Collisions and Transport Theory II: Electrical Conductivity - Spitzer Problem (22.616: Class Notes)

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Collision Operator: Summary

Landau form of operator for collisions of particles of type, a, off particles of type, b.

$$\begin{aligned}
\mathcal{C}_{ab}\left(f_a, f_b\right) &= \frac{1}{2}\Gamma^{ab}\frac{\partial}{\partial \mathbf{v}} \cdot \int d^3\mathbf{v}'\mathbf{U}\left(\mathbf{v}, \mathbf{v}'\right) \cdot \left(\frac{\partial}{\partial \mathbf{v}} - \frac{m_a}{m_b}\frac{\partial}{\partial \mathbf{v}'}\right) f_b\left(\mathbf{v}'\right) f_a\left(\mathbf{v}\right) \\
\Gamma^{ab} &= \frac{4\pi Z_a^2 Z_b^2 e^4 \ln \Lambda}{m_a^2} \\
\mathbf{U}(\mathbf{v}-\mathbf{v}') &= \frac{1}{|\mathbf{v}-\mathbf{v}'|}\left(\mathbf{I} - \frac{(\mathbf{v}-\mathbf{v}')(\mathbf{v}-\mathbf{v}')}{|\mathbf{v}-\mathbf{v}'|^2}\right)
\end{aligned}$$

Before moving on to transport theory it will be useful to develop some approximate versions of these operators that will make actual calculations more tractable. We will consider mostly a two-species, electron-ion plasma with ions having a single charge state. Then the electron collision operator is the sum of two operators,

$$\mathcal{C}_e = \mathcal{C}_{ei}\left(f_e, f_i\right) + \mathcal{C}_{ee}\left(f_e, f_e\right)$$

The electron - ion operator is linear in the electrons and has a very simple form in the limit of infinite ion mass. The full expression is,

$$\mathcal{C}_{ei}\left(f_e, f_i\right) = \frac{2\pi Z_i^2 e^4 \ln \Lambda}{m_e^2}\frac{\partial}{\partial \mathbf{v}} \cdot \int d^3v'\mathbf{U}(\mathbf{v}-\mathbf{v}') \cdot \left(\frac{\partial}{\partial \mathbf{v}} - \frac{m_e}{m_i}\frac{\partial}{\partial \mathbf{v}'}\right) f_e\left(\mathbf{v}\right) f_i\left(\mathbf{v}'\right) \tag{1}$$

As $m_e/m_i \to 0$ clearly the first term dominates. Also, the ion distribution function looks like a delta function, $f_i\left(\mathbf{v}'\right) \simeq n_i\delta\left(\mathbf{v}'\right)$, on the electron velocity scale. In this limit the electron ion collision operator becomes,

$$\begin{aligned}
\mathcal{C}_{ei}\left(f_e, f_i\right) &= \frac{2\pi Z_i^2 e^4 \ln \Lambda}{m_e^2}\frac{\partial}{\partial \mathbf{v}} \cdot \int d^3v'\mathbf{U}(\mathbf{v}-\mathbf{v}')n_i\delta\left(\mathbf{v}'\right) \cdot \left(\frac{\partial}{\partial \mathbf{v}} - \frac{m_e}{m_i}\frac{\partial}{\partial \mathbf{v}'}\right) f_e\left(\mathbf{v}\right) \\
&= \frac{2\pi n_i Z_i^2 e^4 \ln \Lambda}{m_e^2}\frac{\partial}{\partial \mathbf{v}} \cdot \frac{1}{v}\left(\mathbf{I} - \frac{\mathbf{v}\mathbf{v}}{v^2}\right) \cdot \frac{\partial}{\partial \mathbf{v}} f_e\left(\mathbf{v}\right)
\end{aligned}$$

which is now a completely linear operator. Note that, $n_i Z_i = n_e$.

