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8.012 Physics I: Classical Mechanics
Fall 2008

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MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

Physics 8.012

Fall 2008

Exam 3

NAME: _____ **SOLUTIONS**

Instructions:

1. Do all **FIVE (5)** problems. You have **90 minutes**.
2. **SHOW ALL WORK.** Be sure to **circle your final answer.**
3. Read the questions carefully.
4. All work must be done in this booklet in workspace provided. Extra blank pages are provided at the end if needed.
5. **NO** books, notes, calculators or computers are permitted. **A sheet of useful equations is provided on the last page.**

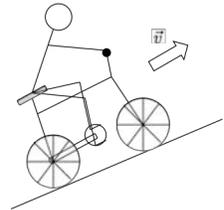
Your Scores

Problem	Maximum	Score	Grader
1	10		
2	20		
3	25		
4	25		
5	20		
Total	100		

Problem 1: Quick Multiple Choice Questions [10 pts]

For each of the following questions circle the correct answer. You do not need to show any work.

- (a) [2 pts] A bicycle rider pedals up a hill with constant velocity v . In which direction does friction act on the wheels?



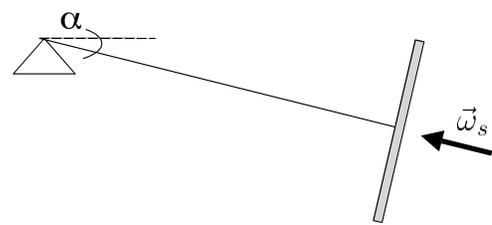
Uphill

Downhill

Friction does not act while the bike moves at constant velocity

Friction provides the only support against falling downhill, so it must point up.

- (b) [2 pts] A gyroscope is set to spin so that its spin vector points back to the mount. The gyroscope is initially set at an angle α below horizontal and released from rest. Gravity acts downward. In which direction does the precession angular velocity vector point once the gyroscope starts to precess?



Up

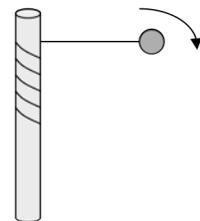
Down

Into the page

Out of the page

The initial fall of the gyroscope causes the spin vector to point up. To conserve angular momentum in the z direction, precession rotation must point downward

- (c) [2 pts] A ball attached to rope is twirled around a stick as shown in the diagram at right. Ignore gravity and friction. Which of the following quantities is conserved in the motion of the ball? Be sure to write down all of the choices below that apply.



Only energy is conserved here.

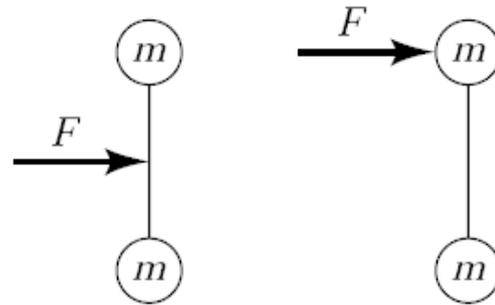
Energy

Momentum

Angular
Momentum

None of these are
conserved

(d) [2 pts] A force F is applied to two identical dumbbells, each comprised of two masses separated by a thin rod. The force is applied at the center of mass for the left dumbbell and directly on one of the masses for the right dumbbell. If the forces are applied for the same duration of time, which dumbbell acquires the greatest center of mass velocity?



Left dumbbell

Right dumbbell

They both have the same velocity

Both dumbbells receive the same impulse, hence their linear momenta are the same, and as their masses are the same they have the same COM velocities.

