cos\frac{\theta}{2}\cos\phi, \cos\frac{\theta}{2}\sin\phi\right) \quad (19)\end{aligned}$$

where the second form has been re-written as an integral over impact parameter. Clearly only the first component, along $\mathbf{u}$, survives. This could have been expected since scatters perpendicular to $\mathbf{u}$ occur with equal probability in either direction. Evaluating eq. (19) for small angles gives

$$\mathbf{A}^{ab} = -\frac{\mu}{m_a}\int d^3\mathbf{v}' f_b(\mathbf{v}')\,2\pi\mathbf{u}\,u\int dbb\frac{\theta^2(b)}{2}$$

or, upon use of eq. (16) in the limit $\theta \ll 1$,

$$\mathbf{A}^{ab} = -\frac{4\pi Z_a^2 Z_b^2 e^4}{\mu m_a} \int d^3\mathbf{v}' f_b\left(\mathbf{v}'\right) \frac{\mathbf{u}}{u^3} \int \frac{db}{b} \tag{20}$$

where the last integral is logarithmically infinite. This brings us to a recurring dilemma with the Coulomb interaction; the divergences that appear at both large and small impact parameters.

Such singularities appear frequently in theoretical physics. They are invariably a signal that the approximation used is breaking down and the inclusion of some additional physics is required to resolve the singularity. For example, the singularity at small b corresponds to the presumed dominance of small angle (large b) scattering throughout the analysis and then integrating over all b in the end. This can be made finite by simply retaining the full expression for $\sigma(\theta)$, eq. (17), in eq. (20); one would have an angular, θ, integral converging at $\theta = \pi$. This would give,

$$\begin{aligned} \int \frac{db}{b} \quad &\rightarrow \quad \frac{1}{4}\int_{\theta_{\min}}^{\pi} d\theta \frac{\sin\theta}{\sin^2\frac{\theta}{2}} = -\frac{1}{2}\int_{\theta_{\min}}^{\pi} \frac{d\theta \sin\theta}{1-\cos\theta} \\ &= \quad \frac{1}{2}\ln\left(1-\cos\theta\right)\Big|_{\theta_{\min}}^{\pi} \simeq \ln\frac{2}{\theta_{\min}} \end{aligned}$$

which is equivalent to cutting off the integral in eq. (20) at $b_{\min} = Z_a Z_b e^2/T$. This is often referred to as the distance of closest approach since a particle at zero impact parameter would approach the scattering center to the point where its initial kinetic energy, T, was equal to the potential energy, $e^2/b_{\min}$, and then reverse, scattering through π radians. This does not completely solve the problem at small b, however, since one should allow for the finiteness of the jumps in $\boldsymbol{\Delta}\mathbf{v}$ (which invalidate a Fokker-Planck description). To emphasize this breakdown of the present description at small b, we traditionally retain the expression (20) and introduce the cutoff, $b_{\min} = Z_a Z_b e^2/T$.

The singularity at large b is more fundamental since it is associated with the infinite range of the Coulomb interaction, not the failure of a Fokker-Planck description. In fact, the singularity appears as $\theta \rightarrow 0$ and $\boldsymbol{\Delta}\mathbf{v} \rightarrow 0$, where the Fokker-Planck equation becomes the most accurate. Here, we argue that the potential around a discrete charge in a plasma is not the *bare* Coulomb field, but rather the Debye shielded field $\approx \exp\left(-r/\lambda_D\right)/r$ with a range of order λ_D. Specifically, we introduce the cutoff, $b_{\max} = 3\lambda_D = 3\sqrt{T/4\pi n e^2}$, in eq. (20) to give,

$$\mathbf{A}^{ab} = -\frac{4\pi Z_a^2 Z_b^2 e^4 \ln\Lambda}{\mu m_a} \int d^3\mathbf{v}' f_b\left(\mathbf{v}'\right) \frac{\mathbf{v}-\mathbf{v}'}{\left|\mathbf{v}-\mathbf{v}'\right|^3}$$

where,

$$\ln\Lambda \equiv \ln\frac{b_{\max}}{b_{\min}} = \ln\left(\frac{3}{2}\frac{T^{3/2}}{\pi^{1/2} n^{1/2} Z_a Z_b e^2}\right)$$

is the Coulomb logarithm. The number Λ is equivalent to $n\lambda_D^3$, the number of particles in a Debye sphere. For fusion and magnetospheric plasmas, $\Lambda \simeq n\lambda_D^3 \simeq 10^7$, so that the Coulomb logarithm is about 16.