This can be written explicitly in spherical coordinates in velocity space, utilizing the expression,

$$\frac{\partial}{\partial \mathbf{v}} = \mathbf{e}_v \frac{\partial}{\partial v} + \frac{\mathbf{e}_\theta}{v}\frac{\partial}{\partial \theta} + \frac{\mathbf{e}_\phi}{v \sin\theta}\frac{\partial}{\partial \phi}$$

We then have,

$$\left(\mathbf{I} - \frac{\mathbf{vv}}{v^2}\right) \cdot \frac{\partial}{\partial \mathbf{v}} = \mathbf{e}_\theta \frac{1}{v}\frac{\partial}{\partial \theta} + \mathbf{e}_\phi \frac{1}{v \sin\theta}\frac{\partial}{\partial \phi}$$

since the tensor $\left(\mathbf{I} - \frac{\mathbf{vv}}{v^2}\right)$ projects onto a two-dimensional space perpendicular to $\mathbf{v}$. Taking the spherical divergence then gives,

$$\begin{aligned} \frac{\partial}{\partial \mathbf{v}} \cdot \frac{1}{v}\left(\mathbf{I} - \frac{\mathbf{vv}}{v^2}\right) \cdot \frac{\partial}{\partial \mathbf{v}} &= \frac{1}{v^3}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\sin\theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial \phi^2}\right] \\ &\equiv \frac{1}{v^3} 2\mathcal{L} \end{aligned}$$

where $2\mathcal{L}$ is the angular part of the Laplacian operator. With this definition, we can write the $m_i \to \infty$ limit of $\mathcal{C}_{ei}$ as,

$$\mathcal{C}_{ei}^L (f_e) = \nu_{ei} \frac{v_e^3}{v^3} \mathcal{L}(f_e) \tag{2}$$

where,

$$\nu_{ei} = \frac{4\pi n Z_i e^4 \ln\Lambda}{m_e^2 v_{Te}^3}$$

is the electron-ion collision frequency. This parameter is closely related to the collision time used in Sigmar & Helander (and also plasma formulary),

$$\nu_{ei} = \frac{3\sqrt{\pi}}{4}\frac{1}{\tau_{ei}}$$

The numbers here appear after we do certain transport calculations. I prefer the original form that appears in the actual structure of the collision operator, particularly since the transport coefficient numbers vary from one transport coefficient to the next.

It is often convenient to write $\mathcal{L}$ in terms of the pitch angle variable, $\mu = \cos\theta$, (not to be confused with reduce mass of previous section) which results in the form,

$$\mathcal{L} = \frac{1}{2}\left[\frac{\partial}{\partial \mu}\left(1-\mu^2\right)\frac{\partial}{\partial \mu} + \frac{1}{1-\mu^2}\frac{\partial^2}{\partial \phi^2}\right]$$

We will sometimes absorb the velocity dependence into the collision frequency to write,

$$\nu_{ei}(v) \equiv \nu_{ei}\frac{v_e^3}{v^3} = \frac{4\pi n Z_i e^4 \ln\Lambda}{m_e^2 v^3}$$

which, of course, has no dependence on the temperature.

The expression (2) is called the Lorentz approximation to the collision operator. Numerically,

$$\begin{aligned}\nu_{ei} &\simeq 0.39\times 10^{-5}\frac{Z_i n\ln\Lambda}{T_e^{3/2}}\\ &= 6.2\times 10^{-5}\frac{Z_i n}{T_e^{3/2}}\end{aligned}$$

if we use the canonically agreed upon number for all plasmas of $\ln\Lambda = 16$.

The main effect of electrons scattering off of ions is angular deflection or diffusion in the pitch angle variable, μ. There is no energy exchange to this order, as one would expect from an infinite mass scattering center. To bring in the energy exchange effects which will be important for transport theory, we must continue the expansion in the mass ratio, $m_e/m_i \ll 1$. We now indicate how that is done and give the result.