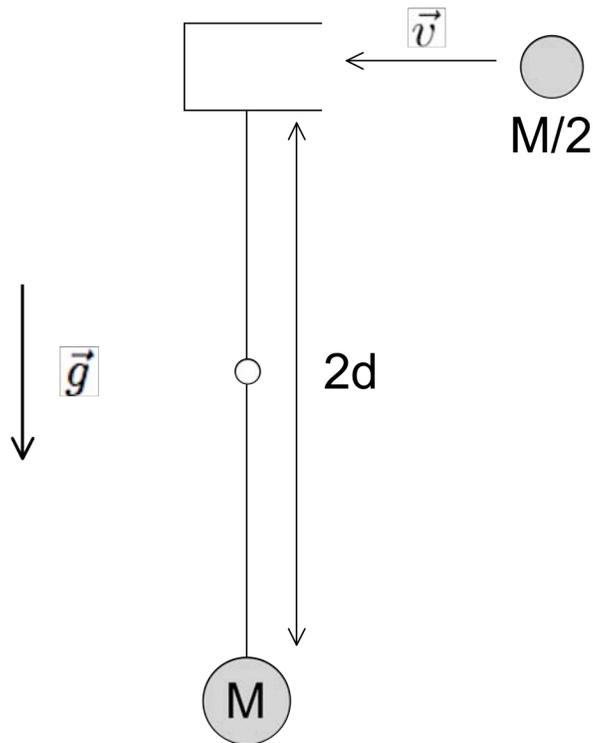
(e) [2 pts] For the same dumbbells in part (d), which one acquires the greatest kinetic energy from the applied force?

Left dumbbell

Right dumbbell

They both have the same kinetic energy

Both dumbbells have the same linear velocity and hence linear kinetic energy, but the right dumbbell will also have rotation kinetic energy associated with its motion. In effect, the force for the right dumbbell acts over a longer distance than the left one and does more work.

Problem 2: Catch Pendulum [20 pts]

A pendulum consists of a ball of mass M attached to the end of a rigid bar of length $2d$ which is pivoted at the center. At the other end of the bar is a container (“catch”). A second ball of mass $M/2$ is thrown into the catch at a velocity v where it sticks. For this problem, ignore the mass of the pendulum bar and catch, and treat the balls as if they were point masses (i.e., neglect their individual moments of inertia).

- (a) [5 pts] What is the initial angular rotation rate of the pendulum after the incoming ball is caught?
- (b) [5 pts] How much total mechanical energy is lost when the incoming ball gets stuck in the catch?
- (c) [10 pts] What minimum velocity does the incoming ball need in order to invert the pendulum (i.e., rotate it by 180°)?

SOLUTION TO PROBLEM 2

(a) The angular momentum of the system *relative to the pivot point* just prior to the upper ball being caught is:

$$L = \frac{M}{2}vd$$

As there are no external torques acting on the system relative to the pivot point during the time of the collision, the angular momentum can be determined as:

$$L = I\omega = \left(Md^2 + \frac{M}{2}d^2\right)\omega = \frac{3}{2}Md^2\omega$$

hence,

$$\omega = \frac{v}{3d}$$

(b) The energy of the upper ball prior to sticking to the catch is:

$$E_i = \frac{1}{2} \frac{M}{2} v^2 = \frac{1}{4} M v^2$$

After the collision, all of the energy can be expressed as pure rotation about the stationary pivot point, hence:

$$E_f = \frac{1}{2} I \omega^2 = \frac{1}{2} \left(\frac{3}{2} M d^2\right) \left(\frac{v}{3d}\right)^2 = \frac{1}{12} M v^2$$

so

$$\Delta E = E_f - E_i = \frac{1}{6} M v^2$$

(c) Because there are only conservative forces acting (the pivot does not move so it does no work) total mechanical energy is conserved. Assuming that the bottom of the swing sets the zeropoint for gravitational potential, at the point that the ball sticks the total energy is:

