8.022 Lecture Notes Class 7 - 09/18/2006

So exactly what is curvilinear? And what's this orthonormal stuff? (=1?)

Gradient in Spherical

Let $f(x) = f(r, \theta, \phi)$

$$\begin{aligned} df = & \vec{\nabla} f \cdot d\vec{x} \\ = & \vec{\nabla} f \cdot (\vec{e_r} dr + \vec{e_\theta} r d\theta + \vec{e_\phi} r \sin\phi d\phi) \\ = & \tfrac{df}{dr} dr + \tfrac{df}{d\theta} d\theta + \tfrac{df}{d\phi} d\phi \end{aligned}$$

So ,

$$\vec{\nabla} f = \vec{e_r} \frac{df}{dr} + \vec{e_\theta} \frac{1}{r} \frac{df}{d\theta} + \vec{e_\phi} \frac{1}{r \sin\theta} \frac{df}{d\phi}$$

Divergence

Let $\vec{v} = v_a \vec{e_a}$ where v_a and A are scalars.

$$\begin{aligned} \vec{\nabla}(A\vec{B}) = & (\vec{\nabla} A) \cdot \vec{B} + A(\vec{\nabla} \cdot \vec{B}) \\ \vec{\nabla} \cdot (\vec{v}) = & (\vec{\nabla} v_a) \cdot \vec{e_a} + v_a (\vec{\nabla} \cdot \vec{e_a}) \\ \vec{\nabla} \cdot \vec{e_r} = & (\vec{e_r} \cdot \tfrac{\partial}{\partial r}) \vec{e_r} + (\vec{e_\theta} \cdot \tfrac{1}{r} \tfrac{\partial}{\partial \theta}) \vec{e_r} + (\vec{e_\phi} \tfrac{1}{r \sin\phi} \tfrac{\partial}{\partial \phi}) \vec{e_r} \end{aligned}$$

$$\frac{\partial}{\partial r} \vec{e_r} = \vec{0} \qquad \frac{\partial}{\partial \theta} \vec{e_r} = \vec{e_\theta} \qquad \frac{\partial}{\partial \phi} \vec{e_r} = \sin\theta \vec{e_\phi}$$

$$\begin{aligned} \vec{\nabla} \cdot \vec{e_r} = & 0 + \vec{e_\theta} \cdot \tfrac{1}{r} \cdot \vec{e_\theta} + \vec{e_\phi} \cdot \tfrac{1}{r \sin\theta} \cdot \sin\theta \vec{e_\phi} \\ = & \tfrac{1}{r} + \tfrac{1}{r} = \tfrac{2}{r} \end{aligned}$$

$$\begin{aligned} \vec{\nabla} \cdot \vec{e_\theta} = & (\vec{e_r} \cdot \tfrac{\partial}{\partial r}) \vec{e_\theta} + (\tfrac{1}{r} \vec{e_\theta} \cdot \tfrac{\partial}{\partial \theta}) \vec{e_\theta} + (\vec{e_\phi} \cdot \tfrac{\partial}{\partial \phi} \vec{e_\theta} \tfrac{1}{r \sin\theta}) \\ = & \vec{e_r} \cdot \tfrac{\partial}{\partial r} \vec{e_r} + \tfrac{1}{r} \vec{e_\theta} \tfrac{\partial}{\partial r} \vec{e_\theta} + \vec{e_\phi} \cdot \cos\theta \vec{e_\phi} \tfrac{1}{r \sin\theta} \\ = & \tfrac{\cos\theta}{r \sin\theta} \end{aligned}$$

$$\vec{\nabla} \cdot \vec{e_\phi} = (\vec{e_r} \cdot \tfrac{\partial}{\partial r})\vec{e_\phi} + (\vec{e_\theta} \cdot \tfrac{\partial}{\partial \theta})\vec{e_\theta} + (\vec{e_\phi} \cdot \tfrac{\partial}{\partial \phi})\vec{e_\phi}$$
$$= \quad \vec{e_r} \cdot \tfrac{\partial}{\partial r}\vec{e_r} + \vec{e_\theta} \cdot \tfrac{1}{r} + \vec{e_\phi}(\tfrac{\partial}{\partial \phi} \cdot \vec{e_\phi}) = \quad 0$$

