8.03SC Physics III: Vibrations and Waves, Fall 2012

Transcript – Lecture 2: Damped Free Oscillations

PROFESSOR: I will start today with a very interesting phenomenon which is known as a beat phenomenon. Suppose you have two simple harmonic motions with the same amplitude but with different frequencies.

So we have x1 is A times cosine omega 1 t. And you have x2, which is A times cos omega two t. And the two are different, same amplitude. If you sum them, x1 plus x2, you get 2A times the cosine of half of the sum omega 1 plus omega 2 divided by 2 times t times the cosine of half the difference.

And when you look at this, you have here a frequency which is high compared to this frequency. You can best see that if omega 1 is very close to omega 2. And you could simply replace it by omega. Then this is simply saying cosine omega t.

And so if you change from the notation of omega in your head to frequency in Hertz, which is a lot easier to think of, omega is 2 pi times the frequency, f. This is in Hertz, and this is in radians per second.

Suppose you have an f1 which is 256 Hertz, and you have an F2 which is 254 Hertz. If you take these numbers now as a working example, then the period of this oscillation is 1/256 of a seconds. But the period of this oscillation is only 1 second because it's the difference between the 2 divided by 2.

And so what happens is that if you make a plot of x as a function of time, I will first sketch this very slow term. So this is very slow compared to this one, which is very fast. I will first make a sketch of the very slow one. So this is the slow one. And I will make also a dotted line here to guide my hands when I'm going to make the plot.

So what you're going to see now is that the fast term goes [MAKES STATIC NOISE], and the slow one goes like this. And the net result is this. This is the fast one, the fast one. And so you see right here is when the slow one is 0. So the slow one kills the amplitude.

You can think of this as an amplitude modulation, that the A is multiplied by that cosine function. We defined-- that is just a matter of definition-- we define this as the beat period when the two are in phase. And here, they are again in phase. The two are here 180 degrees out of phase. We define this as the beat period.

For the working example that we have on the blackboard there, the 256 Hertz and the 254 Hertz, from here to here would be one second. That's the time for the slow one to go 360 degrees. And so the beat periods would, in that case, be half a second.

So if you want to see in your head what is happening, then if one oscillation is like this, then when they are in phase-- each one has an amplitude A-- when they are in phase, you get 2A. But when they are out of phase, you have to add this 180 degrees out of phase. Then, the net result is

0. And that's the case here. And that's the case there. And that's the case there. And this is known in the literature as a beat phenomenon.

And the frequency of the beat, so f beat, is then defined as delta f in our case. 2 Hertz-- indeed, you see that the beat period is half a second. A very nice way to demonstrate this is to make you listen to it.

I have here a tuning fork of 256 Hertz. And I another one which is also 256 Hertz. But I can change this one by putting a little weight on there. I can make the frequency a little lower.

I'll put a weight here. And I will simultaneously make you listen to them. And if somehow I managed to give them the same amplitude, your eardrum will reach this cycle, which is at 256. And then, there comes a time that hear nothing. That means sound plus sounds makes silence, an interesting concept called interference. And you can hear that. So listen carefully.

That's the beat. If I make the frequency different, larger, the beats come faster. Can all of you hear this, also in the back? So now, I will make the offset larger. Unpleasant-- it makes you feel like throwing up or something, very high beat frequency.

If the amplitudes of the two are not the same, then you don't have to be a rocket scientist to anticipate what will happen. You just never get a 0 here. But you always have some residual. And so you'll get something that looks like this. Are you still have an oscillation here.

And I can also make you see that. It's harder to make you hear that because your ears cannot really tell the difference between a little bit of sound and no sound. You hear [MAKES VIBRATIONAL NOISE], and that's all you hear.

But I can make you see that. And I'm going to make you see that right there when we pick up the signals of these two tuning forks with a microphone. And we will display it there with an oscilloscopes. So you will hear it and you will see it at the same time.

And it is unlikely when I strike them that the amplitudes, as received by the microphone, will be exactly the same. So it's not likely you will see it go through 0. But I can, of course, change the distance to the microphone and try to get it as close as I can to 0.

[VIBRATIONAL HUMMING SOUND]

PROFESSOR: Not bad. Now, it's close to 0. I'll do it once more.

[REPEATS SOUND]

PROFESSOR: Now, I purposefully make the amplitudes different. I bring one very close to the microphone. This is what you saw there on the board.

[REPEATS SOUND]

PROFESSOR: And if I make the amplitudes the same, then you get--.

[REPEATS SOUND]

PROFESSOR: --you get the 0 back. Beat phenomenon are actually quite common.

When you sit on an airplane on the runway and the airplane has two engines, perhaps you have noticed something very irritating at times, that there's some crazy humming sound that goes up in volume, and then it goes down, and goes up. That is almost certainly the two engines beating against each other. They're not rotating at exactly the same frequency, but slightly off. That's my explanation why you have that strangely rising volume and then decaying volume. It's very common.

When I was a student, I remember there were two faucets in my bedroom. And no matter how much I tightened the faucets, they were leaking. They were dripping. And they were dripping in such a way that the frequency with which the drops came out was almost the same. So it went like this this-- clunk, clunk, clunk.