$$E_i = \frac{M}{2}(2d)g + \frac{1}{12}Mv^2$$

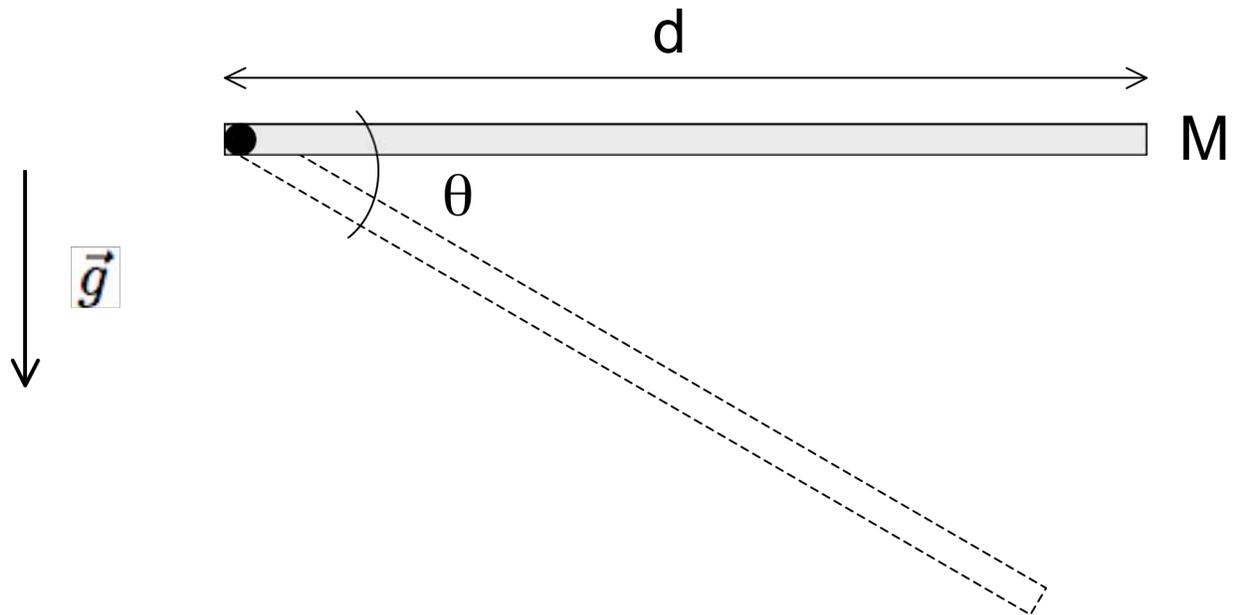
The point at which the pendulum just comes to rest inverted, the bottom mass is at height $2d$:

$$E_f = M(2d)g$$

With $\Delta E = 0$ this gives:

$$2Mdg = Mdg + \frac{1}{12}Mv^2$$

$$\Rightarrow v = \sqrt{12dg}$$

Problem 3: Swinging bar [25 pts]

A uniform bar of mass M and length d is pivoted at one end. The bar is released from rest in a horizontal position and allowed to fall under constant gravitational acceleration.

- (a) [5 pts] How much work does the contact force apply to the system as a function of angle?
- (b) [5 pts] What is the angular speed of the bar as a function of angle?
- (c) [5 pts] What is the angular acceleration of the bar as a function of angle?
- (d) [10 pts] What are the vertical and horizontal forces the bar exerts on the pivot as a function of ~~time~~ **angle**?

SOLUTION TO PROBLEM 3

(a) The contact force is a fixed constraining force, and hence does no net work on the system. The only work done is by gravitational force.

(b) Since gravity (a conservative force) is the only force that produces net work on the system, total mechanical energy is conserved. Energy provides the easiest (but not only) route to velocity. Considering this a problem of pure rotation about the pivot point:

$$E_i = 0$$

$$E_f = -Mg \frac{d}{2} \sin \theta + \frac{1}{2} I \dot{\theta}^2 = E_i = 0$$

$$\Rightarrow \dot{\theta}^2 = \frac{Mgd}{I} \sin \theta = \boxed{3 \frac{g}{d} \sin \theta}$$

where $I = Md^2/3$ about the pivot point has been used.

(c) Angular acceleration is directly tied to torques, so we can solve for the net torque on the system as a function of angle. Choosing again the pivot point as the origin of our coordinate system:

$$\vec{\tau} = \vec{r} \times \vec{F} = \frac{d}{2} Mg \cos \theta \hat{\otimes} = \frac{d\vec{L}}{dt} = I\ddot{\theta} \hat{\otimes}$$

$$\Rightarrow \ddot{\theta} = \boxed{\frac{3}{2} \frac{g}{d} \cos \theta}$$

Note that this could also be solved from the relation:

$$\frac{d}{dt}(\dot{\theta}^2) = 2\dot{\theta}\ddot{\theta}$$

and plugging in the solution to (b).

