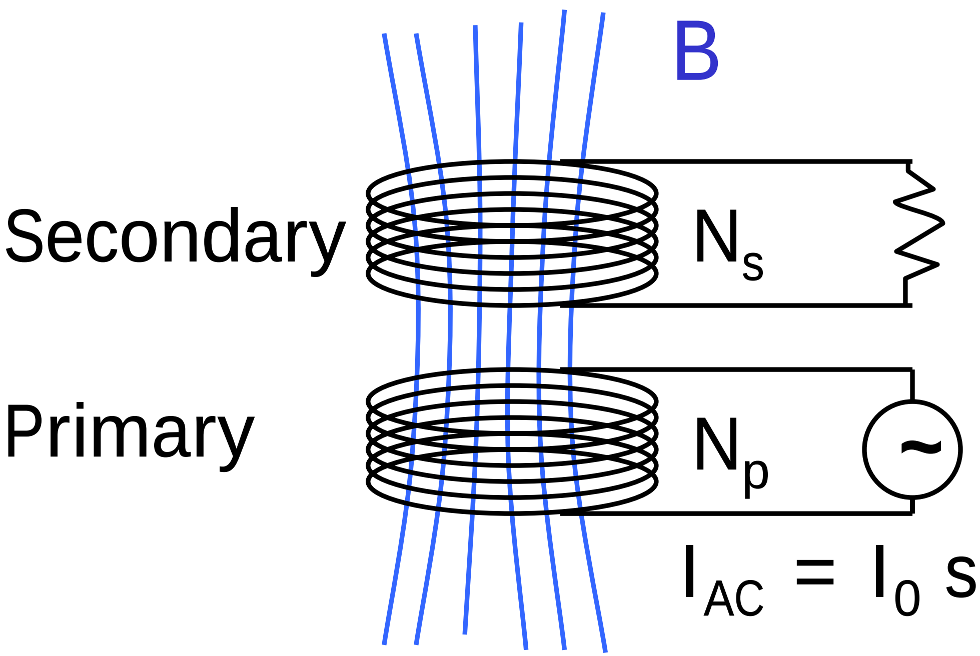
Electricity and Magnetism

- Review
  - Self and mutual inductance
  - Energy in B-Field
  - LR circuit
  - LRC circuits and Oscillations
  - AC circuits
  - Displacement current
  - Maxwell's equations
  - EM waves

Mutual Inductance

- Transformer action

$$\xi_s / \xi_p = N_s / N_p$$



$$I_{AC} = I_0 \sin(\omega t)$$

$$\xi_s = - N_s \, d\Phi_B/dt$$

same

$$\xi_p = - N_p \, d\Phi_B/dt$$

Flux through single turn

Mutual Inductance

$\xi_s / \xi_p = N_s / N_p$

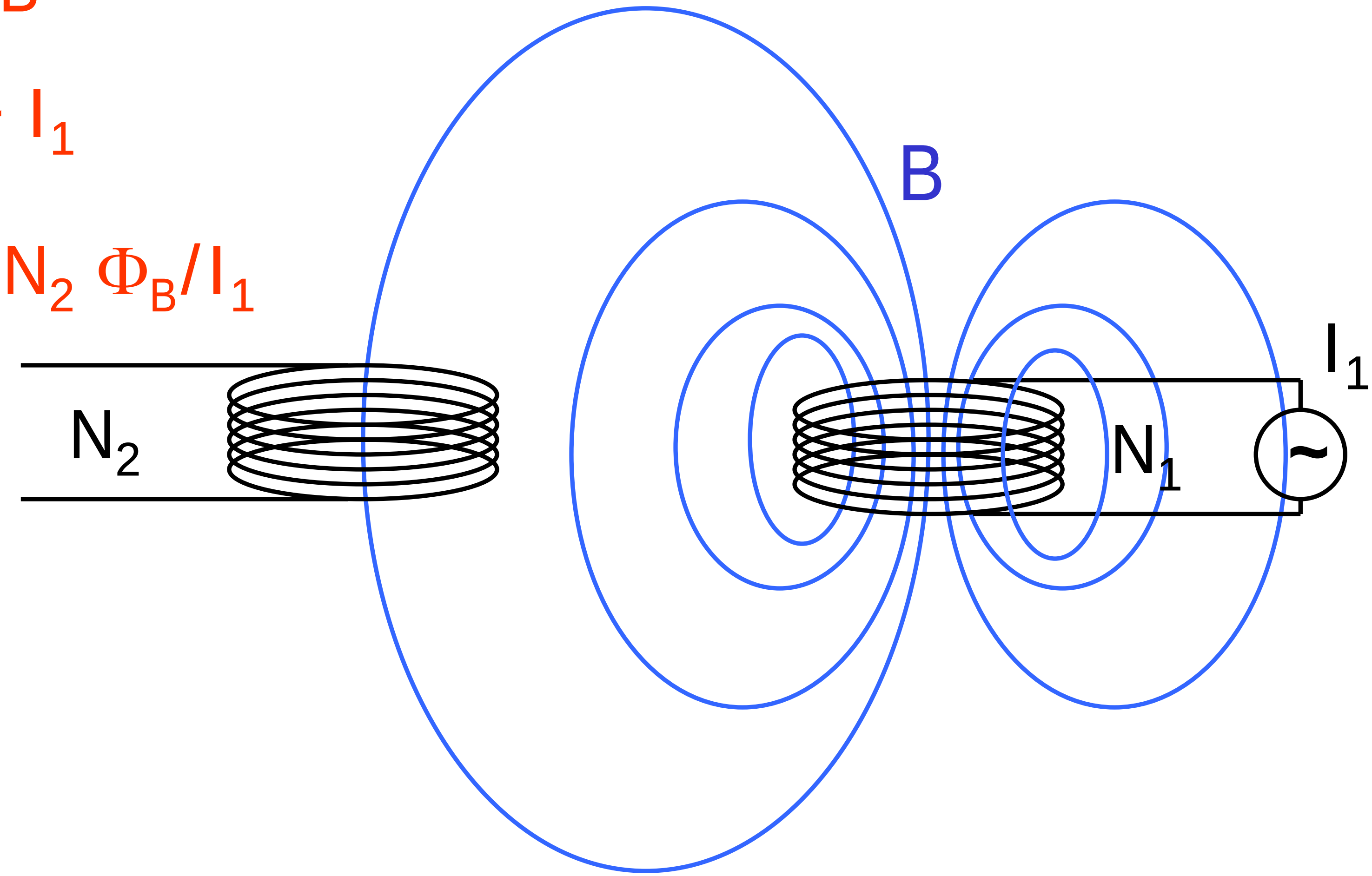
- Transformer action
- Transformers allow change of amplitude for AC voltage
  - ratio of secondary to primary windings
- Constructed such that Φ_B identical for primary and secondary
- What about general case of two coils?

Mutual Inductance

$\Phi_B \sim B$

$B \sim I_1$

Def.: $M_{12} = N_2 \, \Phi_B / I_1$



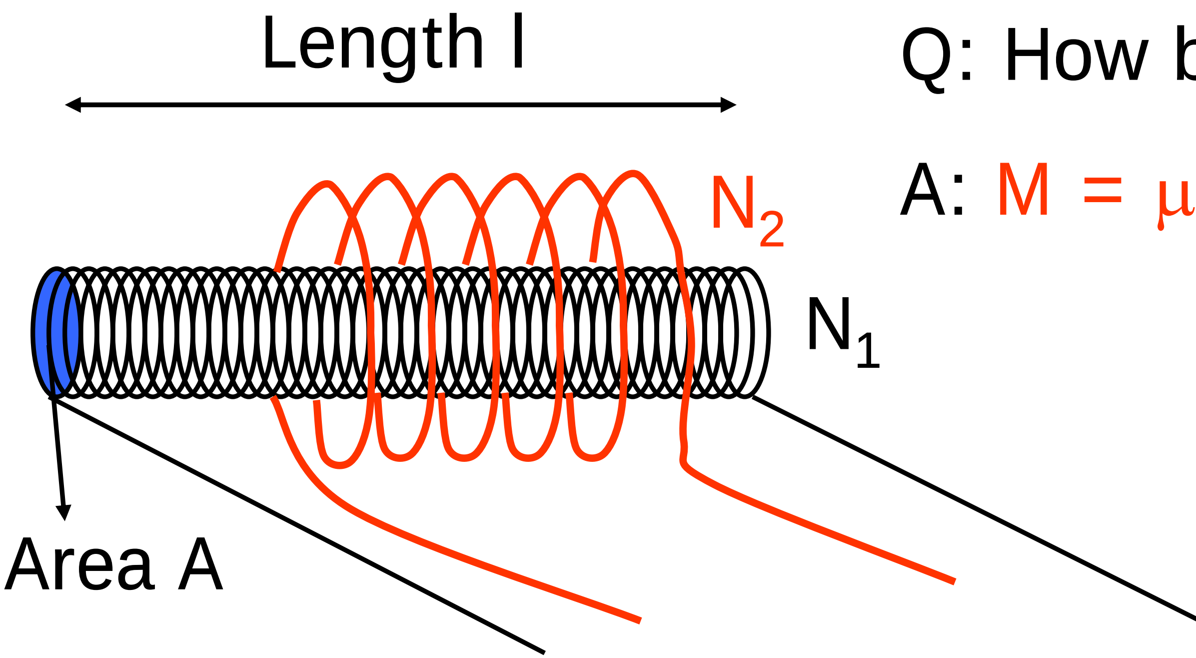
Mutal Inductance

- Coupling is symmetric: $M_{12} = M_{21} = M$
- M depends only on Geometry and Material
- Mutual inductance gives strength of coupling between two coils (conductors):

$$\xi_2 = - N_2 \, d\Phi_B/dt = - M \, dI_1/dt$$

- M relates ξ_2 and I_1 (or ξ_1 and I_2)
- Units: [M] = V/(A/s) = V s /A = H (‘Henry’)