But the frequency was never exactly the same. So after a while, it would go clunk-clunk, clunk-clunk, clunk-clunk. And then after a while, clunk, clunk. It drove me crazy!

But of course, I took pride in knowing that it's a beat phenomena except that there, sound plus sound never made silence because when they were 180 degrees out of phase, you would hear both drops. Irritating, but that's the way it is. Another example of beat phenomenon whereby two oscillators very closely the same period, given enough time, they go out of phase. And then, they go back into phase.

We have made an effort here with these two pendulums, really an effort, they're about 60 centimeters length, to really make these lengths the same. And so when I give them both a certain amplitude, they will go in unison.

They're in phase. But no matter how hard you try, they will never be exactly the same. So as I continue to lecture, you will see that they gradually go out of phase. And it may take five minutes for them to be out of phase. That would be, then, the time from here to here. That would be half a beat period.

So I'll show you that. And I'm not going to wait for them to be 180 degrees out of phase because it can take a long time since their periods are so closely the same. If you look at them now, you would say, boy, are they in unison. And keep an eye on it. And you'll see a few minutes from now they will no longer be exactly in phase.

When you think of a conductor for an orchestra, what the conductor is really doing, he's making sure that all the instruments stay in phase. He's really a phase locker. He's a phase keeper. If he didn't do that, then each person would play at its own pace. Then, of course, you would end up with chaos. So now, perhaps, you know where the expression comes from-- "Keep the faith."

Now, I go to the heart of this lecture. And the heart of this lecture is damping. Oh, you see it's already out of phase a little? No, it's completely out of phase. But they're no longer-- keep an eye on them. You'll see that it will grow.

At the heart of this lecture is damping. When we have a spring or we have a pendulum and we let it oscillate, we all know that the oscillation will come to a halt. That means there is friction or there is air drag. And it will ultimately die out, and it will stop just like these will stop.

The frictional force always opposes velocity. And the frictional force in its most general form as a vector is minus C1 times v, whereby C1 is a positive number. So you see that it opposes the velocity. Minus C2 times v squared. And I put here the unit vector, v, to remind you that it's opposing the velocity.

This term has a name. This is called the viscous term. And the other term has a name, which is called the pressure term. One is proportional with the velocity. And the other one is proportional with the square of that.

In my lecture number 12, which is on OCW, of my 1999 Newtonian mechanics lectures, I spent an entire lecture on this equation. And I discuss when the viscous term is dominant and when the squared term is dominant. I do demonstrations in both domains.

If the velocity is low, this term dominates. Is the velocity is high, if the speed is high, that term dominates. When you drive your car 10, 20 miles an hour, it is the v squared term that dominates, the air drag, that slows you down. A raindrop that falls, it's the v squared term that dominates, not this one.

But if you take ball bearings and you drop them in syrup, then the velocity is always very low. The speed is very low. And then, this term dominates. And that's one of the experiments I do during this lecture to show you that, indeed, the predictions that follow from this part can really be verified quite beautifully. So if you have the patience, I would strongly advise you to watch that lecture.

To manage the math, it becomes almost impossible except where you want to do it numerically to take both terms into account. And so to be nice to you in 803, we always ignore this term. And we only take that one into account. And of course, that is only valid if the velocities are low, which is often the case but of course not always.

I will start with an extremely pedestrian example. I have here a spring on a horizontal surface. And as it moves, there is friction. And we assume that the friction has this form. The spring constant is k. The mass is m. And if I move it a distance, x, away from equilibrium, then there is a spring force that drives it back to equilibrium.

In addition, there is this frictional force. And I have no idea in which direction it is because if the object moves in this direction, the frictional force is like so. If the object at this moment moves in this direction, then the frictional force is so. So I cannot put it in there. I have to know the velocity in order to put in the frictional force. So I won't do that.

But what I can do, I can write down Newton's Second Law which says that mx double dot is minus kx. That is, this force when x is positive, the force is in the negative direction. So the minus sign takes into account that we're dealing with vectors.

And then, we have minus this term. Now, when we do this in 803, we will call this C1 just b. That's the tradition. We don't call it C1. We call it b. In other words, we have minus b times x dot. x dot is a vector, is a velocity. And if the velocity is positive, then the force is in this direction. If the velocity is negative, then the frictional force is in that direction.

So that now is the differential equation that we have to solve. I will introduce some shorthand notations that k/n is omega 0 squared. Remember, that is the angular frequency of a spring oscillating in the absence of any friction. We've seen that last time.

And I will also introduce the symbol b/m equals gamma. That's also a tradition. gamma has the same units as omega, seconds to the power minus 1. And so if I rewrite this differential equation, then I get x double dot plus gamma times x dot plus omega 0 squared times x equals 0. And you see now our task ahead of us, to solve this differential equation. What is x as a function of time?

Without starting the math, you can in your head see what's going to happen. An oscillation that goes on forever and ever and ever is going to come to a halt. So instead of a nice cosine that continues, the cosine will gradually decay away. And then, this thing will stop.

