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8.012 Physics I: Classical Mechanics  
Fall 2008

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MASSACHUSETTS INSTITUTE OF TECHNOLOGY  
Department of Physics

Physics 8.012

Fall 2006

**Midterm Exam 1**  
**Thursday, October 5, 2006**

NAME: \_\_\_\_\_ **SOLUTIONS** \_\_\_\_\_

Instructions:

1. Do all **FIVE (5)** problems. You have **90 minutes**.
2. **SHOW ALL WORK**. Be sure to circle your final answer.
3. Read the questions **carefully**.
4. All work must be done in this booklet. Extra blank pages are provided.
5. No books, notes, calculators or computers are permitted.

**Your Scores**

Problem	Maximum	Score	Grader
1	10		
2	20		
3	25		
4	20		
5	25		
<b>Total</b>	100		

Problem 1: Quick Multiple Choice Questions [10 pts]

For each of the following questions circle the correct answer. Note that each question is worth only 2 points, so do not spend a lot of time on this part!

(a) Compared to the gravitational force with which the Earth pulls you, the gravitational force with which you pull the Earth is

Greater

Less

Equal

You exert no gravitational force on the Earth

(b) Given a force law  $\vec{F} = Ar^2\hat{r}$ , what are the dimensions of A?

 $[M][L][T]^{-2}$  $[M][L]^{-1}[T]^{-2}$  $[L][T]^{-1}$  $[M][L]^{-2}[T]^{-1}$ 

(c) A metal hammer and rubber mallet of identical masses are swung at a nail with identical speeds. Which applies the greater impulse?

Hammer

Mallet

The impulses are the same

(d) A block with mass M and contact area A slides down an inclined plane with friction, covering a distance L in time T. How much time does it take another block with the same mass and composition, but contact area 2A, to slide down the same length?

 $\sqrt{T}$ 

T

 $T^2$  $T^3$ 

(e) A pendulum of length L supporting mass M swings back and forth with period P. If the mass is doubled, what is the new period?