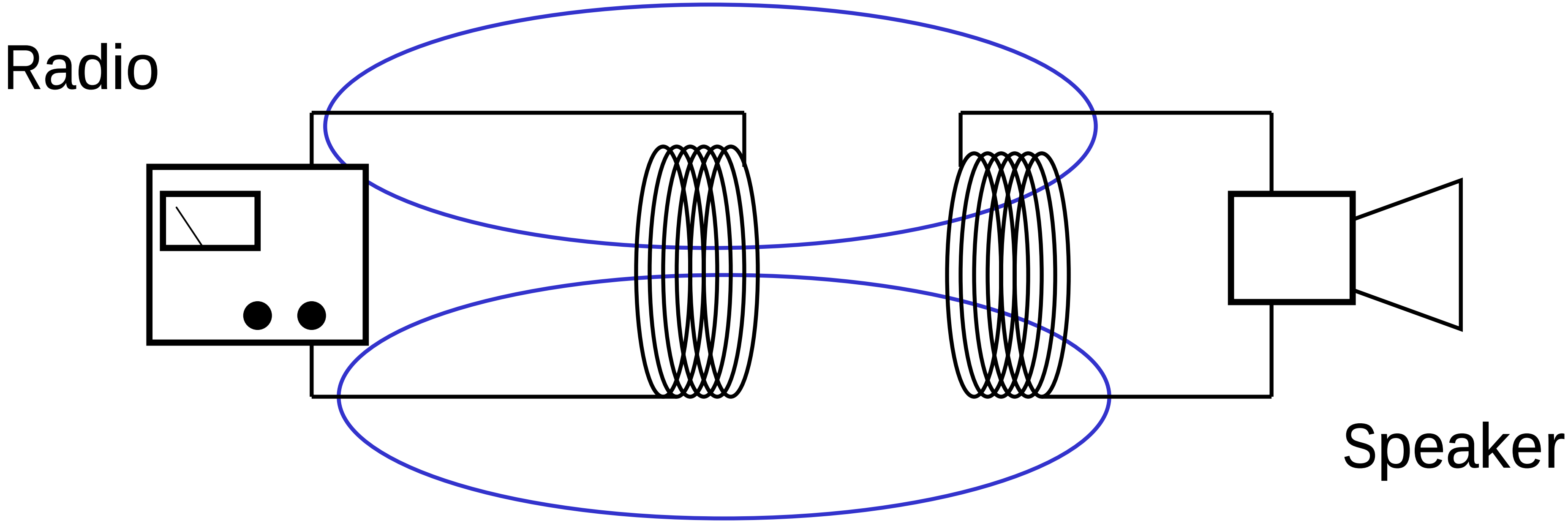
Example: Two Solenoids



Q: How big is $M = N_2 \Phi_B / I_1$?

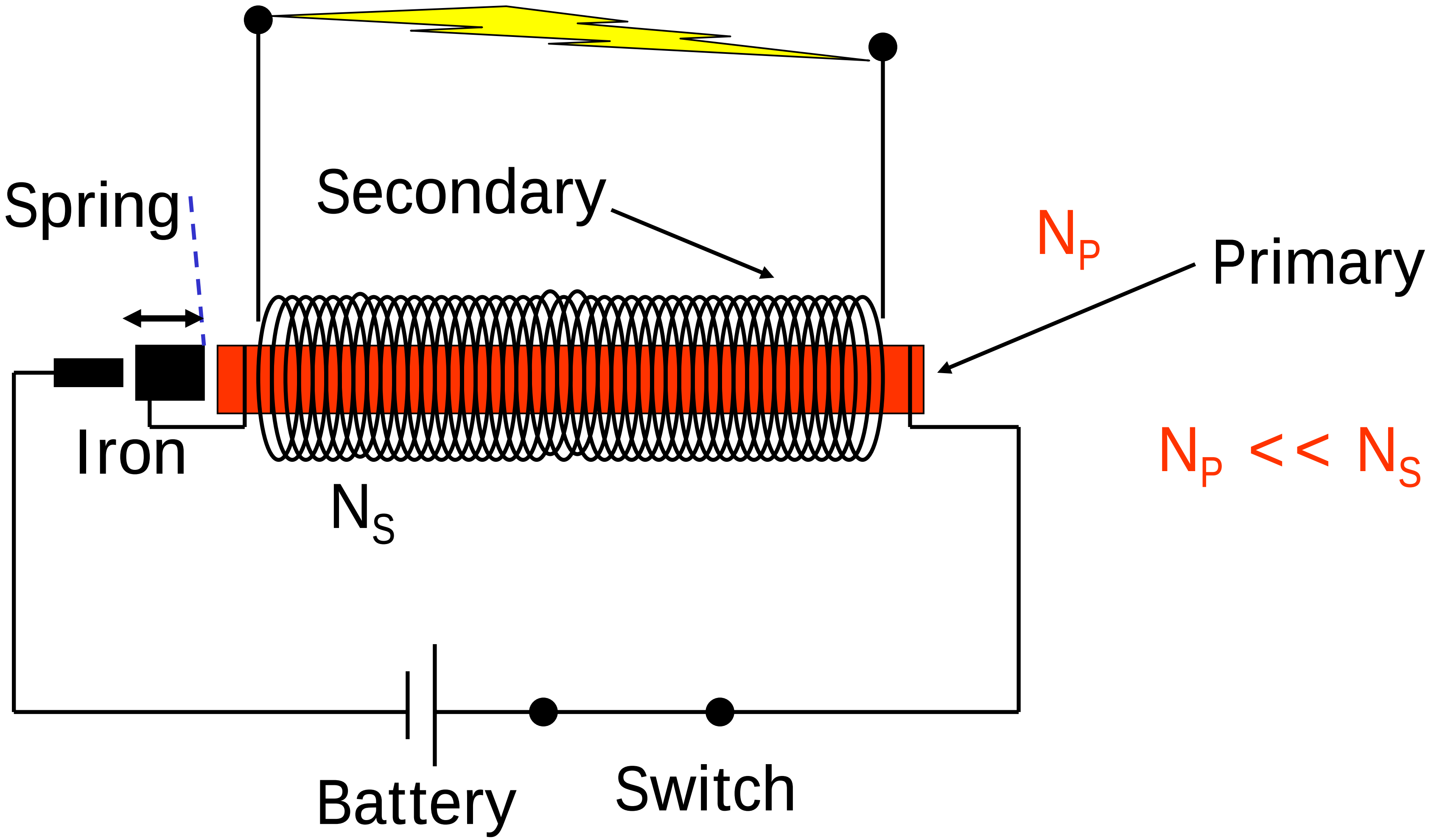
A: $M = \mu_0 N_1 N_2 A/l$

In-Class Demo: Two Coils



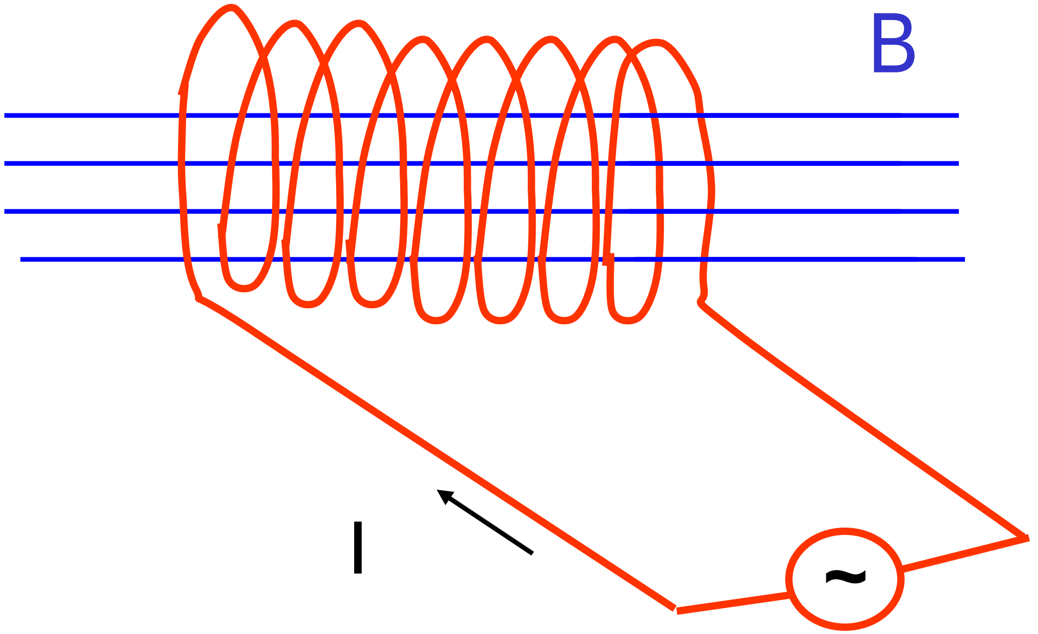
- Signal transmitted by varying B-Field
- Coupling depends on Geometry (angle, distance)

In-Class Demo: Marconi Coils



Self Inductance

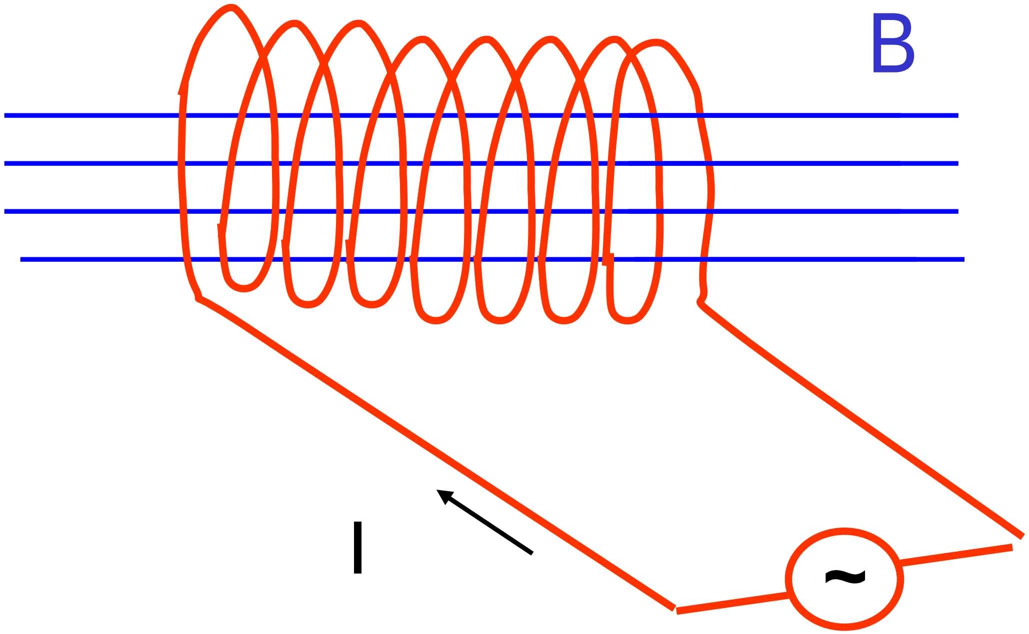
Circuit sees flux generated by it self



Def.: $L = N\,\Phi_B/I$

Self-Inductance

Example: Solenoid



Q: How big is L ?