I take a spring. I Offset it. It starts to oscillate. And it comes to a halt.

So the amplitude must die out, must decay. That's a prediction. Now, let me try your intuition.

Let us assume that the system will oscillate with a unique frequency, omega. Do you think that this omega will be larger than omega 0, smaller than omega 0, or equal to omega 0? Or to put it in the other way, will a whole oscillation under the influence of friction take longer or shorter or the same amount of time than in the absence of friction?

So who thinks that if I have damping that a whole period of a whole oscillation will take longer? Who thinks it will take shorter? Who thinks it will be the same?

OK, we'll see what comes out of it. No one thinks that it takes shorter. Your intuition is very good there. Now, I'm going to give you some advice, some fatherly advice.

During the next 18 minutes, take no notes because during the next 18 minutes, I'm going to solve this differential equation for you. And I have decided, which I don't always do, to do it exactly the way the French does it, very elegant. So if you take notes, you're wasting your time because you have it in your book.

And I would much rather prefer that you follow each step that I make so you have more time to concentrate on the methods. So don't write anything down. That is my recommendation. 18 minutes, you can time me if you don't believe me.

We're going to jump through the complex plane. This is a beautiful example where the complex solutions really come in handy. So I'm going to change this equation to z, which is now a complex notation, z double dot plus gamma z dot plus omega 0 squared times z equals 0.

And then z, my trial function, is some amplitude, A, times e to the power j pt plus alpha. Clearly, this has the smell of an amplitude and this here has the smell of a frequency. Anything e to the power j pt has the smell of a frequency. But you will see there you're in for a surprise, what happens with this p.

Once we have found a proper solution to z, we take the real part. And we are in business. Then, we go back to x.

I'm going to take the second derivative of this equation. So jp comes out twice. So that is j squared times p squared. And so I'm going to see now minus p squared. That is the second derivative of this function. And the z comes later.

Then, I have plus gamma. And then, what pops out is jp because it's only the first derivative, jp. You have to be very careful that you can distinguish your j's from your gammas. And then, you get plus omega 0 squared. And the whole thing times z, and that now must be equal to 0.

Well, if you have an equation like this, you may think it's one equation. But it really is two equations because if something like this has to be 0-- and of course z is 0 is not an acceptable solution because that's the oscillation-- if you have an equation like this which has a real part and it has an imaginary part, then both must independently be 0 for obvious reasons.

These are apples. And these are oranges. And five apples minus five oranges is never zero oranges, is never zero apples. You cannot subtract apples from coconuts. So this has to be 0 and this has to be 0.

Now, if you make gamma 0, you have no problem. There is no damping. So that's ridiculous.

If you make p 0, then you no longer have any oscillation going on because you smell that pt, that's going to be the oscillatory thing. So that's also unacceptable. And so this won't work. Therefore, p itself must be complex.

And so we're going one step further now. And we're going to say the only way that we could make this work is if we say p equals n plus js, whereby n and s, then-- let's hope-- are real. Let's first calculate what p squared is. We go slowly so we don't get into trouble with the algebra.

So p squared is n squared plus this one squared, the j squared's minus 1. So that is minus s squared plus 2n times j times s. So now, I can go back to this form and have minus p squared. So I get minus n squared-- that is this minus p squared-- plus s squared minus 2nj times s. So I'm done here.

Now, I get gamma jp plus gamma times j. And now, I have to multiply that by p, which is n, plus gamma times j times js. And then, I have plus omega 0 squared. And this now has to be 0.

I have here j squared, j squared is minus 1. So I can erase this and replace it by minus gamma s. You cannot do that. But that's the price you pay if you didn't take my advice not to take any notes.

So this is minus gamma s. Now, this has to be 0. That means the apples must add up to zero. These are the apples. And the oranges have to be 0. This is an orange. This is an orange. This is an orange, and this is an orange.

For this to be 0, there's a j here. There's a j here. There's an n here. There's an n here. 2s must be gamma. So gamma over 2 is s. Oh, we found s.

For this to be 0, we find that n squared-- minus n squared-- plus gamma squared over 4, because we know what s is now. So plus gamma squared over 4 minus gamma over 2 times minus s, which is minus gamma squared over 2-- minus gamma squared over 2-- plus omega 0 squared equals 0.

And that means that n squared is omega 0 squared minus gamma squared over 4. So we also have n. So we now have n. And we have s.

And so now I can go back to my trial function. And I can substitute the values that we have found in that complex function. So we get z equals a times e to the power j. And now, we have to put in pt plus alpha.

But p is n plus js. So nt plus jst plus alpha, yeah? I'm going slowly. I don't skip too many steps. But I know that s is gamma over 2.

But I know that j squared is minus 1. So therefore, this part here and this part here can be brought out. So we get A times e to the minus gamma over 2 times t. Here, we have the t. The j squared makes it a minus. And s is gamma over 2.

And then, I get e to the power j times nt plus alpha. The first thing that you see here, which is amazing, this term here is an exponential decay in time. There's nothing oscillatory about it.