First we assume that the ion distribution function is a Maxwellian plus small corrections,

$$f_i = f_{Mi} + \mathcal{O}(\delta)$$

where the $\delta \ll 1$ corrections will not be needed, provided there are no ionic flows to order $\sqrt{m_e/m_i}$. The ion temperature is different from the electron temperature but of the same order. The velocity variables in equation (1) are scaled according to,

$$\frac{v'}{v} \sim \frac{v_{Ti}}{v_{Te}} \sim \sqrt{\frac{m_e}{m_i}}$$

and equation (1) is expanded to second order in $\sqrt{m_e/m_i}$. Thus for the tensor, $\mathbf{U}$,

$$\mathbf{U}\left(\mathbf{v}-\mathbf{v}'\right) = \mathbf{U}(\mathbf{v}) + \mathbf{v}'\cdot\nabla_{\mathbf{v}}\mathbf{U}(\mathbf{v}) + \frac{1}{2}\mathbf{v}'\mathbf{v}' : \nabla_{\mathbf{v}}\nabla_{\mathbf{v}}\mathbf{U}(\mathbf{v}) + \cdots$$

The integrals in equation (1) can then be evaluated to give a correction to the infinite mass result above,

$$\mathcal{C}_{ei}(f_e) = \mathcal{C}_{ei}^{L}(f_e) + \mathcal{C}_{ei}^{E}(f_e)$$

where the order m_e/m_i, correction term,

$$\mathcal{C}_{ei}^{E}(f_e) = \nu_{ei}\frac{m_e}{2m_i}v_{Te}^3\frac{\partial}{\partial\mathbf{v}}\cdot\left[\frac{\mathbf{v}}{v^3}\left(f_e + \frac{T_i}{m_e}\frac{\mathbf{v}}{v^2}\cdot\frac{\partial}{\partial\mathbf{v}}f_e\right)\right]$$

physically represents energy exchange between electrons and ions. In spherical coordinates this part can be expressed in terms of $v = |\mathbf{v}|$ alone,

$$\mathcal{C}_{ei}^{E}(f_e) = \nu_{ei}\frac{m_e}{2m_i}v_{Te}^3\left[\frac{1}{v^2}\frac{\partial}{\partial v} + \frac{T_i}{m_e v^2}\frac{\partial}{\partial v}\frac{1}{v}\frac{\partial}{\partial v}\right]f_e$$

1 Plasma Conductivity: The Spitzer-Härm Problem

As a first example of transport theory we consider the computation of the conductivity of a fully ionized plasma, as done originally by Spitzer and Härm. Here we apply a constant electric field to an infinite homogeneous problem and compute the steady state current. We expect to find a relation of the form,

$$\mathbf{J} = \sigma \mathbf{E}$$

where σ is the conductivity. This will serve to illustrate the basic ideas of transport theory without involving the elaborate machinery and algebra required to treat spatial inhomogeneities of plasma supported by magnetic field.

The governing equation for transport theory is usually taken to be the full Vlasov equation plus collision operator,

$$\frac{\partial}{\partial t} f_e + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}} f_e - \frac{e}{m} \left(\mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B} \right) \cdot \frac{\partial}{\partial \mathbf{v}} f_e = \mathcal{C}_e(f_e)$$

where one argues that for the problems considered in transport theory the force and inhomogeneity terms of the Vlasov operator do not affect the collision operator which was, of course, derived for a homogeneous and force free medium. This should be a good approximation when the time scale of the transport phenomenon exceeds the correlation time, $\tau_c \sim \omega_{pe}$, and the spatial scale exceeds the Debye length. Excepting the condition resulting from the gyroradius scale for electrons, these conditions are well satisfied in practise. Furthermore, one can show that the effect of the electron gyroradius on the collision operator is simply to modify the argument of the Coulomb logarithm, replacing $n\lambda_D^3$ by, $n\rho_e^3$ if, $\rho_e < \lambda_D$. This introduces a negligible error.