A: $L = \mu_0 N^2 A/L$

Self Inductance

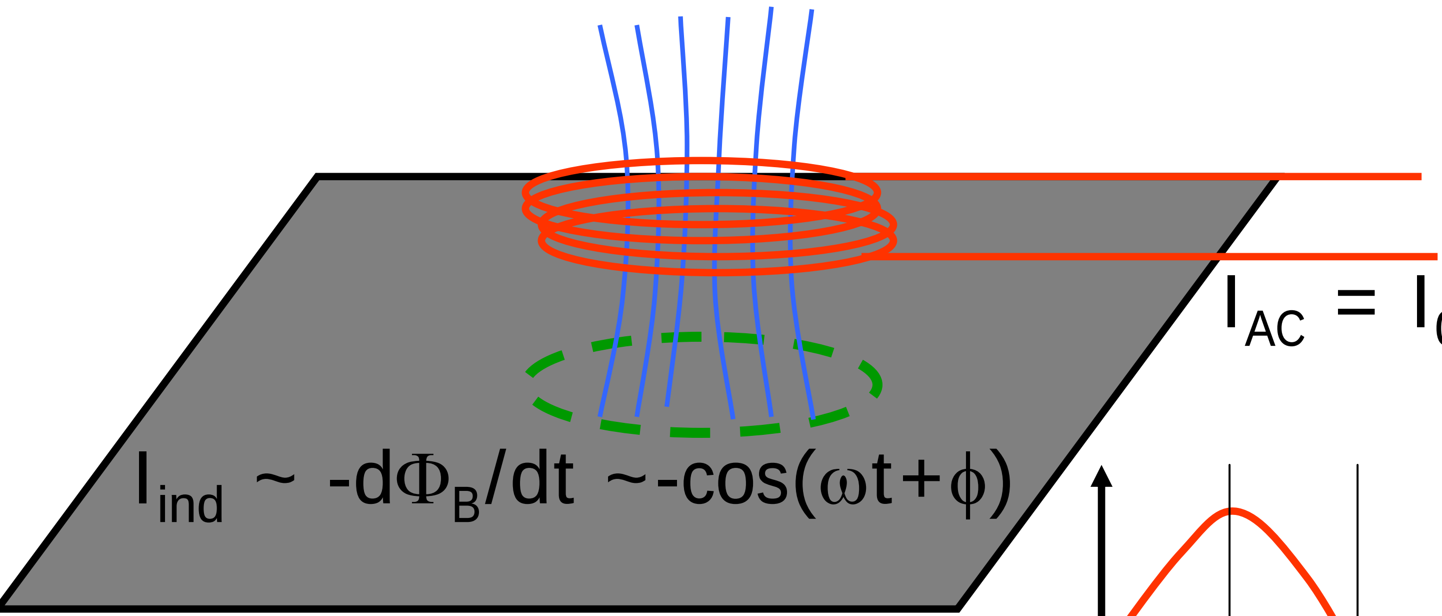
- L is also measured in [H]
- L connects induced EMF and variation in current:

$$\xi = - L\, dI/dt$$

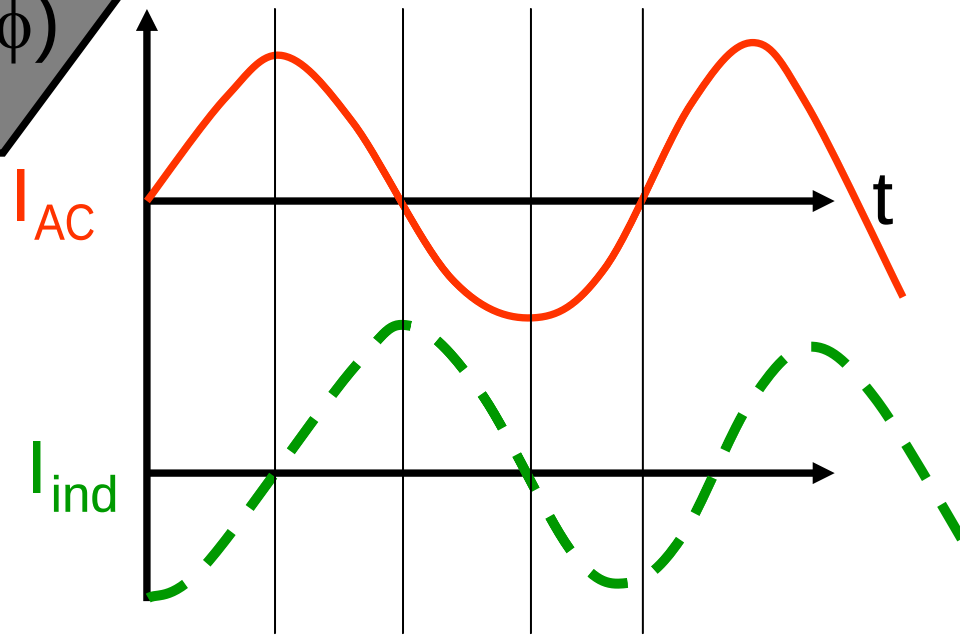
- Remember Lenz' Rule:

  Induced EMF will 'act against' change in current -> effective 'inertia'
- Delay between current and voltage

In-Class Demo: Levitating Coil



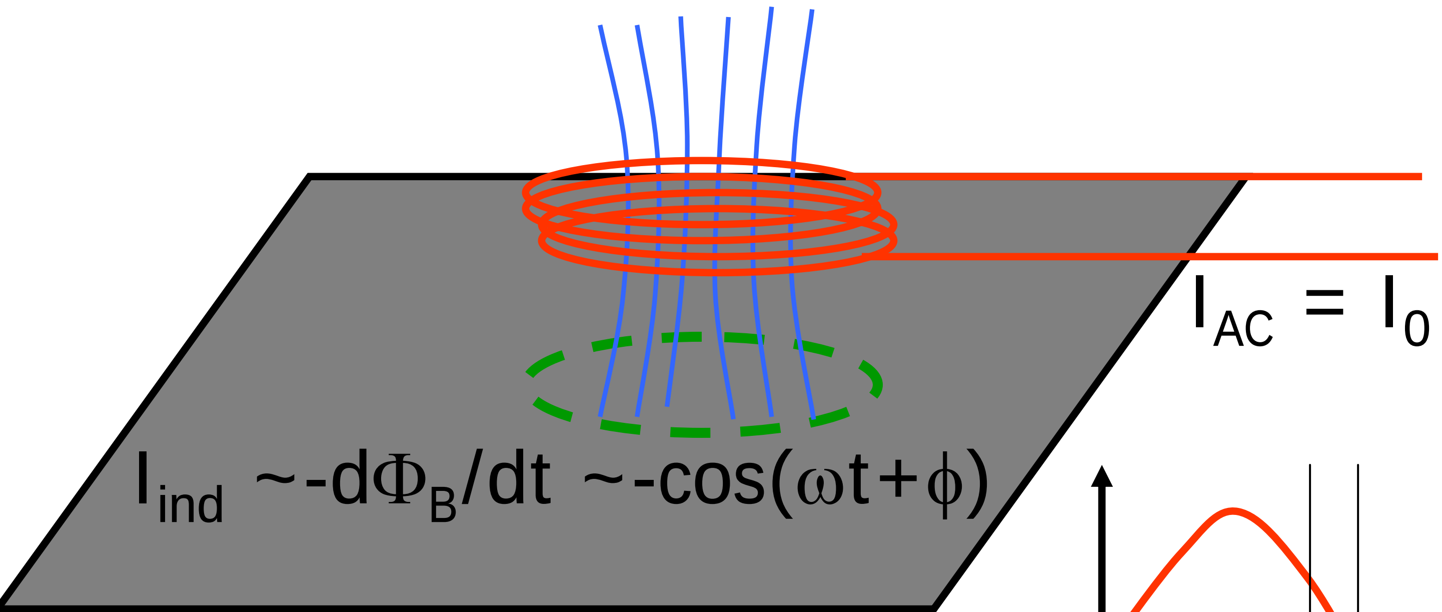
$I_{AC} = I_0 \sin(\omega t)$



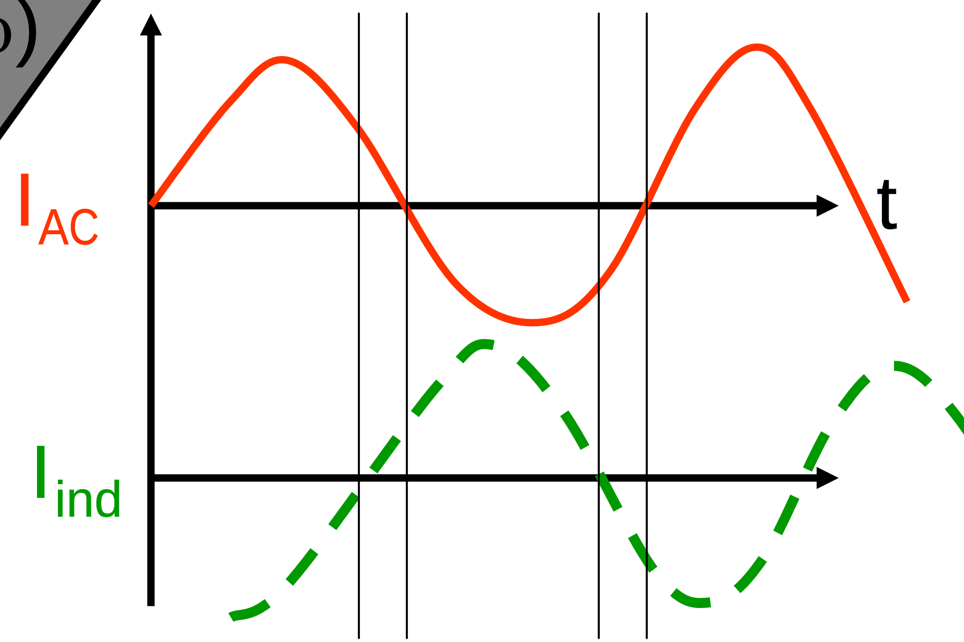
Without delay ($\phi = 0$):