It means that this A, which is the original amplitude, is gradually dying out with a decay time, tau, which is 2 divided by gamma. Remember, gamma has units second minus 1. So this really has units of time. So this one is going to be responsible for that decay that we discussed earlier.

What is this now? e to the power jnt, well n is obviously a frequency. This is the striking example in the complex plane of a rotating vector and the angular frequency is n. But we know what n is. And so this n really is nothing but omega squared. n squared is really omega squared.

And so I can replace this n now by whatever you have here. And so if I write this now, A e minus gamma over 2, I can write now j omega t plus alpha. And I know exactly now what that omega is. That is the square root of this number.

And so therefore, omega squared is omega 0 squared minus gamma squared over 4. So those of you in the audience who intuitively sensed that the frequency would go down because of the friction, they were right. See, omega is lower than omega 0.

Now of course, if gamma is very low, the two could be close together. It depends on gamma. That's obvious. If you have a lot of friction, then omega will be a lot, lot less than omega 0. If you have very little friction, well then, they will be very closely the same.

And so we can now write down in all its glory the real part of this function, which then becomes x equal some amplitude, A, times e to the minus gamma over 2 times t times e to the minus j times omega t plus alpha.

Uh-oh, look what I did. I have an equation in the complex plane. That's perfectly OK. And then, I went back to the real world, going from z, the complex plane, to the real part of z.

And what did I write down? This is fine. But then, I wrote down e to the power minus j omega t plus alpha. First of all, there should not be a minus sign in front of the j. That's not a major problem.

But I do not take it out of the complex plane. I leave it in the complex plane. So clearly what I should have written, which is what I will do now, x equals A times e to the minus gamma over 2 times t times the cosine of omega t plus alpha. Now, we are in the real world.

You're going to see this equation at least for the next five or 10 minutes on the tape. It is a thorn in my side. But there is nothing I can do about it. You just have to live with it and think of it as being this.

And there are two adjustable constants which depends entirely on the initial condition. At time t equals 0, there are two things you can do to the object. You can bring it to a certain point away from equilibrium. And you can give it a kick. We call that a velocity.

And you're free to choose any way you want to do that. And so it's clear that with those two choices that you have that in your final solution, you end up with two adjustable constants which depend on the initial condition.

And so you see that this amplitude is going to die out with the 1/e decay time of 2 over gamma seconds and that the frequency is lower than the frequency of the undamped system. I tried to make a plot. So here is t, and here is x. And to guides my hand, I'm going to first put in this exponential.

And now, I'm going to put in that oscillatory term here whereby the frequency is uniquely determined. Omega is uniquely determined. And the period, t, is that omega divided by 2 pi.

Oops! How could I? Look what I wrote. I wrote t equals omega divided by 2 pi.

It was not my day. Clearly, it should have been t equals 2 pi divided by omega. Now fortunately for me, very shortly afterwards, I erase this from the blackboard. So you won't see it for very long. In any case, this is what it should be.

And therefore, I'm going to put in here zero crossings to guide myself so I don't make the mistake which you often see, that people think that this period is slowly getting shorter in time. That is not true. It is the amplitude that decays away.

And so if we now try to put in the oscillation, then this is what's going to happen. And so you see that the oscillation dies away in time, but the period, t, is uniquely determined. And that depends on how much friction there is.

It is not uncommon to introduce a quality factor, q, that is high if there is little damping and that is low if there is a lot of damping. And that q, which is dimensionless, is omega 0 divided by gamma. You see immediately if gamma is high, then q is low. It's a low quality oscillator.

If you introduce that, you can go back to your omega squared equation. And then, omega squared becomes omega 0 squared times 1 minus 1 divided by 4q squared. Is this any different from that? No. It's just a different way of writing it because you introduced this.

What it tells you is that if q is about 10-- and I bet you the q's are much higher for these pendulums-- then you have here 1/400. That is 1/4 of a percent. But since omega is the square root of that, it's only 1/8 of a percent. So for q of 10, omega is only 1/8 of a percent lower than omega 0.

Even if you make the q as low as 2, the frequencies are only off by about 3.2%. So I want you to appreciate that most of the time, but not always, is omega very close to omega 0.

We can look at the decay in a different way. We cannot do that here. I will do that here on this center board because we don't need that anymore.

Instead of saying I have to wait 2 over gamma seconds for the amplitude to go down by a factor of e, I can ask myself, how many oscillations do I have to wait for the amplitude to go down by a factor e? How many oscillations?

Well, n oscillations will take this long, t being the period of one oscillation. But for reasonable values of q, t and t0 are the same, like omega and omega 0 are closely the same. So I can write for this that this is approximately n times t0. So that's approximately n times 2 pi divided by omega 0.

I can now substitute this time in this t and only concentrate on that decay portion, that early part. So what I find, then, that A, after n oscillations, is a times e to the minus gamma over 2t, e to the minus gamma over 2, times this time after n oscillations, n divided by 2 pi divided by omega 0.

And you lose a 2. But omega 0 divided by gamma is q. And so now, you have it in a forum which is minus n times pi over q. Is this any different from what we had before? No.