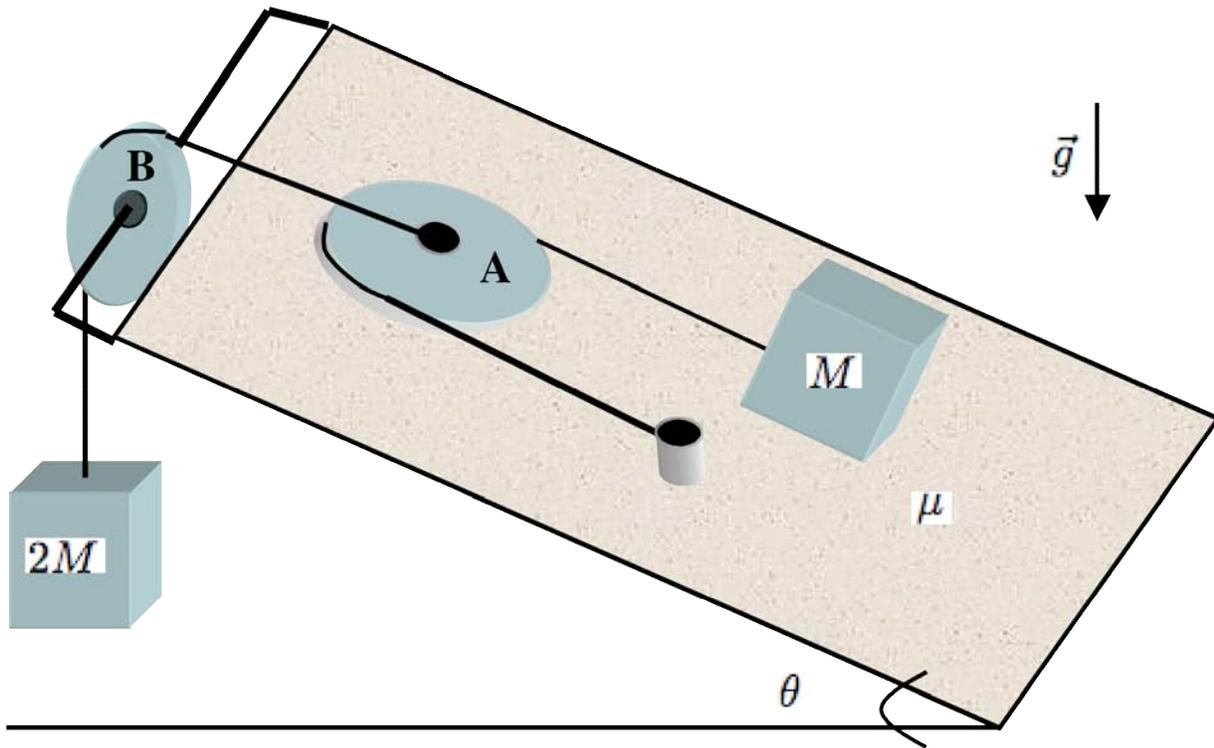
 $\frac{1}{\sqrt{2}}P$ 

P

 $\sqrt{2}P$ 

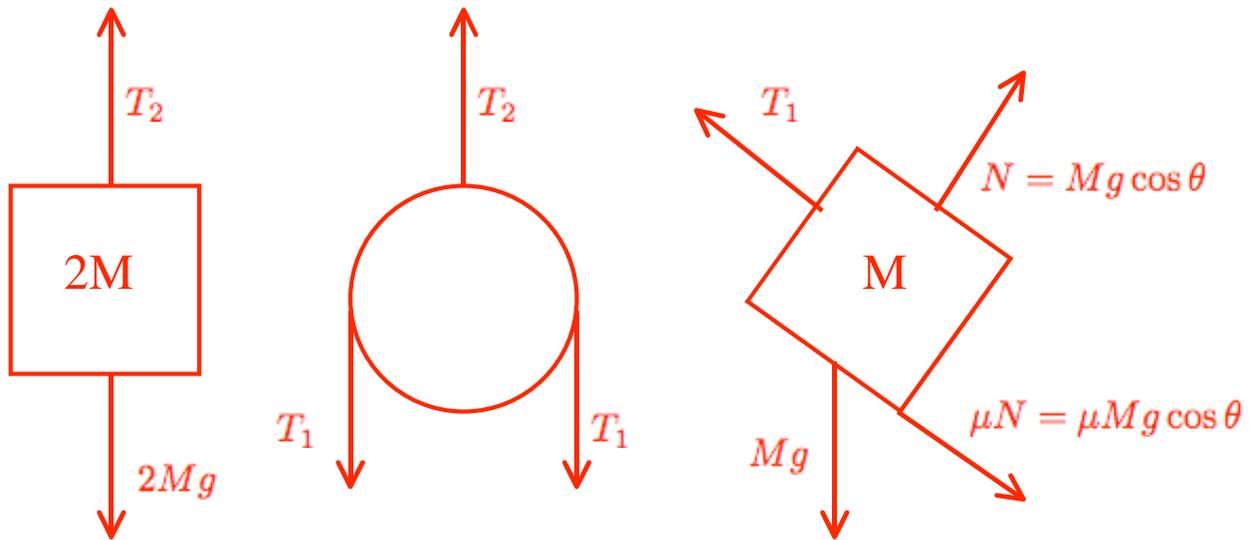
2P

Problem 2: Blocks and Pulley on an Incline [20 pts]



A block of mass  $M$  sits on an inclined plane, and is connected via a massless string through a massless pulley  $A$  (that slide without friction on the plane) to a fixed post. This pulley is in turn connected via a massless string through a second massless pulley  $B$  (attached to the top of the inclined plane and oriented to rotate about a horizontal axle) to a second block of mass  $2M$  that hangs above the ground. The coefficient of static friction between the inclined plane and block resting on the inclined plane is  $0 < \mu < 1$ . Gravity is assumed to be acting in a vertical direction with constant acceleration  $\vec{g} = -g\hat{z}$ . The inclined plane is tilted to an angle  $\theta$  with respect to horizontal, and the masses are assumed to be initially at rest.

(a) [5 pts] Draw force diagrams for both masses and pulley A. Assuming that pulley A is massless, derive a relation between the tensions of both massless strings.