No net force

In-Class Demo: Levitating Coil



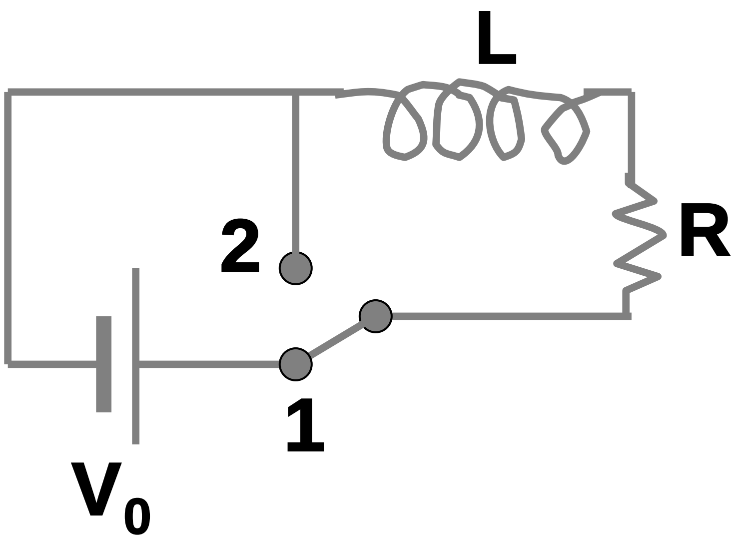
$I_{AC} = I_0 \sin(\omega t)$

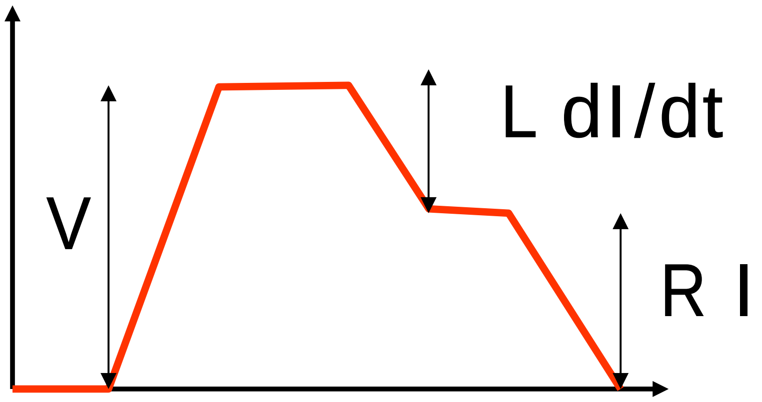


With delay (ϕ >0):

Net repulsion (currents are opposite most of the time)

RL Circuits

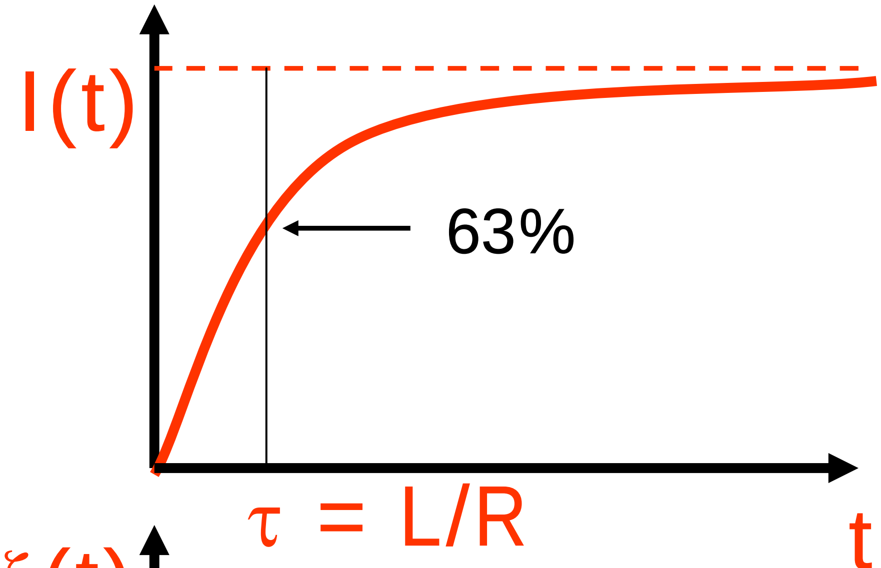




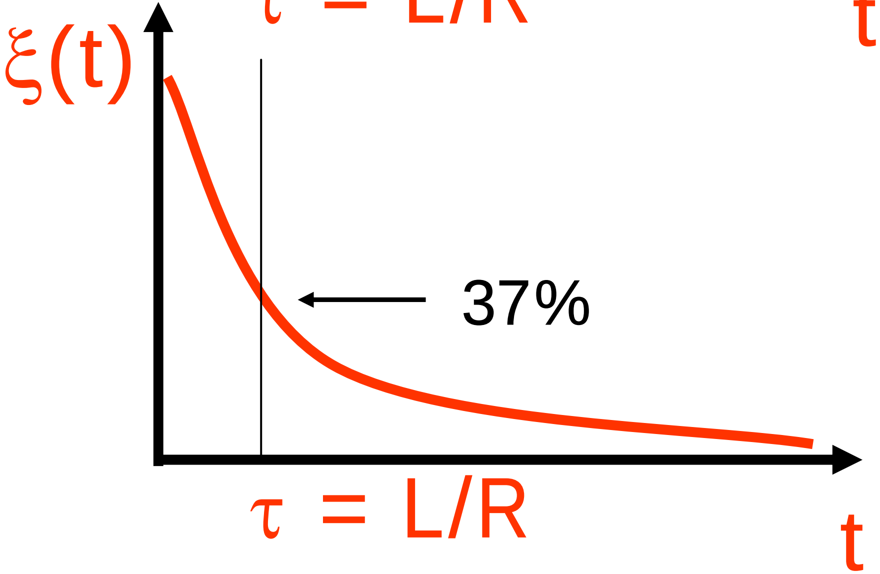
Kirchoffs Rule: $V_0 + \xi_{ind} = R\,I$ -> $V_0 = L\,dI/dt + R\,I$

Q: What is $I(t)$?