(d) One can solve this by computing the polar coordinate forces (being careful to get the changing orientation of the gravitation force):

$$\hat{r} : -Mr\dot{\theta}^2 = F_r + Mg \sin \theta$$

$$\hat{\theta} : Mr\ddot{\theta} = -F_\theta + Mg \cos \theta$$

and then reorienting these forces back into a Cartesian system. An easier way is to rewrite the Cartesian coordinates in terms of θ . For the horizontal force (positive x pointing toward the right):

$$\begin{aligned} F_h &= M\ddot{x} = M \frac{d^2}{dt^2} \left(\frac{d}{2} \cos \theta \right) \\ &= \frac{1}{2} M d \frac{d}{dt} (-\sin \theta \dot{\theta}) \\ &= -\frac{1}{2} M d (\sin \theta \ddot{\theta} + \cos \theta \dot{\theta}^2) \end{aligned}$$

substituting in our expressions above:

$$F_h = -\frac{9}{4} Mg \sin \theta \cos \theta$$

this is the force the pivot exerts on the bar, the force the bar exerts on the pivot is:

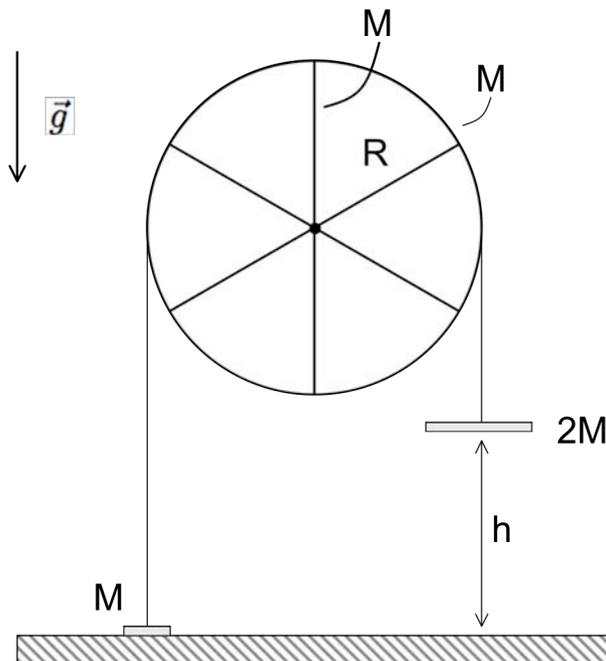
$$F_h = \frac{9}{4} Mg \sin \theta \cos \theta$$

For the vertical force (positive z pointing up):

$$\begin{aligned} F_v - Mg &= M\ddot{z} = M \frac{d^2}{dt^2} \left(-\frac{d}{2} \sin \theta \right) \\ &= \frac{1}{2} M d \frac{d}{dt} (-\cos \theta \dot{\theta}) \\ &= \frac{1}{2} M d (-\cos \theta \ddot{\theta} + \sin \theta \dot{\theta}^2) \\ &= \frac{3}{2} Mg (\sin^2 \theta - \frac{1}{2} \cos^2 \theta) \end{aligned}$$

rearranging, and again flipping sign for the force on the pivot:

$$F_v = Mg \left(\frac{3}{4} \cos^2 \theta - \frac{3}{2} \sin^2 \theta - 1 \right)$$

Problem 4: Atwood machine [25 pts]

An Atwood machine consists of a fixed pulley wheel of radius R , a rim of mass M and 6 spokes each with length R and mass M . An effectively massless string passes around the pulley wheel and connects two plates of mass M and $2M$. The lighter plate is initially on the ground when the system is released from rest. Constant gravitational force acts downwards, and assume that the string never slips.

- (a) [5 pts] Calculate the moment of inertia of the pulley wheel.
- (b) [10 pts] Calculate the speed of the more massive plate when it hits the ground (there are multiple ways to solve this).
- (c) [10 pts] Assume that when it strikes the ground, the heavier plate sticks. Furthermore, the string remains “stuck” to the wheel, so it continues to pull the lighter plate up. How high does the lighter plate go?

SOLUTION TO PROBLEM 4

(a) The moment of inertia of the pulley wheel is simply the sum of the moments of inertia of each spoke rotated about one end ($MR^2/3$) and that of the rim (MR^2).

