Massachusetts Institute of Technology OpenCourseWare

8.03SC **Fall 2012**

Notes for Lecture #3: Forced Oscillations with Damping

As in Lecture 2, a mass m on a spring with constant k damped with a frictional force $b\dot{x}$ is considered, defining parameters $b/m = \gamma$ and $\omega_0^2 = k/m$. What is new is that a force $F_0 \cos\omega t$ is applied to the object. Writing $F = m\ddot{x}$ gives $m\ddot{x} = -kx - b\dot{x} + F_0 \cos\omega t$. Moving to the complex plane, recalling that $\cos\omega t$ is the real part of $e^{j\omega t}$, and rearranging terms gives $\ddot{z} + \gamma\dot{z} + \omega_0^2 z = \frac{F_0}{m} e^{j\omega t}$ (**3:30**). We propose a solution $z = Ae^{j(\omega t - \delta)}$ which includes the assumption that, in the long term, the system must vibrate at the driving frequency ω. This is the *steady state* solution: the only way that the differential equation can be satisfied at all times is if all terms oscillate at this same frequency. Note that "steady state" does not mean static. The δ term encodes the possibility that the motion and driving force may not be exactly synchronized (for example, always both maximum at the same time). Doing the derivatives and substituting, $(-\omega^2 + \gamma j\omega + \omega_0^2)Ae^{j(\omega t - \delta)} = \frac{F_0}{m} e^{j\omega t}$ (**6:30**). Dividing both sides by $e^{j(\omega t - \delta)}$ (note that this would **not** be possible if the oscillation was not at the driving frequency) gives:

$$(-\omega^2 + \gamma j\omega + \omega_0^2)A = \frac{F_0}{m} e^{\delta} = \frac{F_0}{m}(\cos\delta + j\sin\delta)$$

This is really two equations, one for the real part, and one for the imaginary part, both of which must be satisfied. Separating, $(-\omega^2 + \omega_0^2)A = \frac{F_0}{m}\cos\delta$ and $\gamma\omega A = \frac{F_0}{m}\sin\delta$. Whenever we have equations with sin and cos like this, we can eliminate them using the identity $\cos\delta^2 + \sin\delta^2 = 1$ and divide them to find $\tan\delta$. Squaring and adding these equations we have $(-\omega^2 + \omega_0^2)^2 A^2 = (\frac{F_0}{m})^2 \cos^2\delta$ and $(\gamma\omega A)^2 = (\frac{F_0}{m})^2 \sin^2\delta$, giving $(-\omega^2 + \omega_0^2)^2 A^2 + (\gamma\omega)^2 A^2 = (\frac{F_0}{m})^2(\cos^2\delta + \sin^2\delta) = (\frac{F_0}{m})^2$. This can be solved to find the amplitude of driven, damped motion (**9:40**):

$$A = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}}$$

Although this is a complicated function, it should be immediately clear that the amplitude is going to be highest when the driving frequency ω is near ω_0, as we expected. We find δ by dividing $\gamma\omega A = \frac{F_0}{m}\sin\delta$ by $(-\omega^2 + \omega_0^2)A = \frac{F_0}{m}\cos\delta$, giving (**10:20**):

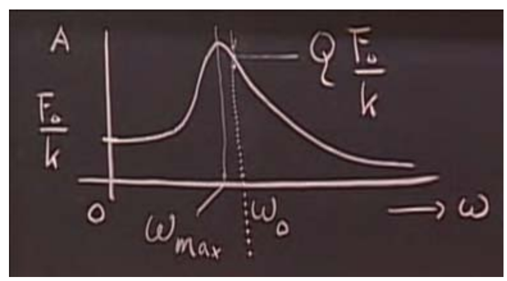
$$\tan\delta = \frac{\omega\gamma}{\omega_0^2 - \omega^2}$$

Again it is clear that something significant happens to the phase shift when the driving frequency ω is near ω_0. This may be a less expected result of resonance, and will be explored below. In the real world, $x = A\cos(\omega t - \delta)$, with A and δ as we just determined. There is no information about the initial conditions in this solution, since the driving force completely dominates the situation.

Some other intuitive aspects of the A equation are that it increases with more forcing, that is to say, F_0 is in the numerator, and decreases with more damping, that is, γ is in the denominator (**12:40**). It is often useful to look at limiting cases, and in this case very slow motions are instructive. In this case, the spring will essentially just be stretched and the inertia of the mass has little effect, so we just have Hooke's Law. This corresponds to near-zero driving frequency, and the force just produces extension, and by Hooke's law $F_0 = kA$ (no minus sign since we are dealing with the force on the spring, not its force back as in Hooke's law in its usual form). So in this limit, $A = F_0/k$. If we put $\omega = 0$ in the A equation, we get $A = \frac{F_0/m}{\sqrt{(\omega_0^2 - 0)^2 + (0)^2}} = \frac{F_0}{m\omega_0^2} = \frac{F_0}{m\frac{k}{m}} = \frac{F_0}{k}$, as expected. Also, in this limit, we are pulling the spring so slowly that its stretch follows exactly the applied force, so we expect zero phase lag (**15:30**). The δ equation gives $\delta = \tan^{-1}(0) = 0$ as expected.