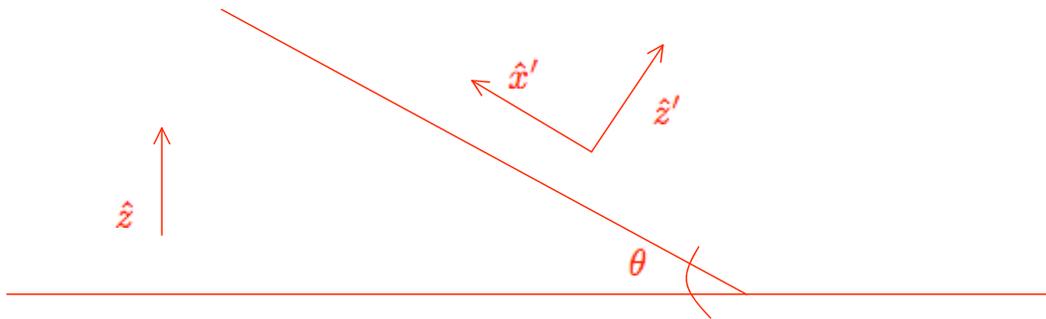


If the pulley is massless, then

$$m_p \ddot{x} = 0 = T_2 - 2T_1$$

$$\Rightarrow T_2 = 2T_1$$

(b) [15 pts] Derive a relation, as a function of  $\mu$  alone, for the minimum angle  $\theta_{min}$  that the inclined plane can be tilted before the blocks start to move. You are not asked to solve explicitly for  $\theta_{min}$  in the final relation.



This problem can be treated as a statics problem; i.e., if none of the blocks are moving, the net forces must equal 0. We first write down the forces acting on the blocks using an inclined coordinate system for the mass on the inclined plane (so  $\hat{x}'$  is parallel to the surface), and a vertical coordinate for the hanging mass ( $\hat{z}$ ). Then:

$$F_x^{(1)} = T_1 - \mu N - Mg \sin \theta = 0$$

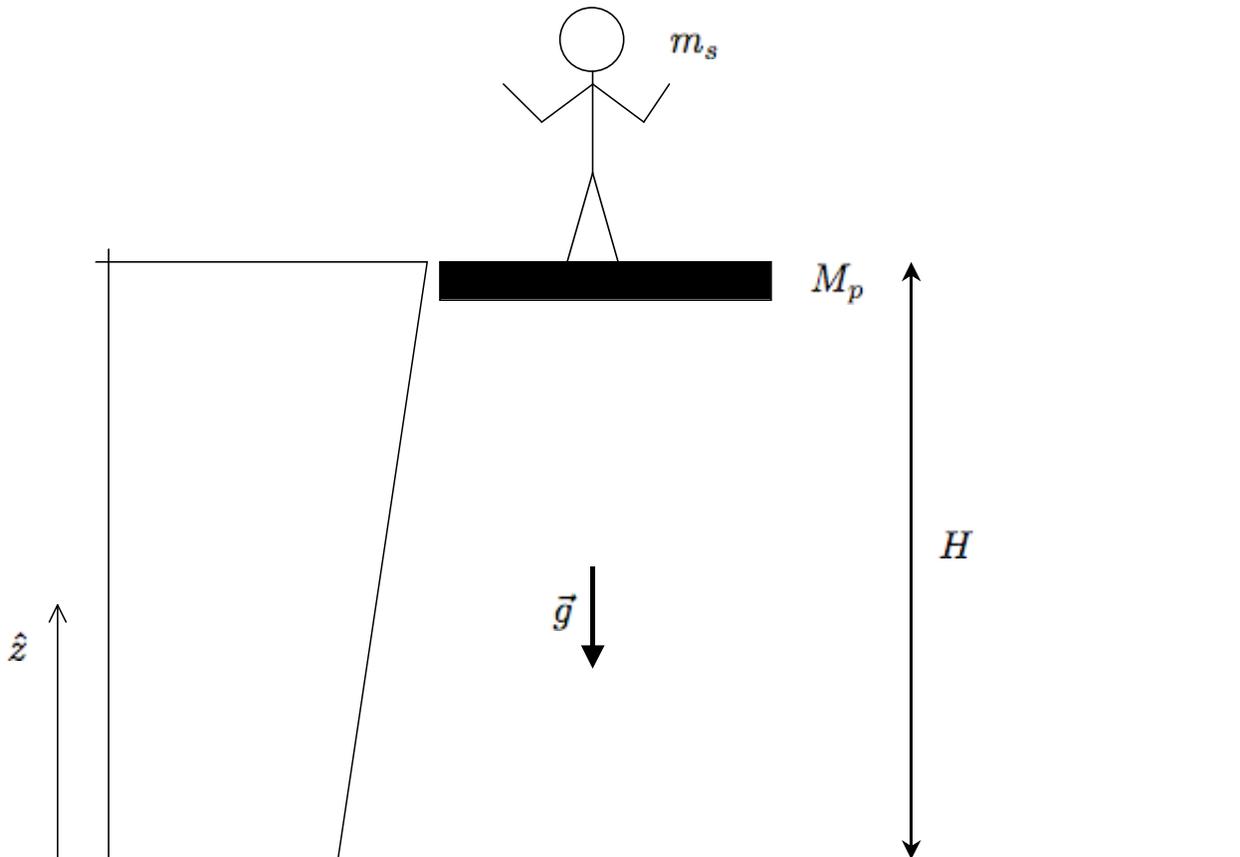
$$F_z^{(1)} = N - Mg \cos \theta = 0 \Rightarrow N = Mg \cos \theta$$