For the conductivity problem then reduces to,

$$-\frac{e}{m} \mathbf{E} \cdot \frac{\partial}{\partial \mathbf{v}} f_e = \mathcal{C}_e(f_e) \tag{3}$$

The electric field, $\mathbf{E}$, is a given constant vector. The object is to solve for the distribution function, f_e, and then to compute the current,

$$\mathbf{J} = -e \int d^3v \mathbf{v} f_e(\mathbf{v}) \tag{4}$$

Now the complexities of the collision operator will prohibit solving this in general. Moreover, the solution in general would not give a linear relation between the current and electric field, as is implied by the notion of a conductivity. Thus we seek a restricted and approximate solution to equation (3) that gives a current linear in the field. This is a specific example of the general transport theory problem, where one finds a set of linear relations between the so-called *thermodynamic forces* and their associated *fluxes*. In the conductivity calculation the "force" is the electric field and the "flux" is the current, so we have a single coefficient.

From another point of view, transport theory concerns systems near local thermodynamic equilibrium. By expanding about thermodynamic equilibrium one solves the kinetic equation. Transport coefficients can then be computed and are found to be linearly related to the forces.

In any case to see how this goes let us first order the two sides of eq.(3). We assume velocity scales on the order of some thermal velocity so that the left hand side of equation (3) is of order $eE/m_e v_e$ times f_e while the right hand side is of magnitude ν_{ei} times f_e. Thus when the electric field strength is equal to,

$$E_R \equiv \frac{m_e v_{Te} \nu_{ei}}{e} \tag{5}$$

the two sides are equal.

The critical field given in eq.(5) is termed the *runaway* field since it corresponds to the value at which collisions cannot restrain the electrons from the accelerating force of the electric field and they speed up or "runaway" indefinitely. The behavior found from a simple fluid picture in which collisions produce a dynamical friction force opposing the electron fluid velocity, $\mathbf{V}$,

$$m_e n \left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) \mathbf{V} = eE - m_e \nu_{ei} \mathbf{V} \tag{6}$$

gives a steady state drift velocity when, $\mathbf{V} \simeq \mathbf{V}_D = e\mathbf{E}/m_e \nu_{ei}$. This equation says that when the electric field is of order the runaway field, $E \sim E_R$, the drift velocity will be of order the thermal velocity. In fact the kinetic picture is quite different as this critical field corresponds to a complete breakdown of the fluid description. We will have more to say about this phenomenon at the end of this section.

In the transport problem we are concerned with situations where the electric field is considerably smaller than the runaway field. Then the left hand side of equation (3) is of order,

$$\frac{E}{E_R} \ll 1$$

with respect to the right hand side and we can use the small parameter, E/E_R , to generate a perturbation theory. Thus we expand the distribution function in powers of E/E_R as follows,

$$f_e = f_e^0 + f_e^1 + f_e^2 + \cdots$$

Inserting into eq.(3) and equating powers of E/E_R order by order gives,

$$\textit{Zero Order} \quad : \qquad \mathcal{C}_e(f_e^0) = 0 \tag{7}$$

$$\textit{First Order} \quad : \qquad \mathcal{C}_e(f_e^1) = -\frac{e}{m}\mathbf{E} \cdot \frac{\partial}{\partial \mathbf{v}} f_e^0 \tag{8}$$

and so on. In writing equations (7, 8) we have assumed the collision operator is a function of the electron distribution alone, which is *not* true if the full electron-ion operator, $\mathcal{C}_{ei}(f_e, f_i)$ is used. To avoid this difficulty now we simply take the Lorentz limit, equation (2). The implications of this assumption, physically, will be taken up when we develop the full set of transport equations later on.