Taken together:

$$I = 6\left(\frac{1}{3}MR^2\right) + MR^2 = \boxed{3MR^2}$$

(b) Again, this can be done using conservation of total mechanical energy, since there are no nonconservative forces in (this part of) the problem. Setting the ground to have zero potential energy, and assuming that the string must have a fixed length so each mass has the same velocity:

$$E_i = 2Mgh$$

$$E_f = Mgh + \frac{1}{2}Mv^2 + \frac{1}{2}(2M)v^2 + \frac{1}{2}I\omega^2$$

The rotation rate of the wheel is connected with the velocities of the masses through the no-slip requirement:

$$R\omega = v$$

hence for $\Delta E = 0$:

$$Mgh = \frac{3}{2}Mv^2 + \frac{1}{2}(3MR^2)\left(\frac{v}{R}\right)^2$$

$$\Rightarrow v = \sqrt{\frac{gh}{3}}$$

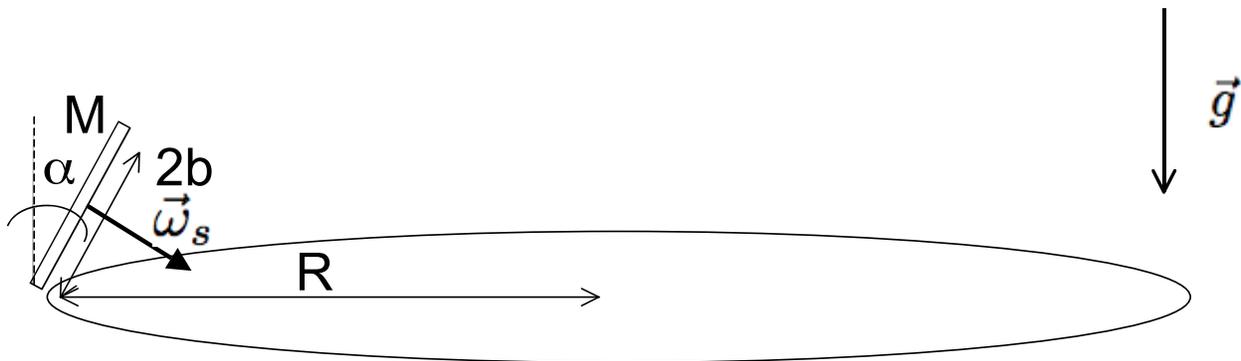
(c) Once the heavy mass strikes (and loses its kinetic energy) the remaining parts of the system will conserve energy, so we can use the same approach:

$$E_f = Mgh' + \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 = Mgh + 2Mv^2$$

$$E_f = Mgh'$$

$$\Rightarrow h' = h + 2\frac{v^2}{g} = \frac{5}{3}h$$

Problem 5: Rolling a coin [20 pts]



A coin (uniform solid circular disk) with mass M and radius b is set to roll in a circular path of radius R on a table surface ($R > b$). The coin is given a spin angular velocity ω_s , and as it rolls it is found to tilt at a small angle α with respect to vertical. Assume that the coin does not slip; constant gravity acts downward.

(a) [5 pts] Determine the “precession” angular velocity of the rolling motion, Ω , both magnitude and direction.

(b) [5 pts] Determine the total angular velocity vector, $\vec{\omega}$, including both spin and precession terms, in a polar coordinate system centered on the center of the coin’s trajectory.

(c) [10 pts] Solve for the angle α in terms of the quantities given in the diagram above. Be sure to examine all equations of motion!

(d) [10 pts BONUS] Determine the total angular momentum vector, \vec{L} , in terms of the quantities given in the diagram above, in a polar coordinate system centered on the center of the coin’s trajectory. For simplicity, assume that α is very small (i.e., only consider first order terms in α).

SOLUTION TO PROBLEM 5

(a) This relation comes straight from the condition that the disk does not slip, which requires that the arclength traversed by the coin along its trajectory is the same arclength covered along the edge of the coin per unit time, otherwise one of the surfaces would be slipping relative to the other. This implies that:

$$R\Omega = b\omega_s \Rightarrow \Omega = \omega_s \frac{b}{R}$$

This vector points upward, which can be inferred from the fact that if the coin does not precess, it would tip over, causing the spin angular momentum in the downward direction to increase. The coin must compensate for this by precessing in such a direction that its angular momentum vector points upwards (you can also infer this from the direction of the rolling coin as well).