$$F_z^{(2)} = T_2 - 2Mg = 2T_1 - 2Mg = 0 \Rightarrow T_1 = Mg$$

Plugging back into the first equation we derive:

$$Mg - \mu Mg \cos \theta - Mg \sin \theta = 0$$

$$\Rightarrow 1 = \mu \cos \theta + \sin \theta$$

Problem 3: Saving Yourself from a Fall [25 pts]

An intrepid student of mass  $m_s$  walks onto a platform of mass  $M_p$  that is attached to the side of a cliff of height  $H$ . When the student reaches the center of the platform the support breaks and both the student and platform plunge to the ground below. However, just before impact, the student jumps off of the platform with sufficient force that she reaches zero velocity with respect to the ground ( $\vec{v}_s = \mathbf{0}$ ), thereby saving herself from injury. In this problem, assume that the acceleration due to gravity is a constant  $\vec{g} = -g\hat{z}$ , and that the viscosity of air is negligible.

(a) [5 pts] What is the speed of the platform just as it is about to hit the ground?  
What is the speed of the student?

The equation of motion for the platform is:

$$\ddot{z}_p = -g \Rightarrow z_p = -\frac{1}{2}gt^2 + v_o t + z_o$$

initial conditions:  $v_o(0) = 0$  and  $z_0 = H$

$$\Rightarrow z_p(t) = -\frac{1}{2}gt^2 + H$$

at impact,  $z_p = 0$  so

$$H = \frac{1}{2}gt_s^2 \Rightarrow t_s = \sqrt{\frac{2H}{g}}$$

hence the speed at impact is

$$\dot{z}_p(t_s) = -gt_s = \boxed{\sqrt{2gH}}$$

As both the platform and student are accelerated by the same constant factor, the speed of the student is the same as that of the platform

(b) [10 pts] Assuming that the student can jump with a velocity  $\mathbf{u}$  with respect to the platform (at the end of her jump), derive an expression for her speed,  $v_s$ , with respect to the (unmoving) ground. Assume that the jump is instantaneous.

The velocity of the student in the frame of rest is the jump velocity relative the velocity of the platform after it has accelerated to its new speed,  $v_p$ ; i.e.,

$$v_s = u - v_p$$

If the jump is instantaneous, then the impulse on the system vanishes and we have momentum conservation:

$$\Delta p = p_{final} - p_{initial} = 0$$

The initial and final momenta are:

$$p_{initial} = -(M_p + m_s)\sqrt{2gH}$$

$$p_{final} = m_s v_s - M_p v_p = m_s u - v_p (M_p + m_s)$$

and hence

$$\Delta p = m_s u - v_p (M_p + m_s) + (M_p + m_s)\sqrt{2gH} = 0$$

$$\Rightarrow v_p = \sqrt{2gH} + \frac{m_s}{M_p + m_s} u$$

Substituting this back in to the equation for  $v_s$  above gives

$$v_s = \frac{M_p}{M_p + m_s} u - \sqrt{2gH}$$

(c) [10 pts] To the limit that the platform is much more massive than the student ( $M_p \gg m_s$ ), what is the maximum height the student can fall and still jump to zero velocity with respect to the ground? Assume that the student can normally jump to a height of 1 foot from the ground.