Alternatively, we can consider the resonant case, $\omega = \omega_0$. Then $A = \frac{F_0/m}{\sqrt{(0)^2 + (\gamma\omega_0)^2}} = \frac{F_0}{m\omega_0\gamma}$, but recall that we defined $Q = \omega_0/\gamma$, so this is $\frac{F_0 Q}{m\omega_0^2} = \frac{F_0}{k}Q$ (**17:10**). So at resonance, the amplitude is Q times what it is at very low frequencies. This can be a very large factor.

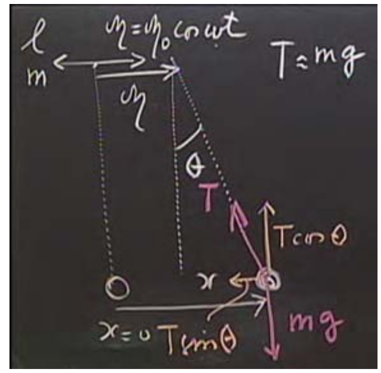
Finally, if one drives at a very high frequency, the inertia of the mass prevents it from responding and the amplitude is essentially zero. The phase at high frequencies is π, i.e. the position is completely out of phase with the driving force (and at resonance it is $\pi/2$ out of phase) (**18:15**).



A sketch of the amplitude versus frequency shows that, although we roughly say the maximum is when the driving is "at" the resonance frequency (although notice we were careful above and said "near"), in fact it is at a slightly lower frequency (**20:15**), referred to as ω_{max}. The phase, however, is $\pi/2$ exactly at ω_0, 0 for very low frequency, and π at high frequency. The way to find a minimum or maximum of a function exactly is to take the derivative and set it to 0. Taking $dA/d\omega$ gives $\omega_{max} = \sqrt{\omega_0^2 - \frac{\gamma^2}{2}}$, which further algebra shows to be $\omega_{max} = \omega_0\sqrt{1 - \frac{1}{2Q^2}}$ (**22:00**). If Q is high, this is very close to (but a little lower than) ω_0. At ω_{max}, $A_{max} = \frac{F_0}{k}\frac{Q}{\sqrt{1 - \frac{1}{4Q^2}}}$. For $Q = 5$ (**24:00**), a low value for many systems, $\omega_{max} = 0.99\omega_0$, and A_{max} is 5.03 times the zero frequency value, rather than 5 times.

There now follows a fairly long discussion, but one which is important (**28:00**). Applying a force to a moving object could be a challenge, especially in the case of a pendulum which is considered. It is much easier to apply the force at the suspension point, and in fact it is much easier to impose a certain motion on the suspension point rather than to apply a specified force. As a result, Prof.

Lewin talks about a new variable, η, which is the displacement of the suspension point from its initial position, for example if it is held in his hand.



The displacement of the pendulum bob itself from the initial equilibrium point is, as usual, denoted x. The motion of the suspension point is considered to be sinusoidal, $\eta = \eta_0 \cos \omega t$. A discussion of the forces on the bob (gravity and tension) ensues, in which the bottom line is that for small motions in which the string remains nearly vertical, the tension is just mg, and the restoring force in the x direction (which arises entirely from tension) is $mg \sin\theta$ (**32:15**). The only difference from what you would study in an introductory mechanics course is that the angle θ is towards a moving, rather than fixed, suspension point, so that $\sin\theta = (x - \eta)/l$. Then $F = ma$ gives $m\ddot{x} = -b\dot{x} - mg\sin\theta$, or $m\ddot{x} = -b\dot{x} - mg\frac{x}{l} + mg\frac{\eta_0 \cos \omega t}{l}$ (**35:30**). As before, we define $\gamma = b/m$ and now use $\omega_0^2 = g/l$ (the frequency of an undriven simple pendulum). Dividing out the m, the differential equation becomes

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \omega_0^2 \eta_0 \cos \omega t$$

which is exactly analogous to that for a mass on a spring, but with $\omega_0^2 \eta_0$ replacing F_0/m (**38:00**). Making this substitution in the solutions, we find that the phase equation does not change and the amplitude is given by (**39:00**):

$$A = \frac{\omega_0^2 \eta_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}}$$

Limiting cases are again useful to consider, and fairly intuitive: for $\omega = 0$, A is η_0, δ is 0; at resonance, A is $Q\eta_0$, δ is $\pi/2$; and at very high frequency, A approaches 0 and δ is π (**41:00**).

Several nice demos follow, which you can now appreciate more fully given the information presented above. In the air track demo, pay careful attention to the comment about a factor of 2 in the amplitude (**47:30**), and also note the initial transient behavior (**49:00**), which will be discussed later in the course. A split screen view (**51:30**) gives a beautiful illustration of the fact that $\delta = \pi/2$ near resonance and π at very high frequency. Unfortunately, the movie of the collapse of the Tacoma Narrows Bridge in Washington State is not included in the video, for copyright reasons. It is recommended that you look for this movie on the internet. It is probably the most famous example of an (unfortunate) driven resonance in a man-made structure. In terms of breaking wine glasses by singing, the popular TV show “Mythbusters” did manage to have a rock star do this. Again, worth looking up on the internet! Prof. Lewin uses a loud speaker and tone generator to accomplish the same thing (**1:04:30**). A strobe light very close to the sound frequency makes it possible to see the oscillations.

8.03SC Physics III: Vibrations and Waves
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These viewing notes were written by Prof. Martin Connors in collaboration with Dr. George S.F. Stephans.