RL Circuits



$$I(t) = V_0/R\,[1 - \exp(-t/\tau)]$$

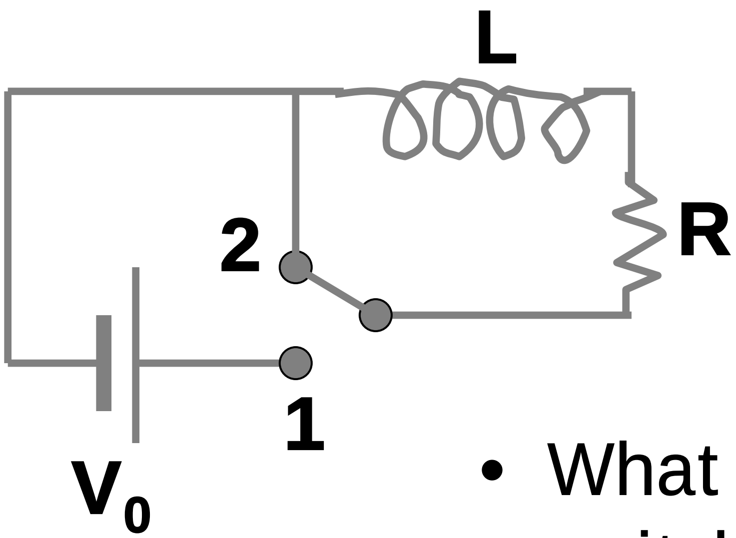


$$\xi(t) = V_0\,\exp(-t/\tau)$$

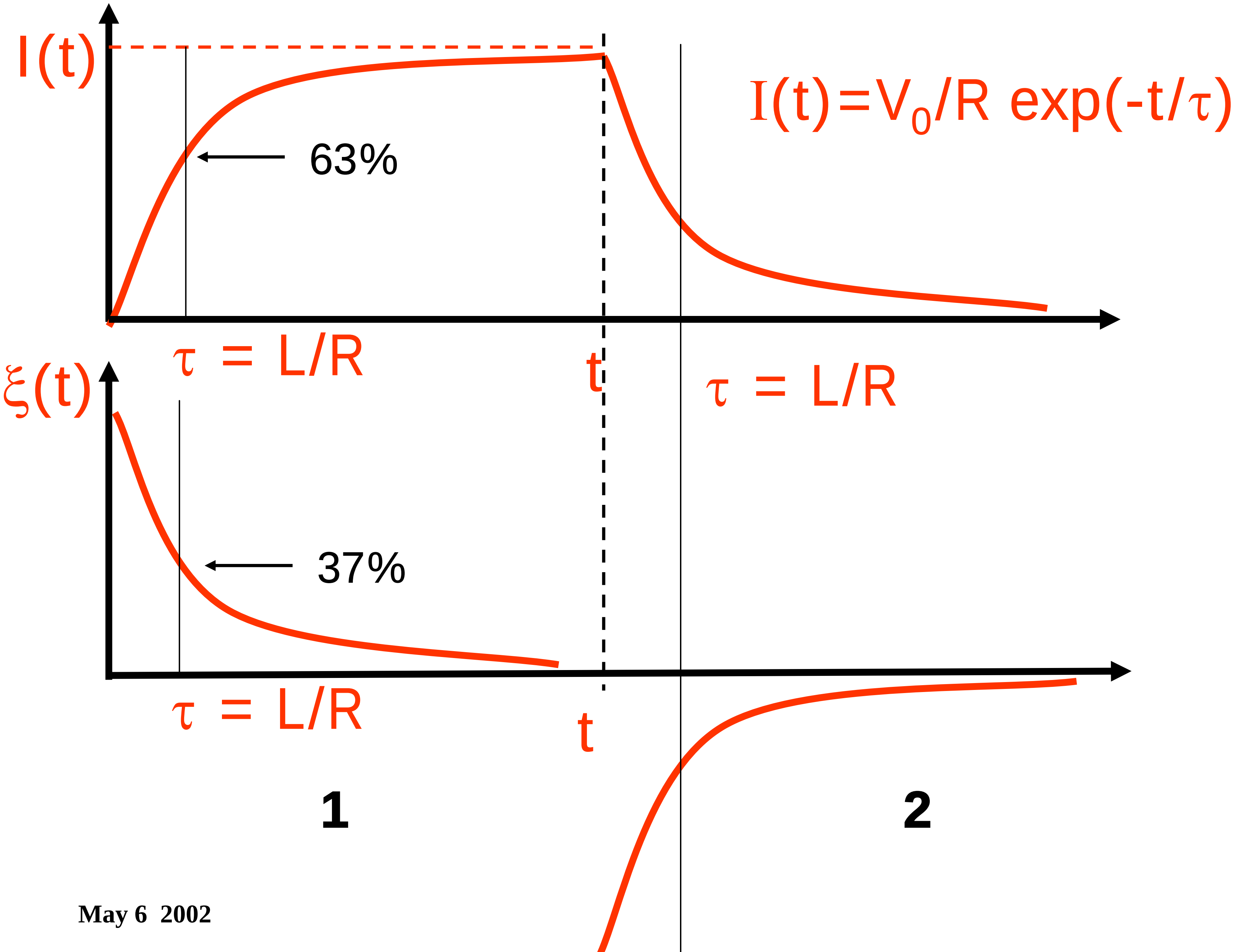
RL Circuits

- Inductance leads to ‘delay’ in reaction of current to change of voltage V_0
- All practical circuits have some L and R
  – change in I never instantaneous

'Back EMF'



- What happens if we move switch to position 2?


I(t)
I(t)=V0/R exp(-t/τ)
63%
τ = L/R
t
τ = L/R
ξ(t)
37%
τ = L/R
t
1
2

RL Circuits

- L counteracts change in current both ways
  - Resists increase in I when connecting voltage source
  - Resists decrease in I when disconnecting voltage source
  - ‘Back EMF’
- That’s what causes spark when switching off e.g. appliance, light

Energy Storage in Inductor

- Energy in Inductor
  - Start with Power $P = \xi I = L\, dI/dt\, I = dU/dt$

  -> $dU = L\, dI\, I$

  -> $U = \frac{1}{2} L I^2$

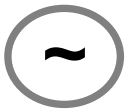
- Where is the Energy stored?
  - Example: Solenoid

  $U/\text{Volume} = \frac{1}{2} B^2/\mu_0$

RLC circuits

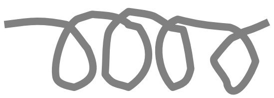
- Combine everything we know…
- Resonance Phenomena in RLC circuits
  - Resonance Phenomena known from mechanics (and engineering)
  - Great practical importance
  - video…

Summary of Circuit Components



V $V(t)$

R $V_R = IR$



L $V_L = L\, dI/dt$

C $V_C = 1/C \int I\, dt$

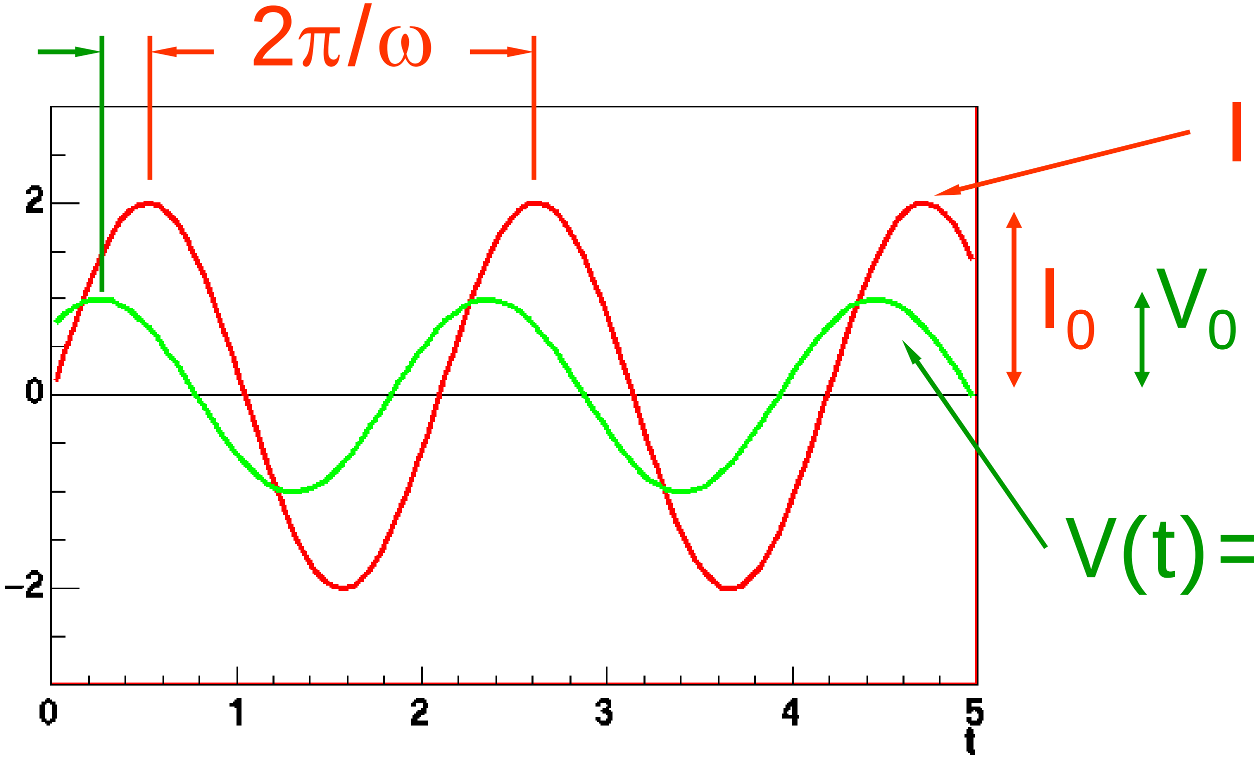
R,L,C in AC Circuit

- AC circuit
  - $I(t) = I_0 \sin(\omega t)$
  - $V(t) = V_0 \sin(\omega t + \phi)$

  } same ω!

- Relationship between V and I can be characterized by two quantities
  - Impedance $Z = V_0/I_0$
  - Phase-shift ϕ