Here, we say I have to wait two over gamma seconds for the amplitude to go down by a factor of e. Here, we say if n is q divided by pi, then the amplitude goes down by 1/e.

So in one case, I ask myself how many seconds do I have to wait. In the other case, I ask myself how many oscillations do I have to wait. And so if q is 10, it tells you that you have to wait about three oscillations, roughly, for the amplitude to go down by a factor of e. And if q was 100, you have to wait more like 32 oscillations for the amplitude to go down by a factor of e.

18 minutes are up. So now, you can start taking notes again. I have here two pendulums. And the pendulums have about the same length. And the objects have about the same radius. That means the b, which is this coefficient in front of the velocity, is about the same.

But gamma is not the same because if they have the same b, but if the mass of the two objects is very different-- this is Styrofoam and this is a billiard ball-- there's a huge difference in gamma. And since the period of the two pendulums is very closely the same-- they have the same length-- you see that the q of the two systems must be very different. Because if b is the same, m is very much higher with a billiard ball. Then, the gamma is much lower. And therefore, the q of this system is way higher than the q of this system.

In fact, if I wanted to wait how many oscillations it would take for that amplitude to go down by a factor of e, I may have to wait five or 10 minutes. So I will not attempt that. But I will attempt to measure q with this pendulum.

If I bring this here, the separation from equilibrium is 27 centimeters. And by the time that it has decayed to this, it's 10 centimeters. So that's about a factor of e. And I want the students in the audience who are sitting here, who can really see it head on, I want them to say stop, scream the word "stop," when the 27 has decayed to 10. And in the meantime, we count the number of oscillations.

So we have then counted the number of oscillations. And therefore, we know what q is because we multiply the number of oscillations by pi. So we've measured q.

We could also have done it with a time measurement. But I prefer to do it this way. Now you guys over there, keep your mouth shut because there's no way you can see it, right? You just can't. You have a projection effect which is awful. So I only want to hear from you. Ready for that? Count, and say stop. Not all of you may say stop at the same moment. Some of you may say stop after 10 oscillations. Others may say stop after 11 oscillations. That's fine. That is part of the uncertainty in a measurement. Ready?

27, there we go. One, two, you see the decay already. Three, four, five, six, seven, eight, nine, ten. My goodness, are you guys crazy?

You're paying a lot of tuition and you can't even pay attention to a demonstration! At 9, I think it was already a 10. When it is here, then it is 10. I will give you a second chance.

You have to scream, "stop!" Oh. One, two, three, four, five, six, seven, eight, nine, 10.

AUDIENCE: Stop.

PROFESSOR: 11, 12. OK, it is somewhere around 10, 11. And I don't know why you didn't scream stop, but that's your problem. So n is about 11, maybe plus or minus 1, 10% uncertainty. And so q, then, is about pi times higher. It's a crude measurement. But very roughly, you would get then times 11. You get about 35.

35, that means the frequency damped is almost identical to the frequency undamped because, remember, this equation, the 1/4-- oh, I erased it perhaps. But you have it up there. If q is 35, you can see how close omega is to omega 0.

On problem set two, your very first task is to do a take home experiment with something similar to what I did today. Make sure you pick up a kit today. You can do that between 11:00 and 1:00 and 3:00 to 5:00 in room 4.335. Or you can do it tomorrow between 2:00 and 5:00. You share one kit, so you chose a partner. And you can do all these experiments together with your partner.

I want to see in your solutions uncertainties. Any timing measurements that you have to do has to come with an estimate of your uncertainty. And all the conclusions that you draw must carry on these uncertainties just as we did last time when we were exploring the possibility whether the equation of the spring was indeed accurate. We were only able to come to the conclusion that it was not accurate because we had our proper uncertainties in there. So I want to see uncertainties in there.

Now comes the mini quiz. Five minute break. It's a bit early, but that's the best point today to do. So if some of you can help me handing it out and also at the end of the five minutes bringing it back to me. Shouldn't take you more than one minute. That leaves you still with four minutes to stretch your legs.

I will now take you back to the good old days of 802. And I will take you back to an RLC circuit whereby I have a battery. This is the plus side, and this is the minus side. Here, I have a resistor, R. I have here a pure self inductor, L. And here, I have a capacitor, C.

And then, I have a switch here. And I can throw that switch. And then, the battery is going to charge up the capacitor. And then, you will get oscillations. This is a wonderful example of damped oscillations.

The way that I solve these problems is a strict discipline. I'd assume that when I started that there is a current going in this direction, I. That current will make this side of the capacitor more positive than that side. So I equals dq dt, q being the charge on this side of the capacitor.

And now, I have to do the closed loop integral of e dot dL. Kirchhoff's law does not hold. The closed loop integral is not 0 because we have a magnetic flux going through a surface attached to this closed loop. So the closed loop integral E dot dL is minus d phi dt, phi being that magnetic flux going through that surface.

And so I want to know what the E vectors are in each one of these components. If the current is in this direction, then the electric field in the resistor is in this direction. The self inductor is made of super conducting wire. So there's never any E field inside the self inductor.