Using this operator the zero order equations, written out in detail, becomes,

$$0 = \mathcal{C}_{ei}^L \left(f_e^0 \right) + \mathcal{C}_{ee} \left(f_e^0, f_e^0 \right)$$

To solve this first note that without the Lorentz part the solution can be inferred from the H-theorem to be a Maxwellian,

$$f_e^0(\mathbf{v}) = \frac{n}{\left(2\pi v_e^2\right)^{3/2}} \exp\left(-\left|\mathbf{v}-\mathbf{V}\right|^2 / v_{Te}^2\right)$$

where $\mathbf{V}$ is the mean or drift velocity of the distribution. If the ion scattering operator is included, the Maxwellian can still solve the zero order equation but the drift velocity must be zero. Clearly a finite value drift would introduce angular velocity dependences to the zero order distribution, f_e^0, and would tend to make $\mathcal{C}_{ei}^L\left(f_e^0\right)$ non-zero. An easy way to show that $\mathbf{V}$ must actually vanish is by taking the velocity moment of eq.(7) to give,

$$0 = \int d^3v \mathbf{v} \mathcal{C}_{ei}^L\left(f_e^0\right) = \int d^3v \mathbf{v} \frac{1}{2}\nu_{ei} v_{Te}^3 \frac{\partial}{\partial \mathbf{v}} \cdot \frac{1}{v}\left(\mathbf{I} - \frac{\mathbf{vv}}{v^2}\right) \cdot \frac{\partial}{\partial \mathbf{v}} f_e^0$$

since the electron-electron operator conserves momentum. Integrating by parts we find,

$$\begin{aligned} 0 &= \frac{1}{2}\nu_{ei} v_{Te}^3 \int d^3v \mathbf{I} : \frac{1}{v}\left(\mathbf{I} - \frac{\mathbf{vv}}{v^2}\right) \cdot \frac{(\mathbf{v}-\mathbf{V})}{v_{Te}^2} f_e^0 \\ &= \frac{1}{2}\nu_{ei} v_{Te} \int d^3v \frac{1}{v}\left(-\mathbf{V} + \mathbf{v}\frac{\mathbf{v}\cdot\mathbf{V}}{v^2}\right) f_e^0 \end{aligned}$$

which cannot be satisfied unless $\mathbf{V} = 0$. In physical terms scattering of the electrons from the ions involves a transfer of momentum and will account for the absorption of the momentum input from the electric field. The zero order equilibrium condition therefore requires that the electrons be at rest with respect to the ions. The momentum transfer will occur at next order along with the transport coefficient.

Knowing the zero order distribution, f_e^0, the first order equation becomes,

$$\frac{2e}{m_e v_{Te}^2}\mathbf{E}\cdot\mathbf{v} f_e^0 = \mathcal{C}_e\left(f_e^1\right)$$

Mathematically, the solution for the first order distribution, f_e^1, requires inverting the collision operator. Formally, the solution can be expressed,

$$f_e^1 = \mathcal{C}_e^{-1}\left(\frac{2e}{m_e v_{Te}^2}\mathbf{E}\cdot\mathbf{v} f_e^0\right)$$

This will result in contribution linear in the electric field with a nonvanishing first moment. From the velocity integral in eq.(4) one can evaluate the current and thus the conductivity. It remains to carry out this operator inversion for specific operators.

Taking the collision operator to be the Lorentz limit for electron-ion collisions, with the bilinear electron-electron operator evaluated to first order, or linearized, we have,

$$\mathcal{C}_e\left(f_e^1\right) = \mathcal{C}_{ei}^L\left(f_e^1\right) + \mathcal{C}_{ee}\left(f_e^1, f_e^0\right) + \mathcal{C}_{ee}\left(f_e^0, f_e^1\right)$$

This is the problem solved by Spitzer and Härm. One can imagine the complexities of this calculation by recalling the integro-difierential nature of the like particle collision operator which must be inverted. Generally, calculations of this accuracy require numerical computations. To illustrate the procedure without becoming too involved in algebra, we will carry out the calculation using the Lorentz operator alone. At the end of this section, the consequences of the complete calculation will be discussed.