AC circuit



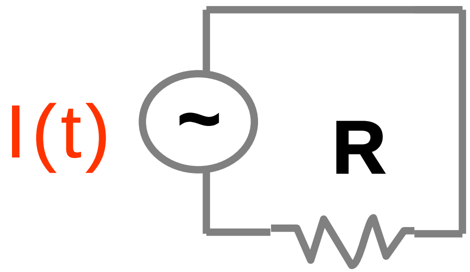
$I(t) = I_0 \sin(\omega t)$

$V(t) = V_0 \sin(\omega t + \phi)$

Impedance $Z = V_0/I_0$

Phase-shift ϕ

First: Look at the components

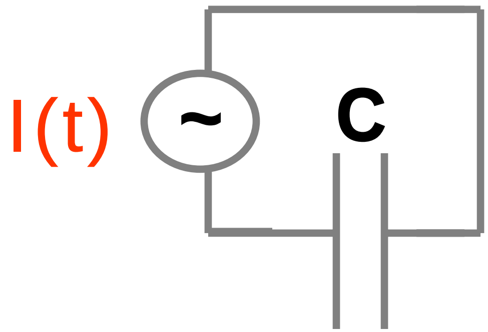


$V = I R$

$Z = R$

$\phi = 0$

V and I in phase

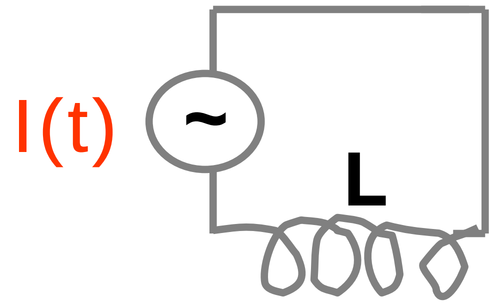


$V = Q/C = 1/C \int I dt$

$Z = 1/(\omega C)$

$\phi = -\pi/2$

V lags I by 90°



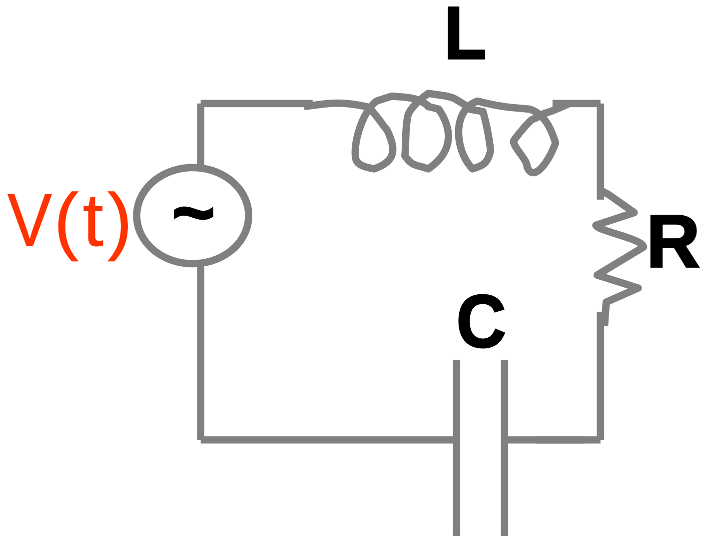
$V = L\, dI/dt$

$Z = \omega L$

$\phi = \pi/2$

I lags V by 90°

RLC circuit

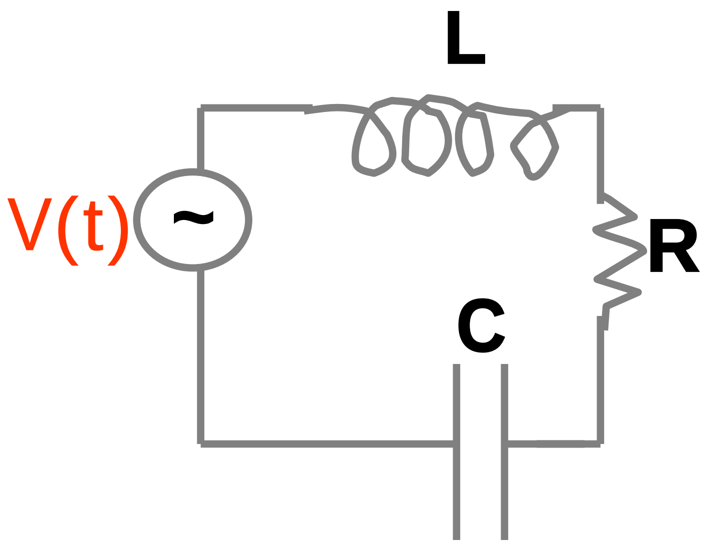


$V - L\, dI/dt \;-\; IR \;-\; Q/C = 0$

$L\, d^2Q/dt^2 = -1/C\, Q - R\, dQ/dt + V$

2nd order differential equation

RLC circuit



$$V - L\, dI/dt - IR - Q/C = 0$$

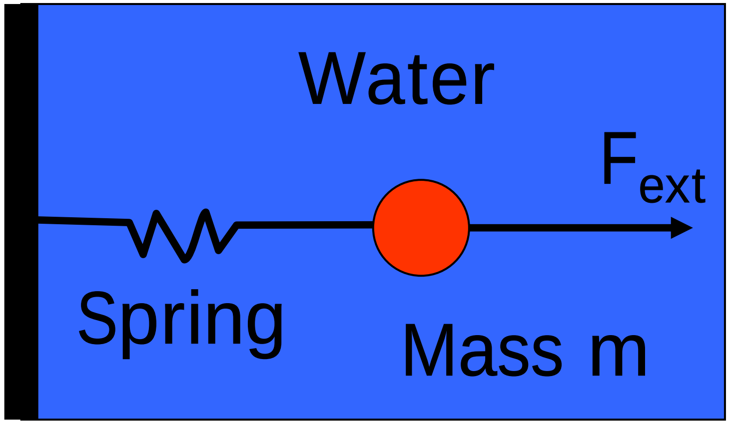
$$L\, d^2Q/dt^2 = -1/C\, Q - R\, dQ/dt + V$$

'Inertia'

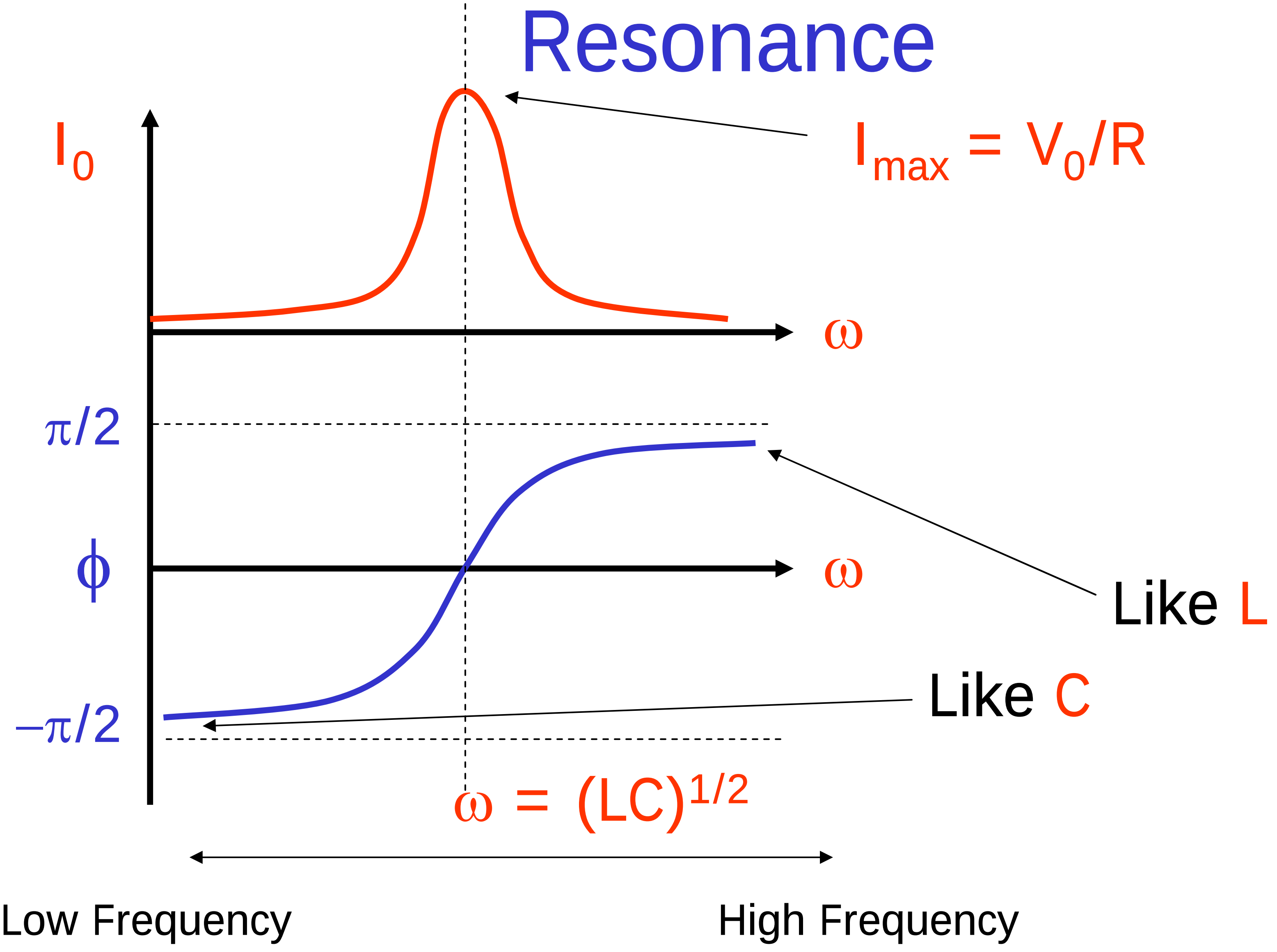
'Spring'

'Friction'

$$m\, d^2x/dt^2 = -k\, x - f\, dx/dt + F_{ext}$$



Resonance



I_0
$I_{max} = V_0/R$
ω
π/2
φ
ω
Like L
Like C
−π/2
ω = (LC)^1/2
Low Frequency
High Frequency

RLC circuit

$V_0 \sin(\omega t) = I_0\{[\omega L - 1/(\omega C)] \cos(\omega t - \phi) + R \sin(\omega t - \phi)\}$

Solution (requires two tricks):

$I_0 = V_0/([\omega L - 1/(\omega C)]^2 + R^2)^{1/2} = V_0/Z$

$\tan(\phi) = [\omega L - 1/(\omega C)]/R$

-> For $\omega L = 1/(\omega C)$, Z is minimal and $\phi = 0$

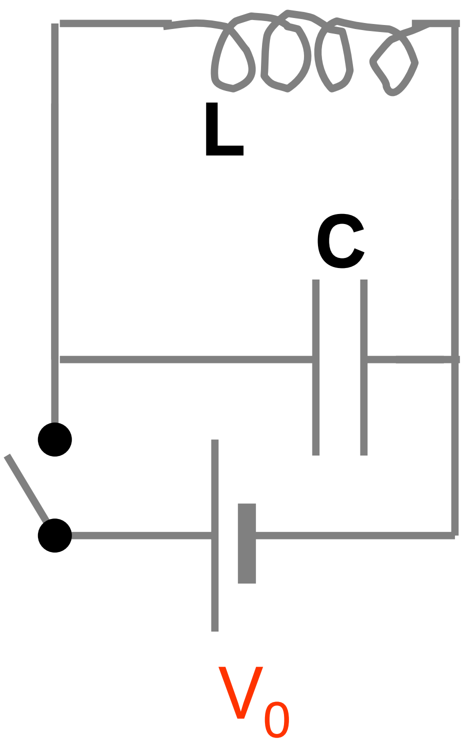
i.e. $\omega_0 = 1/(LC)^{1/2}$ <u>Resonance Frequency</u>

Resonance

- Practical importance
  - ‘Tuning’ a radio or TV means adjusting the resonance frequency of a circuit to match the frequency of the carrier signal

LC Circuit

- What happens if we open switch?