Unlike what many EE people think, there is never any electric field inside a self inductor. So the E here is 0. This is plus, and this is minus. So the potential, the electric field, is in this direction. This is plus, and this is minus. And so the electric field is opposing me if I go around clockwise.

So if now I go around clockwise and I call the integral E dot dL from this side to this side of the capacitor, if I call that V of C, which is q/C. That's the definition of capacitor. If you're ready for this, and I start here, and I go clockwise around.

Then from here to here, I get plus IR. Going through the self inductor, the integral E dot dL from here to here is 0 because there is no electric field inside the self inductor. Going from here to here, I get plus VC. That is what we call the potential difference over the capacitor.

Here, the electric field is opposing me. I walk into the electric field. So E dot dL is negative. And this is minus 0. And now, I go to Mr. Maxwell to Faraday, and he says this now is minus L dI dt. The hell with Kirchhoff, minus L dI dt. And now I have my equation correct.

So now, I'm going to replace I by dq dt. And I'm bringing L to the left. So I get L times q double dot plus R times q dot-- because I is q dot-- plus VC, which is q/C. And that now equals V0.

And I'm going to divide by L. And so I get q double dot plus R/L times q dot plus q over LC equals V0 divided by L. I'm going to replace R/L by gamma. You will see very shortly why we do that. And 1/LC is omega 0 squared.

And when we do that, we get an equation which is almost identical to the one we had on the blackboard for the spring. We get q double dot plus gamma times q dot plus omega 0 squared times q equals V0 divided by L. And now, we have to solve this differential equation.

You'll recognize that gamma is the damping. R/L has the same function as the damping had in the case of the spring. In the case of the spring, it was b/m. Here, it is R/L. The larger R is, the more heat dissipation there is. Heat dissipation goes in terms of I squared R. And clearly, that means you take energy out of the system. So that means there is damping.

And this is, then, the natural frequency of the RLC circuit if there is no R. If there's only an L and a C, this is the square or the frequency. If only this were a 0, then I know the solution because we had it on the board there. We still have the complex notation there.

Then, q would be some q1 for which we have an A there, times e to the minus gamma over 2 times t-- and gamma is this value-- times the cosine of omega t plus alpha. And omega squared is omega 0 squared minus gamma squared over 4.

That would be the solution if this were 0. Now, it's not a 0. So you can take 1803 and try to solve it if it's not 0. Or you could think of it like a physicist and say, I don't need 1803!

What is the difference between this differential equation and the one before with the spring that we had a 0 here? In the case of the 0, it means that the oscillation would end up at x equals 0 whereas here, the oscillation in the charge will end up, if we wait long enough, with a fully charged capacitor. Because when the oscillation has died out, the capacitor has been fully charged.

And so clearly, if you add here this charge on that capacitor, then you must be OK. That must take into account that it is not 0. And this q max, q max, is simply V0 times C, which means the capacitor is fully charged.

So do it your 1803 way. Or do it with brains. And then, you will immediately agree to this solution to the differential equation.

So you see here the decay. You see the oscillation. And then when you end up, you must end up there with a fully charged capacitor.

You now have to solve, depending upon the initial conditions, for alpha. And you have to solve for q1. So that means you have to your initial conditions. And the initial conditions of this problem, for instance, would be that at t equals 0 when I throw the switch that there is no charge on the capacitor. That is one way I could do it. And there is no current flowing. That could be my initial condition. Not the only one possible, but that is a possible initial condition.

And now, I will leave you with a little bit of work. First of all, you substitute in this equation q equals 0 when t is 0. So you still have q1, and you still have the cosine of alpha.

Then, you have to take the derivative of this equation, which gives you the current. And then, you have to make that 0. Now, you get two terms when you do the derivative. So be careful. You get the derivative of this one times this and the derivative of this times this.

So you have to do it slowly. When you do that, you will find that q1 equals minus q max divided by the cosine of alpha. And you'll find that the tangent of alpha equals minus gamma over 2 omega. I don't have a very good feeling for these numbers.

But what I do have a good feeling for is that I know when q, say, is larger or equal to 5, then I know that omega is very close to omega 0, extremely close. So that means I can write for this minus gamma divided by 2 omega 0, which is minus 1 over 2 Q. And if Q is 5 or larger, my goodness, this is 1/10 if we forget the minus sign now. So that means that phi is only something like 5.7 degrees-- with a minus sign-- and the cosine of 5.7 degrees-- PFFT-- is 1, for physicists at least.

So therefore, for all systems whereby q is not absurdly low, you can say that q1 equals minus q max. And I'll put here a wiggle. But it's an extremely good approximation. And so therefore, my solution here can now be changed by simply replacing the q1 minus q max. And truly, it should be an approximately, but the approximation is extremely close. It depends on q.

If now you want to plot this, you get, of course, something completely similar to what you have here except that you do not end up at 0. The q is not 0. But it's offset by this q max.

So if this is t, if this is 0, and if this is q max, which is where ultimately the charge will be if you wait long enough, then to plot now this equation will then give me a curve like this. Be very careful.