By a very similar calculation one can obtain the diffusion tensor, $\mathbf{D}^{ab}$. Introducing definition for the constant involving the Coulomb Logarithm,

$$\Gamma^{ab} = \frac{4\pi Z_a^2 Z_b^2 e^4 \ln \Lambda}{m_a^2} \tag{21}$$

our final expressions for the friction and diffusion coefficients become,

$$\mathbf{A}^{ab} = -\Gamma^{ab}\left(1+\frac{m_a}{m_b}\right)\int d^3\mathbf{v}' f_b\left(\mathbf{v}'\right)\frac{\mathbf{v}-\mathbf{v}'}{\left|\mathbf{v}-\mathbf{v}'\right|^3} \tag{22}$$

$$\mathbf{D}^{ab} = \frac{1}{2}\Gamma^{ab}\int d^3\mathbf{v}' f_b\left(\mathbf{v}'\right)\frac{1}{\left|\mathbf{v}-\mathbf{v}'\right|}\left(\mathbf{I}-\frac{\left(\mathbf{v}-\mathbf{v}'\right)\left(\mathbf{v}-\mathbf{v}'\right)}{\left|\mathbf{v}-\mathbf{v}'\right|^2}\right) \tag{23}$$

With these formulas as definitions of the coefficients, equation (11) gives the Coulomb collision operator for the scattering of species a from species b. Please note the Sigmar and Helander use a different constant. Their, $L^{ab} = 4\pi\Gamma^{ab}$, is defined to simplify the collision operator written in terms of Rosenbluth potentials, while ours is written to simplify the Landau form of the operator.

At this point, the Debye cutoff procedure probably raises as many questions as it answers. If two discrete charges interact at a distance of order λ_D there are about a million or so other particles intervening. What makes those two "collide" without hitting the others? How can this interaction possibly be considered two body, as assumed above, when literally millions of particles are involved? These questions are subtle and will not be answered in detail here. What one can say is that each plasma particle, simultaneously, plays two very different roles. It can on the one hand act *independently* as a discrete charge while at the same time participating *collectively* in the continuous, Vlasovian, response. The collective response accounts for the Debye shielding while in their independent role the particles undergo scattering. This is the physical interpretation of the Balescu-Lenard equation derived in Chapter ??, which properly resolves the singularity at small scattering angles while still having the divergence at large angles. To resolve both small and large angle divergences in a single equation is a formidable task, which, nonetheless, has been accomplished by Bernstein and Ahern [ ] to give a *completely convergent* operator. This operator takes the form of a Boltzmann equation type operator,

$$\mathcal{C}_{ab}\left(f_a, f_b\right) = \int d^3\mathbf{v}' \int d\phi d\theta \sin\theta\sigma\left(\theta,\phi\right)\left|\mathbf{v}-\mathbf{v}'\right|\left\{f_a\left(\mathbf{v}+\frac{\mu}{m_a}\Delta\mathbf{u}\right)f_b\left(\mathbf{v}'-\frac{\mu}{m_a}\Delta\mathbf{u}\right)-f_a\left(\mathbf{v}\right)f_b\left(\mathbf{v}'\right)\right\} \tag{24}$$

with an effective cross section, σ, that depends on the distribution function. It reduces to the usual Boltzmann equation for the Coulomb interaction at large angles and to the Balescu-Lenard equation at small angles. Finally, one can show that the dominant contributions to this collision integral give the coefficients (22) and (23) for operator (11), with corrections of order unity in comparison to $\ln\Lambda$. In this sense the Fokker-Planck equation describes the dominant effects of Coulomb scattering to an accuracy of order $1/\ln\Lambda \lesssim 5\%$. A corollary to this is that the precise numerical factors in Λ are irrelevant since the resulting corrections are comparable to the intrinsic errors in the operator.

3 Properties of the Collision Operator: The Landau form and Boltzmann's H-Theorem

The collision operator for Coulomb scattering is a very complicated object. Since the friction and diffusion coefficients are given as integrals over the field particle distribution functions, it is a nonlinear, in fact bilinear, integro-differential equation. This is not something that is amenable to analytic solutions, and numerical solutions are made expensive by the field particle integrals. This makes it essential to establish general properties of the operator to aid in making analytic expansions and testing numerical solutions. Perhaps more important, because of the length and complexity of the derivation, it is necessary to establish essential physical properties that lend more credibility to the operator's validity.