Let us pick an electric field vector orientation along the z-axis and use a spherical velocity space coordinate system accordingly. We then have $\mathbf{E} \cdot \mathbf{v} = Ev\cos\theta = Ev\mu$,, where μ, is often referred to as the pitch angle variable. The Lorentz operator of equation (2) can be easily written in terms of μ, to give our first order equation,

$$\frac{E}{E_R}\frac{2v}{v_{Te}}\mu f_e^0 = \frac{v_{Te}^3}{2v^3}\left[\frac{\partial}{\partial\mu}\left(1-\mu^2\right)\frac{\partial}{\partial\mu} + \frac{1}{1-\mu^2}\frac{\partial^2}{\partial\phi^2}\right] f_e^1 \tag{9}$$

Since the left hand side is independent of azimuthal angle, ϕ, so will be f_e^1, and the ϕ derivative can be dropped. The dependence on v is algebraic. We thus have a very simple ordinary differential equation in the pitch angle variable, μ. This can be solved by the well known technique of expansion in Legendre polynomials. We write the first order distribution as,

$$f_e^1(v,\mu) = \sum_l a_l(v) P_l(\mu)$$

expand the left side of eq.(9) similarly and utilize the equation,

$$\frac{\partial}{\partial\mu}\left(1-\mu^2\right)\frac{\partial}{\partial\mu}P_l(\mu) = -l(l+1)P_l(\mu)$$

to eliminate the differential operator. Then, from the orthogonality of the Legendre polynomials, the coefficients a_l, actually functions of velocity, v, result. Here the expansion is rather trivial since,$\mu = P_l(\mu)$. The expansion has only one term. The result is,

$$f_e^1 = -2\frac{E}{E_R}\left(\frac{v}{v_{Te}}\right)^4 \mu f_e^0 \tag{10}$$

The current is now easily computed,

$$\begin{aligned} J &= -e\int d^3v v_z f_e^1 \\ &= -e\int_0^\infty dv v^2 \int_{-1}^{+1} 2\pi d\mu v\mu f_e^1 \\ &= \frac{2\pi e E n}{E_R}\frac{4}{3}\int_0^\infty dv\left(\frac{v}{v_{Te}}\right)^7 \frac{e^{-v^2/v_{Te}^2}}{(2\pi)^{3/2}} \\ &= \frac{e v_{Te} n}{E_R}\frac{8}{\sqrt{\pi}}E \end{aligned}$$

From this we infer the Lorentz conductivity,

$$\sigma^L = \frac{8}{\sqrt{\pi}} \frac{e^2 n}{m_e \nu_{ei}} \tag{11}$$

Equation (11) illustrates an important lesson about transport theory. All the parametric dependence of this equation was known already. It could have been written down from a fluid equation like (6). All the kinetic theory has done is to provide the numerical coefficient. Note, however, that it is a number greater than four and therefore far from trivial in applications. It is the purpose of transport theory to provide these numbers. We should mention that not all transport problems are this simple and more than just numerical coefficients can be found in some cases.

The complete calculation includes the electron-electron collisions and these actually reduce the conductivity above by almost a factor of two. This may seem surprising since the electron-electron collisions cannot produce a net friction force or momentum transfer as this result would apparently imply. What actually occurs is a kind of symbiotic effect where electron self collisions enhance the momentum transfer to the ions. The numerical factor that results is $\sim .58$, which we choose to represent analytically as $\sim 3\pi/16$.