The period is uniquely determined. The period does not change. That's determined by that omega. 2 pi divided by this omega is the period. And so you're going to get-- you start with q0. And you see if you wait long enough, you end up with q max, which is this one, whereas in the case of the spring if you wait long enough, you end up with nothing-- x equals 0.

And this is something that I can demonstrate to you in a very, very nice way. I must say, I find is one of the most intriguing demonstrations. Instead of having a battery and throw the battery, in which case you would only for an extremely short amount of time-- for an amount of time which is roughly 2 divided by gamma seconds which may only be milliseconds, you would see this thing.

But we want to give you more than a few milliseconds. And therefore, what we do, we replace that battery by seven milliseconds constant voltage. And we repeat that. And every time we show you this portion. And so you will see it on the Tektronix. We trade it with every time that it comes up here, and then it comes up here. And you will see then this will be staring you in the face.

I'm going to give you the values that I will be using so that you can digest at home exactly what you are going to see and what all the constants are in this problem. The R is going to be 50 ohms. The L is going to be 50 millihenries, 10 to the minus 3. So that tells me that gamma is 1,000, is R/L.

That tells me that 2 divided by gamma, which is the 1/e decay time, is then 2 milliseconds. C is going to be 0.06 microfarads. So that would give you an omega 0, if you were interested in omega 0. I never had a good feeling for omegas. I like frequencies much better. But if you're interested in omega 0, which is 1 over the square root of LC, that will be 18.3 times 10 to the 3. And that is radians per second.

The frequency in Hertz, which is 2 pi lower, is then 2906 Hertz. And so now, you have also q, which is omega 0 divided by gamma. And that q is now 18.3. And that means q divided by pi, which is the-- is there a problem, Michael-- q divided by pi, which is the number of oscillations that you have to wait for this function to go down by fact of e. Remember, we did that with the pendulum. That is now about 5.8.

So we're going to take a look at a curve like that. And then, we're going to indeed check that after roughly six oscillations, the amplitude is indeed down by factor of e. However, while we're at it, I'm going to kill two birds with one stone.

I'm then going to make this 100 millihenries a little later, which changes the gamma. The gamma becomes 500. The decay time then becomes four milliseconds. I'm not going to change C.

So the frequency goes down by the square root of 2. That becomes 2,055. And now, the q also changes. The q goes up by a factor square root of 2, becomes 25.8. And now, you have to wait about 8.2 oscillations for the amplitude to go down by a factor of e. And that's what I want to show you now.

I have to choose my channel, unless Mark already did that for me. Ah, the surprise is already there. So you see now here the oscillation was about 2,900 Hertz. This is the location where the capacitor is fully charged. That is that q max and q 0 is right here at the left side at the bottom. That is where q is 0.

And let's now count the 5.8 oscillations, roughly six-- one, two, three, four, five, six-- and if you look now at this amplitude, after six oscillations, and you compare that with this, you can very easily see that it is roughly a factor of 2.7. So that works exactly as predicted. You see this beautiful exponential decay.

If now I change the self inductance to 100 millihenries, then the frequency goes down from here, 2,906 to 2,055. The q goes up, and now you have to wait eight oscillations. So q is 0 is on the left side at the bottom. This is, again, fully charged q max. And now, we go one, two, three, four, five, six, seven, eight.

And the amplitude now is roughly the same as what it was before when we had six oscillations. So you see that this works like a charm if you do it with an LRC circuit. Really quite remarkable. So dying oscillation, and the damping is the result of the resistor which is dissipating energy.

If you want sleepless nights, which is healthy occasionally, you may want to try to find for yourself a convincing reason why the L also comes in the damping. The L can never dissipate any energy. Only the R can. Why is it that the gamma is R/L?

I'll be very honest with you. When I search for one, it took me 10 minutes to come up with the right answer. At first, you say this is weird. But if you give it a little bit more thought and if you got a B+ or higher for 802, you can do it.

All right, all this is very nice and dandy. But what now happens if gamma squared over 4 is larger than omega 0 squared? What now happens if gamma squared over 4 is larger than omega 0 squared?

So our solution, omega squared equals omega 0 squared minus is gamma squared over 4, is utter nonsense because omega would then be imaginary. So now, we have to go back to our original differential equation and find a new solution.

We are closer to a solution than you may think. Remember, we did have at one point in time n squared equals omega 0 squared minus gamma squared over 4. And then, we recognized that n really is omega.

Well, n is therefore also j times gamma squared over 4 minus omega squared because if you square this, you get that.

Uh-oh, I did something weird again. I told you it's not my day. We had this earlier in the lecture, which is fine. And then I said, well, if you write n this way, that's also perfectly OK because when you square this, you get this back.

But that's not true. In fact, there are two mistakes in here. It's not so easy to make two mistakes in one equation. For one, there is a 0 here. So this should have been omega 0 squared. That's mistake number one.

But second, obviously you have to take the square root of this. If now you square this, you get this. Sorry.

If now you substitute this back in the equation that we had earlier when we had the equation that z was A times e to the power j pt plus alpha and then p was n plus js, remember, if you do that now and you turn the crank, then you get the valid solution for z. And you take the real part. And you get the valid solution for x. And I'll write it down immediately in terms of x.