(b) The total angular velocity vector is the sum of the spin and precession vectors. Choosing a polar coordinate system centered at the center of the orbit with the radial vector outward and the z-axis pointing up, these can be written as:

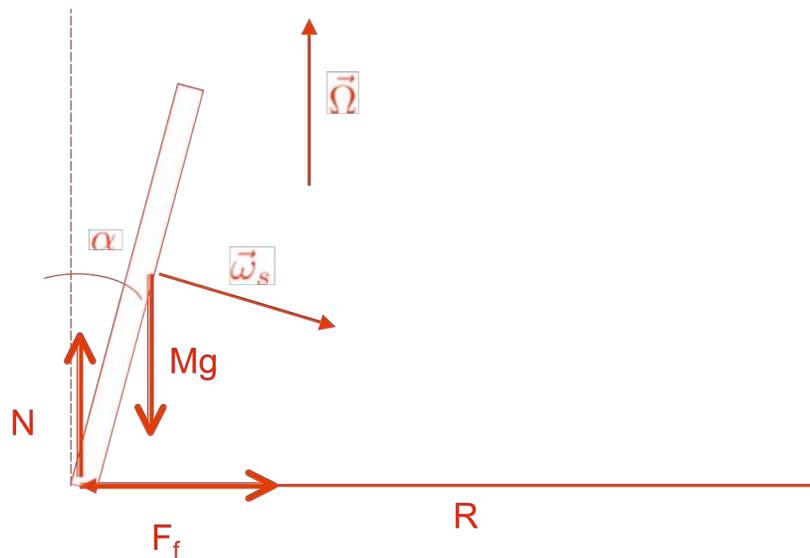
$$\vec{\omega}_s = \omega_s(-\hat{r} \cos \alpha - \hat{z} \sin \alpha)$$

$$\vec{\Omega} = \omega_s \frac{b}{R} \hat{z}$$

hence

$$\vec{\omega} = \vec{\omega}_s + \vec{\Omega} = \omega_s \left[\hat{z} \left(\frac{b}{R} - \sin \alpha \right) - \hat{r} \cos \alpha \right]$$

(c) Now we want to consider the torque acting on the coin. We will consider our reference point to be the center of mass of the coin, since for this point torques are not affected by non-inertial forces in the coin's moving frame of reference. The possible forces acting on the coin (see figure) are its weight (through the center of mass, which produces no torque), a vertical normal force which is equal in magnitude to weight (since the coin is not falling in the z direction), and a tangential frictional force which provides the necessary radial force to keep the coin moving in a circular trajectory, equal to



$$F_f = M(R - b \sin \alpha)\Omega^2 \approx MR\Omega^2$$

to within our $R \gg b$ assumption. The total torque about the center of mass is therefore:

$$\begin{aligned} \vec{\tau} &= \vec{r} \times \vec{F} = bMg \sin \alpha \hat{\otimes} + bMR\Omega^2 \cos \alpha \hat{\odot} \\ &= (Mbg \sin \alpha - Mb^2\omega_s\Omega \cos \alpha) \hat{\otimes} \end{aligned}$$

This torque acts to change the direction of the total angular momentum vector, or more precisely the radial component of the angular momentum vector that rotates as the coin goes along its trajectory. The spin component of this is simply:

$$\vec{L}_s \cdot \hat{r} = \frac{1}{2}Mb^2\omega_s \cos \alpha$$

The precession component is somewhat more complicated (see below), but is going to be smaller than the spin component in the radial direction by a factor of b/R because of the smaller precession rate. Hence, we can use the gyroscopic approximation and consider only the change in direction of the spin angular momentum vector, namely the change in the radial component:

$$\frac{dL}{dt} = (\vec{L}_s \cdot \hat{r})\Omega = |\vec{\tau}|$$

$$\Rightarrow Mbg \sin \alpha - Mb^2\omega_s\Omega \cos \alpha = \frac{1}{2}Mb^2\omega_s\Omega \cos \alpha$$