$$-L\, dI/dt - Q/C = 0$$

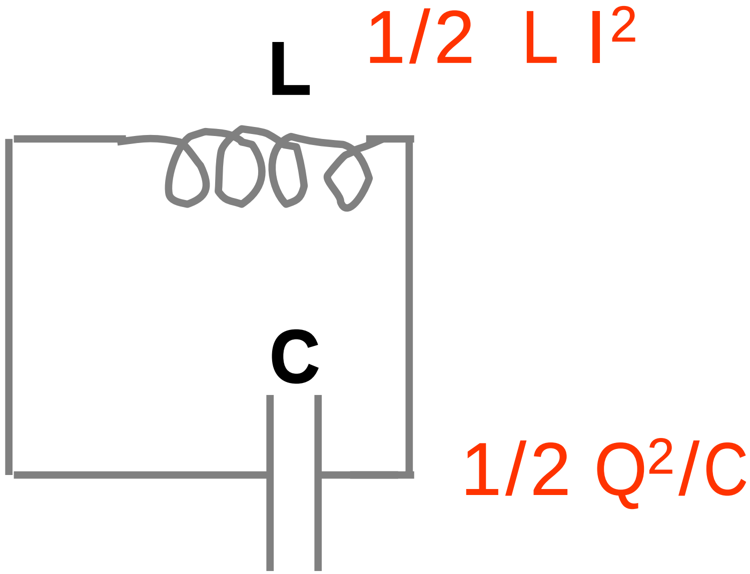
$$L\, d^2Q/dt^2 + Q/C = 0$$

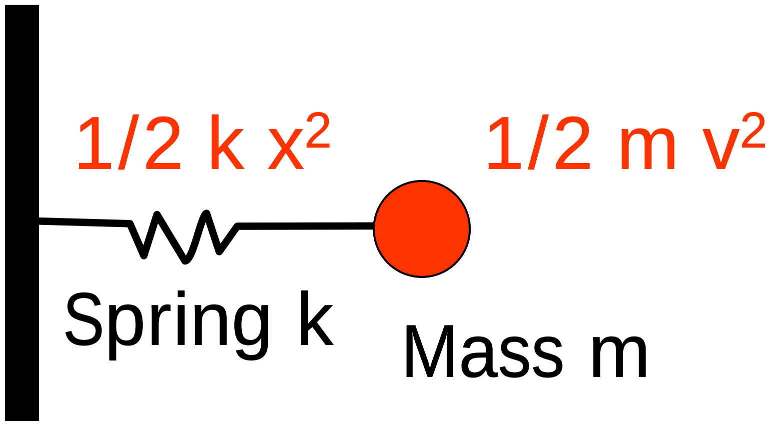
$\updownarrow$

$$d^2x/dt^2 + \omega_0^2 x = 0$$

Harmonic Oscillator!

LC Circuit





Energy in E-Field

↕ Oscillation

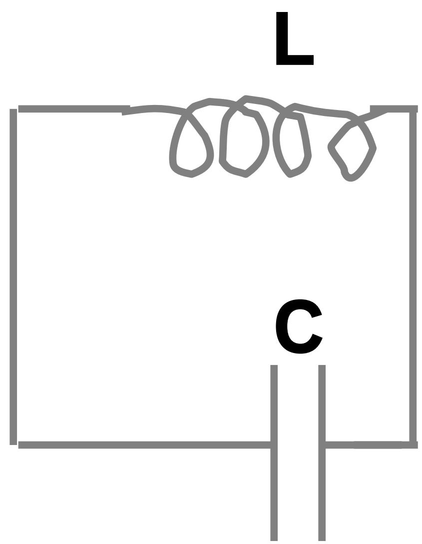
Energy in B-Field

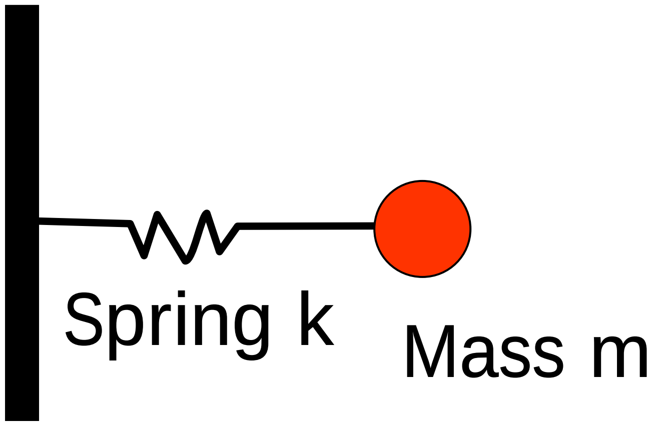
Potential Energy

↕ Oscillation

Kinetic Energy

LC Circuit





$d^2Q/dt^2 + 1/(LC)\,Q = 0$

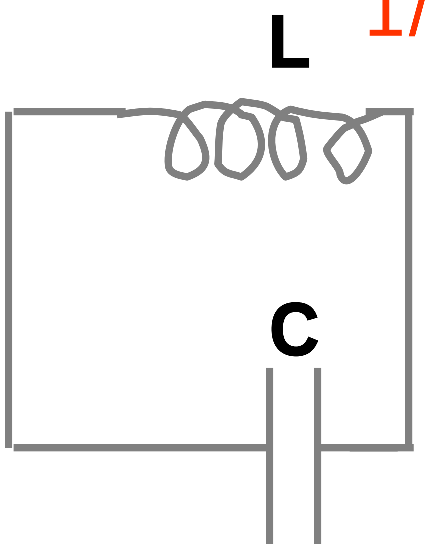
$d^2x/dt^2 + k/m\,x = 0$

$\omega_0^2 = 1/(LC)$

$\omega_0^2 = k/m$

LC Circuit

$\frac{1}{2} L I^2$



$\frac{1}{2} Q^2/C$

Energy in E-Field

↕ Oscillation

Energy in B-Field

- Total energy U(t) is conserved:

$Q(t) \sim \cos(\omega t)$

$dQ/dt \sim \sin(\omega t)$

$U_L \sim (dQ/dt)^2 \sim \sin^2$

$U_C \sim Q(t)^2 \sim \cos^2$

$\cos^2(\omega t) + \sin^2(\omega t) = 1$

Electromagnetic Oscillations

- In an LC circuit, we see oscillations:

Energy in E-Field

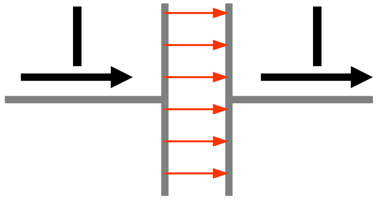
↕

Energy in B-Field

- Q: Can we get oscillations without circuit?
- A: Yes!
  - **Electromagnetic Waves**

Displacement Current

- Ampere's Law broken – How can we fix it?

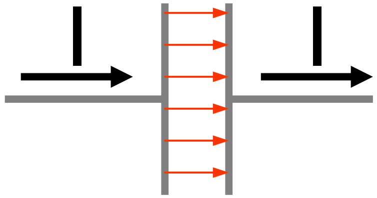


$Q = C\,V$

Displacement Current $I_D = \varepsilon_0\, d\Phi_E/dt$

Displacement Current

- Extension of Ampere's Law:



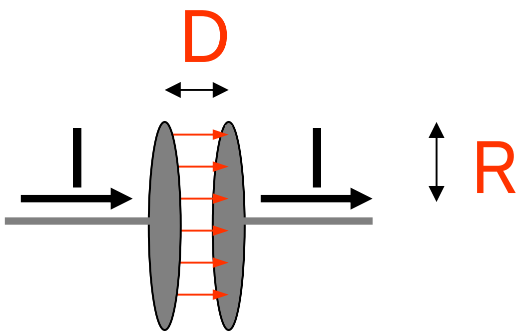
$Q = C\ V$

Displacement Current $I_D = \varepsilon_0\ d\Phi_E/dt$

Changing field inside C also produces B-Field!

Displacement Current

- Example calculation: B(r) for r > R



Q = C V

-> $B(r) = R^2/(2rc^2)\ dV/dt$

Maxwell's Equations

$$\oint_{A_{closed}} \vec{E} \cdot d\vec{A} = \frac{Q_{encl}}{\epsilon_0}$$

$$\oint_{L_{closed}} \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt}$$

$$\oint_{A_{closed}} \vec{B} \cdot d\vec{A} = 0$$

$$\oint_{L_{closed}} \vec{B} \cdot d\vec{l} = \mu_0 I_{encl} + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$$

- Symmetry between E and B
  - although there are no magnetic monopoles
- Basis for radio, TV, electric motors, generators, electric power transmission, electric circuits etc

Maxwell's Equations

$$\oint_{A_{closed}} \vec{E} \cdot d\vec{A} = \frac{Q_{encl}}{\epsilon_0}$$

$$\oint_{L_{closed}} \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt}$$

$$\oint_{A_{closed}} \vec{B} \cdot d\vec{A} = 0$$

$$\oint_{L_{closed}} \vec{B} \cdot d\vec{l} = \mu_0 I_{encl} + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$$

$1/c^2$

- M.E.'s ***predict*** electromagnetic waves, moving with speed of light
- Major triumph of science

Electromagnetic Waves

- Until end of semester:
  - What are electromagnetic waves?
  - How does their existence follow from Maxwells equations?
  - What are the properties of E.M. waves?
- Prediction was far from obvious
  - No hint that E.M. waves exist
  - Involves quite a bit of math

Reminder on Waves

- Types of waves
  - Transverse
  - Longitudinal
    - compression/decompression

Reminder on Waves

- For a travelling wave (sound, water)