$$\vec{\nabla} \cdot \vec{v} = \underbrace{\tfrac{\partial v_r}{\partial r} + \tfrac{2}{r}v_r}_{\text{product rule w/}\frac{\partial}{\partial r}} + \underbrace{\tfrac{1}{r}(\tfrac{\partial v_\theta}{\partial \theta} + \tfrac{\cos\theta}{\sin\theta}v_\theta)}_{\text{product rule w/}\frac{\partial}{\partial \theta}} + \underbrace{\tfrac{1}{r\sin\theta} \cdot \tfrac{\partial v_\phi}{\partial \phi}}_{\text{product rule w/}\frac{\partial}{\partial \phi}}$$

$$d\vec{x} = \Sigma_a(h_a dx_a)\vec{e_a} \qquad \vec{e_a} \cdot \vec{e_b} = \delta_{ab}$$

$$d(vol) = \Pi_a(h_a dx_a) = \begin{cases} dxdydz \\ r^2\sin\theta dr d\theta d\phi \text{ (depending on coordinate system)} \\ .... \\ .... \end{cases}$$

Jacobian... okay...??

$$\begin{cases} da_{||}\vec{e}_a = h_b dx_b \cdot h_i \cdot dx_i \qquad \vec{e_b} \times \vec{e_i} \\ da_r = r^2 d\Omega \vec{e_r} \\ d\Omega = \sin\theta d\theta d\phi \end{cases}$$

$$d(vol) = dxdydz = \frac{dvdudw}{||J||} \quad \text{where } ||J|| \text{ is the Jacobian.}$$

Spherical Coordinates::

$$||J|| = \frac{1}{r^2\sin\theta}$$

$$d(vol) = r^2\sin\theta dr d\theta d\phi$$

$$\begin{aligned}
\delta^3(\vec{x}-\vec{x_0})\cdot d(vol) &= \delta(v-v_0)\delta(u-u_0)\delta(w-w_0)\cdot dudvdw \\
&= \delta(u-u_0)\delta(v-v_0)\delta(w-w_0)\cdot dudvdw \\
\delta^3(\vec{x}-\vec{x_0}) &= \tfrac{\delta(u-u_0)\delta(v-v_0)\delta(w-w_0)}{d(vol)}\cdot dvdudw \\
&= ||J||\delta(u-u_0)\delta(v-v_0)\delta(w-w_0) \\
&= \tfrac{1}{r^2\sin\theta}\delta(r-r_0)\delta(\theta-\theta_0)\delta(\phi-\phi_0)
\end{aligned}$$

$$\nabla\cdot\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3} = \nabla\cdot\frac{\hat{r}}{r^2} = \delta^3(\vec{x}-\vec{x}')$$

$$\int\delta^3(\vec{x}-\vec{x}')drd\theta d\phi \qquad \int\sin\theta d\theta d\phi = 4\pi$$

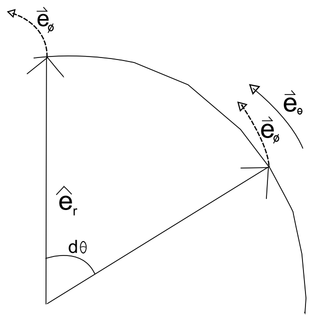
$$\nabla\cdot E = \frac{\rho}{\epsilon_0} \quad \text{Differential form of Gauss's Law}$$

$$\begin{aligned}
\int_V \nabla\cdot E &= \textstyle\int_S \vec{E}d\vec{A} \\
&= \textstyle\int_V \frac{\rho}{\epsilon_0} = \frac{1}{\epsilon_0}\int_V\rho
\end{aligned}$$

$$\int_S \vec{E}\cdot d\vec{A} = \frac{1}{\epsilon_0}Q_{enclosed}$$

Integral form of Gauss's Law (useful when there is symmetry)

- $\frac{\partial}{\partial r}\vec{e_r} = \vec{0}$
- $$\begin{aligned}
\frac{\partial}{\partial\phi}\vec{e_r} &= \\
&= \tfrac{\vec{e_r}'-\vec{e_r}}{d\phi} \\
&= \tfrac{(\cos d\phi\cdot\vec{e_r}+\sin d\phi\cdot\vec{e_\phi})\sin\theta-\vec{e_r}\sin\theta}{d\phi} \\
&= \tfrac{(\vec{e_r}\sin\theta+\sin\theta d\phi\vec{e_\phi}-\vec{e_r}\sin\theta)}{d\phi} \\
&= \sin\theta\vec{e_\phi}
\end{aligned}$$
- $\frac{\partial}{\partial\theta}\vec{e_\phi} = \vec{0}$


$\vec{e}_\phi$
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