This gives the Spitzer-Harm answer as,

$$\sigma^{SH} = 3\frac{\sqrt{\pi}}{2} \frac{e^2 n}{m_e \nu_{ei}} \tag{12}$$

This can be expressed very simply in terms of the collision time defined in Sigmar and Helander (their equation (3.31)),

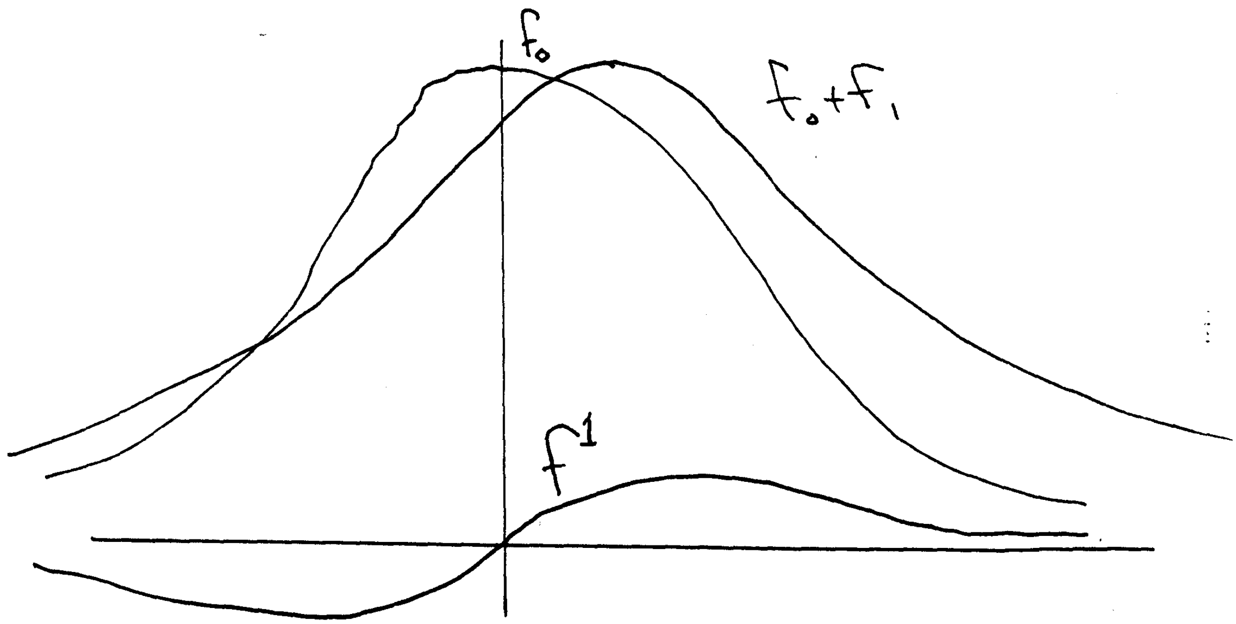
$$\frac{1}{\tau_{ei}} \equiv \frac{4}{3\sqrt{\pi}} \nu_{ei}$$

to give,

$$\sigma^{SH} = 2\frac{e^2 n \tau_{ei}}{m_e}$$

which is the formula used in the plasma formulary for expressing the electrical conductivity. This factor of 2 is interesting but certainly *not* the effect of electron-electron collisions, which *reduces* the conductivity by the factor ~ 1.72.

Although the calculation is involved, the reason for the *reduction* of the conductivity from the Lorentz case can be understood quite simply in physical terms. Note that the first order distribution, f_e^1, in equation (10), because of the factor v^4 has very enhanced tails as compared to the Maxwellian. This is shown schematically in the figure.



Distribution function for classical resistivity calculation.

These tails result from the reduction of the collisional drag at large velocities, apparent in the v^{-3} dependence of the Lorentz operator. It is because of these tails that the numerical coefficient in eq.(11) is so large. Now electron self collisions will try to restore the distribution to a Maxwellian. They cannot eliminate the antisymmetric perturbation driven by the electric field, but they will pull down the tails, and thereby reduce the conductivity. Note that all the momentum transfer is still to the ions, and the velocity integral of the electron-electron collision operator is identically zero.

Now discuss runaways . . .