Jumping from the ground, the equation of motion for a student is again ballistic:

$$z_s = -\frac{1}{2}gt^2 + v_0t + z_0$$

the initial conditions are  $v_0 = u$  and  $z_0 = 0$ , so

$$z_s = -\frac{1}{2}gt^2 + ut$$

At the maximum height, the vertical velocity vanishes so

$$u = gt_s \Rightarrow t_s = \frac{u}{g}$$

In relation to the maximum height  $h$  the student can jump:

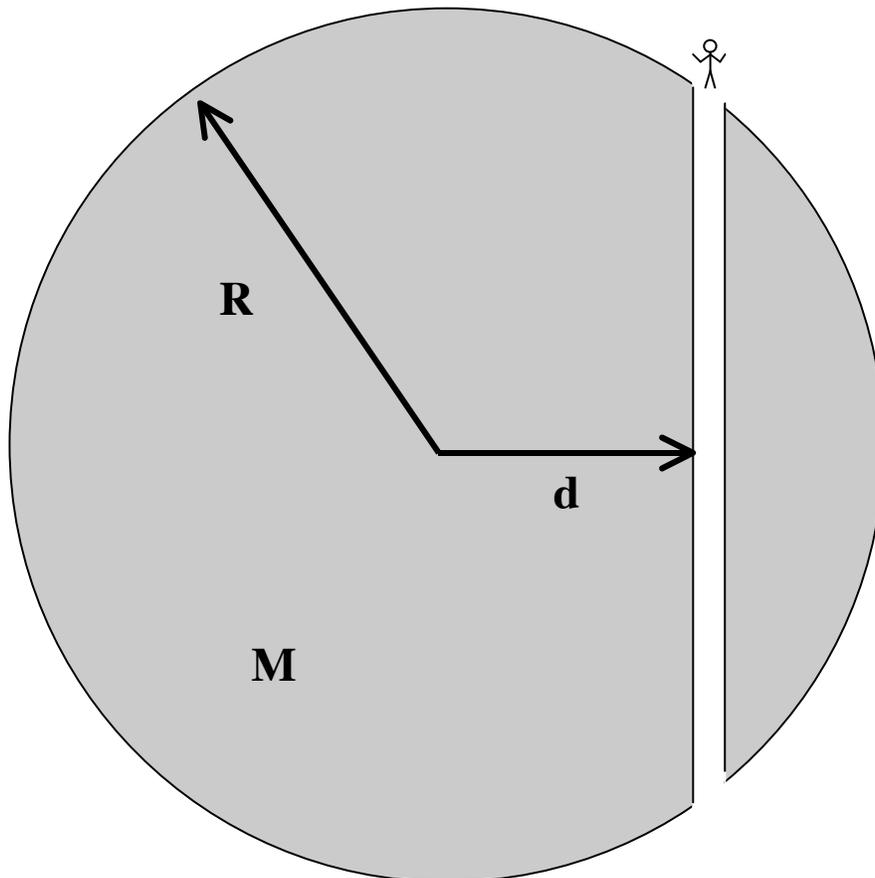
$$h = -\frac{1}{2}gt_s^2 + ut_s = \frac{u^2}{2g}$$

$$\Rightarrow u = \sqrt{2gh}$$

Plugging this into the above solution for  $v_s = 0$  and

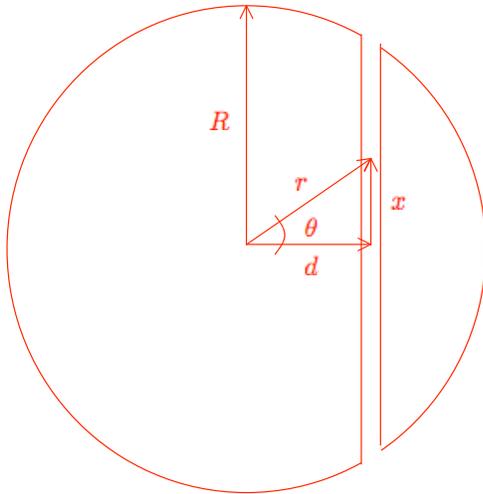
$$\frac{M_p}{M_p + m_s} \approx \frac{M_p}{M_p} = 1 \quad \text{implies:}$$

$$\sqrt{2gH} = u = \sqrt{2gh} \Rightarrow H = h = \boxed{1 \text{ foot}}$$

Problem 4: Falling through the Center of the Earth (again) [20 pts]

A tunnel is built connecting two spots on the Earth that follows a straight trajectory through the interior, but does not intersect the center of the Earth. Our same poor intrepid student wanders too close to the tunnel and falls in. Assume that the Earth is a uniform spherical mass  $M$  with radius  $R$ , and that the medial point of the tunnel passes to within a distance  $d$  of the center. Further assume that the walls of the tunnel are frictionless, and that the mass excavated from the tunnel is negligible as compared to the mass of the Earth.