The last two sections derived the collision operator due to Coulomb collisions in the form of a Fokker-Planck equation essentially following the derivation first given by Rosen-bluth, MacDonald and Judd [3]. Actually, the same operator, although in a different form, was derived some twenty years earlier by Landau 4]. Landau started from the general Boltzmann equation with the Rutherford cross section, (17) and expanded for increments $\Delta\mathbf{v}$ small compared to the thermal velocity. This calculation is straightforward and will be left as an exercise to the enterprising student. In the present section, we derive the Landau form by transforming fhe Fokker-Planck operator, thereby demonstrating the equivalence of the two operators. The Landau form of the operator is the most useful for proving the conservation laws for particles, energy and momentum and for deriving the H-theorem.

We consider then, eq. (11), with coefficients (22) and (23). To obtain the Landau form it is convenient to define the tensor that appears in the diffusion coefficient,

$$\mathbf{U}\left(\mathbf{v},\mathbf{v}'\right) \equiv \frac{1}{u}\left(\mathbf{I}-\frac{\mathbf{u}\mathbf{u}}{u^2}\right)$$

where, $\mathbf{u}=\mathbf{v}-\mathbf{v}'$, and $u=|\mathbf{u}|$. One can easily show that the tensor, $\mathbf{U}\left(\mathbf{v},\mathbf{v}'\right)$, has the property,

$$\frac{\partial}{\partial\mathbf{v}}\cdot\mathbf{U}\left(\mathbf{v},\mathbf{v}'\right) = -\frac{\partial}{\partial\mathbf{v}'}\cdot\mathbf{U}\left(\mathbf{v},\mathbf{v}'\right) = \frac{\partial}{\partial\mathbf{u}}\cdot\mathbf{U}\left(\mathbf{v},\mathbf{v}'\right) = -2\frac{\mathbf{u}}{u^3}$$

Therefore the friction term in eq. (11) can be rewritten as,

$$\begin{aligned} -\frac{\partial}{\partial\mathbf{v}}\cdot\mathbf{A}^{ab} &= \frac{1}{2}\Gamma^{ab}\left(1+\frac{m_a}{m_b}\right)\frac{\partial}{\partial\mathbf{v}}\cdot\int d^3\mathbf{v}' f_b\left(\mathbf{v}'\right)2\frac{\mathbf{u}}{u^3}f_a\left(\mathbf{v}\right) && (25)\\ &= \frac{1}{2}\Gamma^{ab}\left(1+\frac{m_a}{m_b}\right)\frac{\partial}{\partial\mathbf{v}}\cdot\int d^3\mathbf{v}' f_b\left(\mathbf{v}'\right)\left(\frac{\partial}{\partial\mathbf{v}'}\cdot\mathbf{U}\left(\mathbf{v},\mathbf{v}'\right)\right)f_a\left(\mathbf{v}\right) \\ &= -\frac{1}{2}\Gamma^{ab}\left(1+\frac{m_a}{m_b}\right)\frac{\partial}{\partial\mathbf{v}}\cdot\int d^3\mathbf{v}'\mathbf{U}\left(\mathbf{v},\mathbf{v}'\right)\cdot\frac{\partial}{\partial\mathbf{v}'}f_b\left(\mathbf{v}'\right)f_a\left(\mathbf{v}\right) && (26)\end{aligned}$$

If we also expand the diffusion term to two terms,

$$\frac{\partial^2}{\partial\mathbf{v}\partial\mathbf{v}}:\left(\mathbf{D}^{ab}f_a\right) = \frac{\partial}{\partial\mathbf{v}}\cdot\mathbf{D}^{ab}\cdot\frac{\partial}{\partial\mathbf{v}}f_a+\frac{\partial}{\partial\mathbf{v}}\cdot\left(\left(\frac{\partial}{\partial\mathbf{v}}\cdot\mathbf{D}^{ab}\right)f_a\right)$$

the first term now being in self-adjoint form, we now notice that the last term can be written,

$$\begin{aligned} & \frac{1}{2}\Gamma^{ab}\frac{\partial}{\partial \mathbf{v}} \cdot \int d^3\mathbf{v}' f_b\left(\mathbf{v}'\right)\left(\frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{U}\left(\mathbf{v},\mathbf{v}'\right)\right) f_a\left(\mathbf{v}\right) \\ = & -\frac{1}{2}\Gamma^{ab}\frac{\partial}{\partial \mathbf{v}} \cdot \int d^3\mathbf{v}' f_b\left(\mathbf{v}'\right)\left(\frac{\partial}{\partial \mathbf{v}'} \cdot \mathbf{U}\left(\mathbf{v},\mathbf{v}'\right)\right) f_a\left(\mathbf{v}\right) \\ = & \frac{1}{2}\Gamma^{ab}\frac{\partial}{\partial \mathbf{v}} \cdot \int d^3\mathbf{v}' \mathbf{U}\left(\mathbf{v},\mathbf{v}'\right) \cdot \frac{\partial}{\partial \mathbf{v}'} f_b\left(\mathbf{v}'\right) f_a\left(\mathbf{v}\right) \end{aligned}$$