And the outcome is very surprising. We call this an overdamped system, whereas a system whereby this is positive is an underdamped system. So the examples that I've shown you are underdamped systems. And this is overdamped, overdamped. And the solution that you're going to find is not at all obvious.

I have no intuition to that solution. A1, which is determined by the initial conditions, e to the minus, and now we get gamma over 2 plus gamma squared over 4 minus omega 0 squared to the power 1/2 times t plus A2 times e to the minus gamma-- not minus gamma-- minus gamma over 2 minus gamma squared over 4 minus omega 0 squared to the power 1/2 times t.

You have to take my word for it that this is the solution. And when you look at this solution, nothing is oscillating anymore. This is a decaying amplitude, A1, and this is a decaying amplitude, A2. They have different decay times. This determines one decay time, and this determines another decay time. And depending upon your initial conditions, will you be able to find A1 and A2? At t equals 0, you can specify what x is and you can specify what x dot is.

No oscillations, that means I can make some guess of what the response will be. Let this be a spring system. And this is x equals 0. And I offset the spring to position x0. And I let it go with speed. That's my initial condition, zero speed.

So that means when I let it go, it must go in this direction, right, in that diagram because dx dt must be 0. But there's no oscillation. So what will happen is this. And it will asymptotically go to 0 with two different decay times.

Now, if you released it here with a high speed in this direction, then you might see this. But not an oscillation-- an overshoot, yes, but not an oscillation. And you will get a chance in one of your problems-- I think it is 1.7 for this week-- where you can play with this a little bit.

There's one more case which is a very special case. And that is when omega 0 is exactly gamma over 2. And that's called critical damping. So we have three domains, underdamping. We did that in detail. Then, you have overdamping, which gives this solution. And then, you have critical damping. And with critical damping, the solution-- and I give you this without any derivation-- is A plus B times t times e to the minus gamma over 2 times t.

Only one decay times, one exponential decay time. But there's something funny here. There's also a t here. So this Bt is linearly growing in the beginning. But of course, this one kills it. And again, your two adjustable parameters, A and B, follow from the initial conditions.

So now, I will show you the effect of damping. In the case of a mechanical system, we have a torsional pendulum there. And the torsional pendulum has extremely high q, very little damping. I'll show it to you were there before we show it there. Just concentrate on this. Without any damping, it will oscillation. And the amplitude-- the torsional pendulum is one that has a spring like a clock-- this will die out very, very slowly. Very high q, very little damping.

But we have here an electromagnet. And we can change the current through the electromagnet. And the copper plate that moves through the electromagnet will now develop Eddy currents because there's a change in the magnetic flux through that metal. It causes Eddy currents. And your Eddy currents will oppose the motion of the metal. You've seen that with 802 when you throw a coin through a very strong magnetic field. The coin slows down enormously. Instead of going clunk, the coin can go down this slowly. It's a very nice demonstration in 802.

So we can also, using the Eddy currents, we can introduce artificial damping. And that's what you're going to see. One, two, three, this should be it. Yeah, oh, we have to change the position. But that's not a big deal. We may have moved it during-- oh, we also need some light on it. I turned the light off. Thank you.

Let's first make sure that we-- yeah, there we see it. So I will give it a large amplitude. The amplitude now is in terms of angle. No damping at all. And if you look closely at the turning point, you can say yeah, it is decaying. Give it a little bit of time. But very slowly, yeah, you can see it decaying. You have to wait a long time to see the amplitude go down by a factor of e because you have such a very high q system.

But now, I'm going to give it damping. It's still underdamped. First, give it a high amplitude. There we go. Damping is in now. And look how fast it's going now.

With this system, I cannot show you an example of overdamping. We did that, and then the current has to be so high that we began to smell smoke. So we decided not to do that. But if I go to the point where you just don't smell smoke, then you're still underdamped. But you get a huge amount of damping. And the reason why I can say with confidence that it's still underdamped, I release it at 0 speed.

And if I release it at 0 speed, and if it is overdamped, it can never overshoot. And yet, you will see that there is a minute overshoot. So it is still a little underdamped, but it's getting very close to being critically damped. And so I will jack this up to the maximum possible.

I can go way beyond 15. Is that smoking domain?

AUDIENCE: It's OK.

PROFESSOR: It's OK. Don't go to the smoking domain, right? So I will first put in the damping. And then, I will offset it and let it go. Damping is in now. I will offset it and let it go. Three, two, one, zero.

You see overshoot? Just a teeny weeny little? That tells you that it's still underdamped because it's not supposed to overshoot since I released it at 0 speed. Of course, if I give it a push in that direction, of course it'll overshoot. That's obvious. But if I do it at 0 speed, that solution that we have on the blackboard dictates that it cannot overshoot.

I'll show it once more. You will see overshoot. That's it.

So that is the whole story about damping. We have underdamping. We have critical damping and overdamping. And that does not put a damper on it. I will see you again on Thursday. And make sure you pick up your 803 kits.