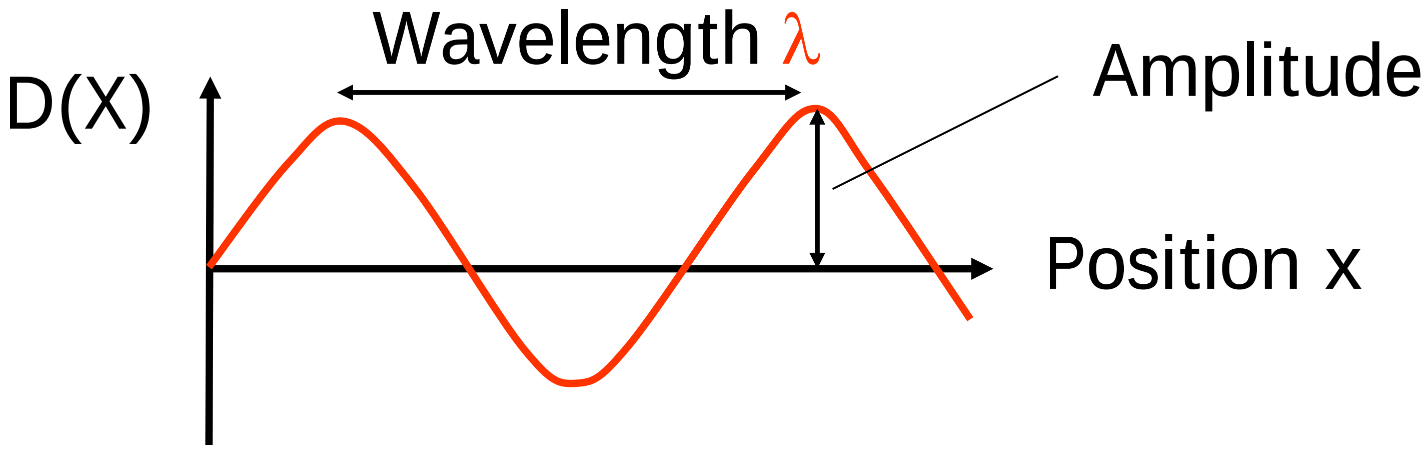
  Q: What is actually moving?
- **-> Energy!**
- Speed of propagation: $v = \lambda f$
- Wave equation:

$$\frac{\partial^2 D(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 D(x,t)}{\partial t^2}$$

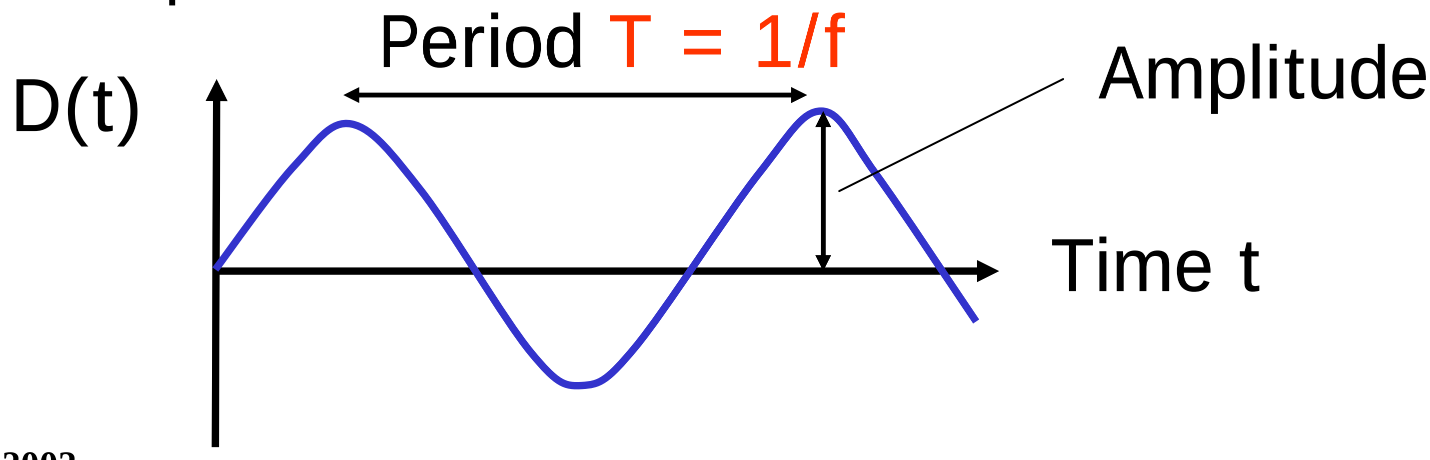
Couples variation in time and space

Reminder on Waves

At a moment in time:



At a point in space:



Wave Equation

- Wave equation:

$$\frac{\partial^2 D(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 D(x,t)}{\partial t^2}$$

Couples variation in time and space

- Speed of propagation: $v = \lambda f$
- *How can we derive a wave equation from Maxwells equations?*

Wave Properties

- What do we want to know about waves:
  - Speed of propagation?
  - Transverse or longitudinal oscillation?
  - What is oscillating?
  - What are typical frequencies/wavelengths?

Differential Form of M.E.

$$\oint_{A_{closed}} \vec{E} \cdot d\vec{A} = \frac{Q_{encl}}{\epsilon_0}$$

$$\oint_{L_{closed}} \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt}$$

$$\oint_{A_{closed}} \vec{B} \cdot d\vec{A} = 0$$

$$\oint_{L_{closed}} \vec{B} \cdot d\vec{l} = \mu_0 I_{encl} + \mu_0\ \epsilon_0 \frac{d\Phi_E}{dt}$$

Gauss, Stokes

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \mu_0\ \vec{j} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

Differential Form of M.E.

Flux/Unit Volume

Charge Density

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \, \vec{j} + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}$$

Loop Integral/Unit Area

Current Density

Maxwell's Equations in Vacuum

- Look at Maxwell's Equations without charges, currents

$$\vec{\nabla} \cdot \vec{E} = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

Now completely symmetric!

Maxwell's Equations in Vacuum

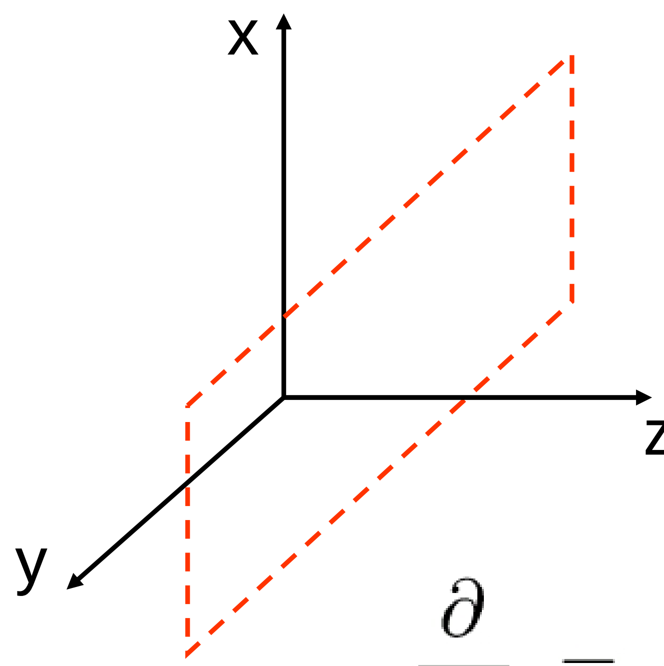
I. $$\vec{\nabla} \cdot \vec{E} = 0$$

II. $$\vec{\nabla} \cdot \vec{B} = 0$$

III. $$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

VI. $$\vec{\nabla} \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

Solve for a simple geometry



Allow variations only in z-direction:

$$\frac{\partial}{\partial x} = 0$$

$$\frac{\partial}{\partial y} = 0$$

Electromagnetic Waves

- We found wave equations:

$$\frac{\partial^2 B_y}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 B_y}{\partial t^2}$$
$$\frac{\partial^2 E_y}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 E_y}{\partial t^2}$$

same for E_x, B_x

v = c

E and B are oscillating!

Electromagnetic Waves

- Note: (E_x, B_y) and (E_y, B_x) independent:

E_y, B_x

$$\frac{\partial B_x}{\partial z} = \frac{1}{c^2}\frac{\partial E_y}{\partial t}$$

$$\frac{\partial B_y}{\partial z} = -\frac{1}{c^2}\frac{\partial E_x}{\partial t}$$

E_x, B_y

$$\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t}$$

$$\frac{\partial E_y}{\partial z} = \frac{\partial B_x}{\partial t}$$

$\vec{E} \perp \vec{B}$

Plane waves

- Example solution: Plane waves

$$\begin{aligned} E_y &= E_0 \cos(kz - \omega t) \\ B_x &= B_0 \cos(kz - \omega t) \\ \text{with } k &= \frac{2\pi}{\lambda}, \omega = 2\pi f \text{ and } f\lambda = c. \end{aligned}$$

- We can express other functions as linear combinations of sin,cos
  - 'White' light is combination of waves of different frequency
  - In-Class Demo…

Plane waves

- Example solution: Plane waves

$$E_y = E_0 \cos(kz - \omega t)$$
$$B_x = B_0 \cos(kz - \omega t)$$
$$\text{with } k = \frac{2\pi}{\lambda}, \omega = 2\pi f \text{ and } f\lambda = c.$$

$$\frac{\partial B_x}{\partial z} = \frac{1}{c^2}\frac{\partial E_y}{\partial t}$$
$$\frac{\partial B_y}{\partial z} = -\frac{1}{c^2}\frac{\partial E_x}{\partial t}$$

$$\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t}$$
$$\frac{\partial E_y}{\partial z} = \frac{\partial B_x}{\partial t}$$

$$\frac{\partial^2 B_y}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 B_y}{\partial t^2}$$
$$\frac{\partial^2 E_y}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 E_y}{\partial t^2}$$

$$-kE_0 \sin(kz - \omega t) = \omega B_0 \sin(kz - \omega t)$$
$$\Rightarrow \frac{|E_0|}{|B_0|} = \frac{k}{\omega} = c$$

E.M. Wave Summary

- $\vec{E} \perp \vec{B}$ and perpendicular to direction of propagation
- Transverse waves
- Speed of propagation $v = c = \lambda f$
- $|\vec{E}|/|\vec{B}| = c$
- E.M. waves travel without medium

Typical E.M. wavelength

- FM Radio:
  - f ~ 100 MHz
  - $\lambda = c/f \sim 3$m
  - Antenna ~ O(m)
- Cell phone
  - Antenna ~ O(0.1m)
  - $f = c/\lambda = 3$ GHz

Energy in E.M. Waves

- Remember:
  - Energy/Volume given by $1/2\ \varepsilon_0\ E^2$ and $1/2\ B^2/\mu_0$

- Energy density for E.M. wave:

$$u = \varepsilon_0\ E^2$$

- What about power?