[20 pts] Show that the student will execute simple harmonic motion as she falls through the tunnel. Find the time it takes the student to return to her point of departure and compare this to the time needed for a satellite to circle the Earth in a low orbit with  $r \approx R$ . Ignore any effects due to the Earth's rotation (N.B.: This is similar to problem 2.26 in K&K).



The gravitational force acting on the person is Newton's inverse square law:

$$m\ddot{r} = -G \frac{mM_r}{r^2}$$

where

$$M_r = \rho \frac{4\pi r^3}{3} = M \frac{r^3}{R^3}$$

This acts toward the center, but as we are interested in the motion along the tube, we need to use the variable

$x = r \sin \theta$  (see figure). The component of the gravitational force in the direction of the tube is also dependent on  $\sin \theta$  so we get:

$$m\ddot{x} = -\frac{GMmr}{R^3} \sin \theta = -\frac{GMm}{R^3} x \Rightarrow \ddot{x} = -\omega^2 x$$

where  $\omega^2 = \frac{GM}{R^3}$  (since G, M and R are all positive numbers). This is the equation for simple harmonic motion.

The period of the motion (which equals the time it takes for someone who has fallen in to come back to the same spot) is:

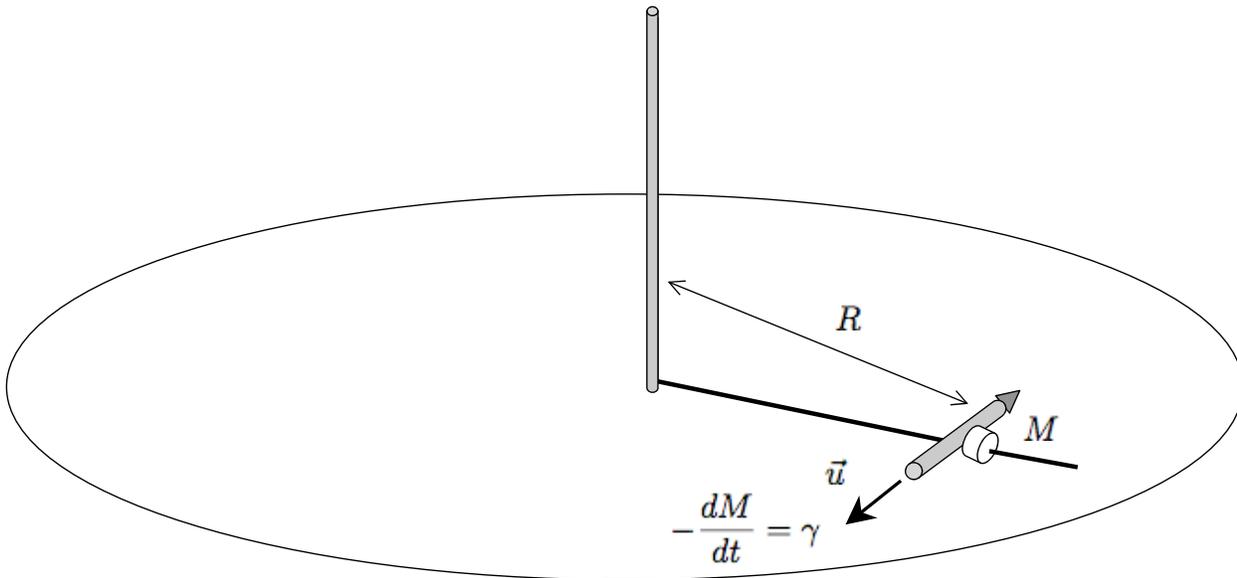
$$P = \frac{2\pi}{\omega} = \boxed{2\pi \sqrt{\frac{R^3}{GM}}}$$