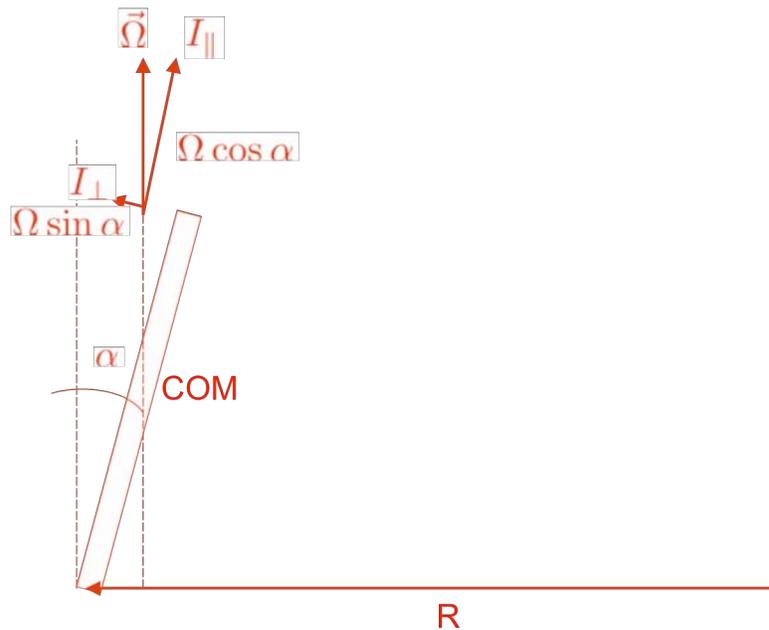
$$\Rightarrow \tan \alpha = \frac{3 b^2 \omega_s^2}{2 g R}$$

(d) The total angular momentum of the coin about its COM is the sum of the spin angular momentum and precession angular momentum vectors:

$$\vec{L} = \vec{L}_s + \vec{L}_\Omega$$

The first is straightforward:

$$\vec{L}_s = \frac{1}{2} M b^2 \omega_s (-\hat{r} \cos \alpha - \hat{z} \sin \alpha)$$



The second is not because of the tilt of the coin, which leads to off-axis terms in the moment of inertia tensor (products of inertia). We can get an approximate handle on this by first breaking up the precession rotation vector into components parallel and perpendicular to the disk plane through the center of mass, as shown in the above diagram. Along these principal axes we compute the component angular momenta:

$$L_\parallel = I_\parallel \Omega \cos \alpha = \frac{1}{4} M b^2 \Omega \cos \alpha$$

$$L_{\perp} = I_{\perp} \Omega \sin \alpha = \frac{1}{2} M b^2 \Omega \sin \alpha$$

we now have to project these axes back along the polar coordinates:

$$\hat{\parallel} = \hat{z} \cos \alpha - \hat{r} \sin \alpha$$

$$\hat{\perp} = \hat{z} \sin \alpha + \hat{r} \cos \alpha$$

which gives:

$$\vec{L}_{\Omega}^{COM} = \frac{1}{4} M b^2 \Omega [\hat{z} (\cos^2 \alpha + 2 \sin^2 \alpha) + \hat{r} \sin \alpha \cos \alpha] \approx \frac{1}{4} M b^2 \Omega [\hat{z} + \alpha \hat{r}]$$

where the latter comes from our small angle approximation. Combining this with the spin angular momentum (and keeping the same approximation):

$$\vec{L} \approx \frac{1}{4} M b^2 \omega_s [\hat{z} (\frac{b}{R} - 2\alpha) + \hat{r} (\frac{b}{R} \alpha - 2)]$$

$$\begin{aligned} &\approx \frac{1}{2} M b^2 \omega_s [\hat{z} (\frac{b}{2R} - \alpha) - \hat{r}] \\ &\approx \frac{1}{2} M b^2 \omega_s [\frac{b}{2R} (1 - \frac{3b\omega_s^2}{g}) \hat{z} - \hat{r}] \end{aligned}$$

Note that we retain the assumption made in (c), that the radial angular momentum is primarily from the spin, and that because $b \ll R$ this component dominates the whole angular momentum vector, making the gyroscopic approximation a realistic one.