This cancels the first part of the friction term in equation (26) so that when all terms are combined, there results,

$$\mathcal{C}_{ab}\left(f_a, f_b\right) = \frac{1}{2}\Gamma^{ab}\frac{\partial}{\partial \mathbf{v}} \cdot \int d^3\mathbf{v}' \mathbf{U}\left(\mathbf{v},\mathbf{v}'\right) \cdot \left(\frac{\partial}{\partial \mathbf{v}} - \frac{m_a}{m_b}\frac{\partial}{\partial \mathbf{v}'}\right) f_b\left(\mathbf{v}'\right) f_a\left(\mathbf{v}\right)$$

which is the Landau form for the collision operator. The distribution function arguments to the collision operator are indicated to emphasize its bilinear nature.

This operator has several very important properties which give it a credibility that transcends any uncertainties one might have about the derivation. For example, with a one-species plasma, evolving according to collisions alone,

$$\frac{\partial f}{\partial t} = \mathcal{C}\left(f, f\right) \tag{27}$$

The collision operator, $\mathcal{C}$, and the distribution function, f, evolving according to eq. (27), have the following properties:

(1) **Conservation of Particles**: $\int d^3v\,\mathcal{C}\left(f,f\right) = 0$

(2) **Conservation of Momentum**: $\int d^3v\, m\mathbf{v}\,\mathcal{C}\left(f,f\right) = 0$

(3) **Conservation of Energy**: $\int d^3v \frac{1}{2}mv^2\,\mathcal{C}\left(f,f\right) = 0$

(4) **Positivity**: if $f > 0$ at $t = 0$, then $f > 0$ for all times.

(5) The **Maxwellian** distribution, f_M, is **stationary**: $\frac{\partial f}{\partial t} = 0 = \mathcal{C}\left(f_M, f_M\right)$

The collision operator annihilates the Maxwellian distribution function.

(6) All solutions approach f_M as $t \to \infty$. Thus the **Maxwellian is the only stationary solution**.

The last three properties follow from a theorem stated and proved by Boltzmann:

BOLTZMANN'S H-THEOREM

For a system evolving according to equation (27),

The entropy change is POSITIVE INDEFINITE,

$$\frac{dS}{dt} \equiv -\frac{d}{dt}\int d^3v\, f \ln f \;\; \geq 0$$

The entropy does not change if and only if f becomes Maxwellian.

In other words, the system evolves in the direction of increasing entropy until the maximum entropy state (consistent with particle, momentum and energy conservation) is achieved. That the maximum entropy state is a Maxwellian was proved in Chapter 1. The significance ofeq. (27) is that it actually evolves the distribution function to this state.

The implications of this collision operator, particularly the $H-$theorem, deserve emphasis. From a consideration of the microscopic dynamics in the collision process we have obtained an equation for the one particle distribution, f. This equation will evolve any initial distribution, in such a way that the entropy increases monotonically, until the thermal equilibrium, Maxwellian, distribution is achieved. In a sense, from a microscopic theory, we have derived the second law of thermodynamics! This cannot, unfortunately, be a "derivation" in the rigorous sense since at some point chaos from outside the laws of physics must be injected if the microscopic laws of dynamics are to yield irreversible behavior. There are many objections and paradoxes that can be raised at this procedure. The crucial step can be examined in detail and made increasingly plausible (as we found in a related problem in Chapter 6). It can never be *proven* in a mathematical sense. Nonetheless, one does have the overwhelming evidence of observation that macroscopic laws do exhibit this irreversible behavior. In the same manner as with all physical laws, the correctness of the collision operator can be verified - from experiment. The Stossahlansatz provides the key link between the microscopic reversible world and the macroscopic irreversible world of of our existence. Culminating in the famous $H-$theorem that bears his name, the theory stands as a tribute to the profundity and creative genius of the man who first made this connection, Ludwig Boltzmann. The ideas involved were so far in advance of the times in which Boltzmann lived that they engendered little acceptance and were intensely criticized. Boltzmann never knew they would form a major branch of modern physics. He died by his own hand a frustrated and depressed man.