For a satellite in low earth circular orbit, the gravitational acceleration balances centripetal acceleration:

$$m\ddot{r} = G\frac{Mm}{R^2} - mR\omega^2 = 0 \Rightarrow \omega^2 = \frac{GM}{R^3}$$

So we derive the same period:

$$\Rightarrow P = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{R^3}{GM}}$$

Problem 5: Rocket on a Rod [25 pts]

A rocket initially with mass  $M_0$  is resting on a frictionless table and is connected to a rigid rod that passes perpendicularly through its center of mass. The rod is attached to a vertical post that spins freely without friction. The rocket is initially at rest a radius  $R$  from the post, which is much greater than the length of the rocket (you can therefore neglect any torque acting on the rocket). At  $t=0$ , the rocket

begins firing and shoots out mass at a constant rate  $-\frac{dM}{dt} = \gamma$  with velocity relative to the rocket  $\vec{u} = -u\hat{\theta}$ . The rocket is prevented from sliding outward by a bolt fixed to the rod that exerts a normal force  $\vec{N}$ . For this problem, ignore any forces due to gravity and assume that the rod is effectively massless.

(a) [10 pts] Find the equations of motion for the rocket in polar coordinates, using the parameters specified above. Which component of momentum is conserved (assuming no net torque on the rocket)?

The rocket equation in vector notation is:

$$\frac{d\vec{p}}{dt} = M(t) \frac{d\vec{v}}{dt} - \frac{dM}{dt} \vec{u} = \vec{F}_{ext} = -N\hat{r}$$

We need to use the form of acceleration in polar coordinates:

$$\frac{d\vec{v}}{dt} = \ddot{r}\hat{r} + r\ddot{\theta}\hat{\theta} + 2\dot{r}\dot{\theta}\hat{\theta} - r\dot{\theta}^2\hat{r} = R\ddot{\theta}\hat{\theta} - R\dot{\theta}^2\hat{r}$$

plugging this in we get:

$$M(t)R\ddot{\theta}\hat{\theta} - M(t)R\dot{\theta}^2\hat{r} - \gamma u\hat{\theta} = -N\hat{r}$$

Where I have used  $\vec{u} = -u\hat{\theta}$ . The time dependent mass is  $M(t) = M_0 - \gamma t$ , and breaking the solution by components gives:

$$\hat{\theta} : (M_0 - \gamma t)R\ddot{\theta} - \gamma u = 0$$

$$\hat{r} : (M_0 - \gamma t)R\dot{\theta}^2 = N$$

The  $\hat{\theta}$  component of momentum is conserved, as the external force (N) acts only in the  $\hat{r}$  direction.

(b) [10 pts] What is the speed of the rocket,  $v$ , as a function of time?

Using the  $\hat{\theta}$  component of the equations of motion gives

$$\ddot{\theta} = \frac{d\dot{\theta}}{dt} = \frac{\gamma u}{R(M_0 - \gamma t)}$$

$$\Rightarrow \int_0^{\dot{\theta}} d\dot{\theta}' = \dot{\theta}(t) = \int_0^t \frac{\gamma u dt'}{R(M_0 - \gamma t')}$$

Now we do a substitution to solve the integral

$$\tau = M_0 - \gamma t', \quad d\tau = -\gamma dt'$$

And arrive at:

$$\dot{\theta}(t) = -\frac{u}{R} \int_{M_0}^{M_0 - \gamma t} \frac{d\tau}{\tau} = -\frac{u}{R} \ln \frac{M_0 - \gamma t}{M_0} = \frac{u}{R} \ln \frac{M_0}{M_0 - \gamma t}$$

Since  $v = R\dot{\theta}$ , we get

$$v = u \ln \frac{M_0}{M_0 - \gamma t}$$

(c) [5 pts] What is the magnitude of the normal force on the rocket from the bolt after the rocket has expended its fuel, equal to 1/2 of its initial mass?

Using the  $\hat{r}$  equation of motion gives:

$$N(t) = M(t)R\dot{\theta}^2 = \frac{M(t)v^2}{R}$$

When  $M(t) = M_0/2$ ,  $v = u \ln 2$  so

$$N = \frac{M_0(u \ln 2)^2}{2R}$$