Massachusetts Institute of Technology OpenCourseWare

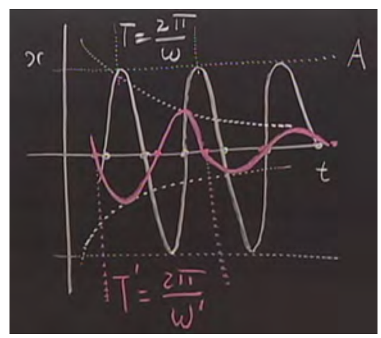
8.03SC **Fall 2012**

Notes for Lecture #4: Forced Oscillations: Resonance & Transient Phenomena

In the last lecture, the steady state solution to a driven oscillator with damping was shown to have no adjustable constants. Initial conditions appear to be irrelevant. For a mass on a spring, the undamped oscillation frequency would be $\omega_0^2 = k/m$, and the damping constant is $\gamma = b/m$. The undriven solution is $x = Xe^{-\gamma t/2}\cos(\omega' t + \alpha)$ where X and α come from initial conditions (**2:30**). Here $\omega' = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$, slightly less than the undamped natural frequency. In contrast, with a driving force of $F_0 \cos\omega t$, we found $x = A\cos(\omega t - \delta)$, with $\tan\delta = \dfrac{\omega\gamma}{\omega_0^2 - \omega^2}$ and $A = \dfrac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2}}$. This solution has no adjustable parameters (**4:30**). The driven equations of motion are:

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \begin{cases} \dfrac{F_0 \cos\omega t}{m} & \text{If the mass is driven with an applied force} \\ \eta_0\omega_0^2 \cos\omega t & \text{If the end of the spring is driven with } \eta_0 \cos\omega t \end{cases}$$

The solution in the latter case would have F_0/m replaced by $\eta_0\omega_0^2$, i.e. $x = A\cos(\omega t - \delta)$, with $\tan\delta = \dfrac{\omega\gamma}{\omega_0^2 - \omega^2}$ and $A = \dfrac{\eta_0\omega_0^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2}}$ (**7:00**). While this latter situation is not worked out in detail, the procedure is similar to what was done for a pendulum in the previous lecture.



Since the solution to the undriven case $x = Xe^{-\gamma t/2}\cos(\omega' t + \alpha)$ satisfies the differential equation $\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$ (note the zero on the right side), it can be added to the solution for the driven case and still satisfy the more complicated differential equation. So, the complete solution in the driven case is $x = A\cos(\omega t - \delta) + Xe^{-\gamma t/2}\cos(\omega' t + \alpha)$ (**9:00**), with the second term being the transient, the first the steady state solution. One could, in principle, solve for the constants, but it would be algebraically challenging. Since the transient and steady state terms oscillate at different frequencies, you can get beat phenomena, especially if Q is high so that the transient behavior dies out slowly (**15:00**). This is demonstrated using the air track.

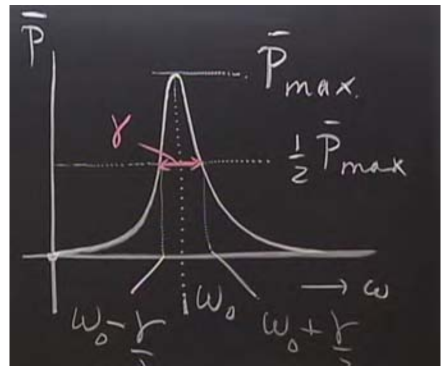
The mechanics definitions of work $dW = \vec{F} \cdot d\vec{x}$ and power $P = dW/dt$, so $P = \vec{F} \cdot \vec{v}$, can be applied to this situation (**21:20**). Since these systems are are 1-dimensional, the dot product is a simple multiplication and $P = Fv = F_0 \cos\omega t[-\omega A \sin(\omega t - \delta)] = -F_0\omega A \cos\omega t[\sin\omega t \cos\delta - \cos\omega t \sin\delta]$. Usually only the average power is of interest (**25:00**). Time averaging, $\cos\omega t \sin\omega t$

gives 0; $\cos\omega t\cos\omega t$ gives 1/2. The average power is therefore:

$$\bar{P} = \frac{1}{2}\vec{F}_0\omega A\sin\delta = \frac{F_0^2\omega^2\gamma}{2m[(\omega_0^2-\omega^2)^2+(\omega\gamma)^2]} = \frac{F_0^2\gamma}{2m[(\frac{\omega_0^2}{\omega}-\omega)^2+\gamma^2]}$$

plugging in the value of A and using the formula for $\tan\delta$ to find $\sin\delta$ (**27:30**).

If damping is very large (**31:00**), the average power is 0, due to the factor of γ in the denominator. If m is very large, no force can get the mass going, and the power again goes to 0. If F_0 is zero, nothing moves so the power also goes to 0. If the driving frequency is 0, again nothing moves, power is again 0. If ω is very high, inertia prevents much motion and again power goes to 0. If ω approaches ω_0 (**34:00**), maximum power is used, and we get $\bar{P}_{max} = \dfrac{F_0^2}{2\gamma m} = \dfrac{QF_0^2}{2m\omega_0}$ since $Q = \omega_0/\gamma$. The power shows a resonance curve with a width at half-maximum very close to γ so that a highly damped system has a wide resonance curve and a high-Q system has a very narrow resonance peak.



The same formalism is applied to an RLC circuit driven by an AC voltage supply $V = V_0\cos\omega t$ (**41:30**). In the circuit, charge q is on the capacitor plate and the current is $I = dq/dt$; at the capacitor $V_C = q/C$. Faraday's Law applies so $\oint\vec{E}\cdot d\vec{l} = -\dfrac{d\phi}{dt}$ where ϕ is the magnetic flux enclosed, most of which is in the inductor L, defined by $\phi = LI$. Applying Faraday's Law around the whole circuit, $IR + V_C - V_0\cos\omega t = -L\frac{dI}{dt}$. Taking another time derivative:

$$L\ddot{I} + R\dot{I} + \frac{I}{C} = -V_0\omega\sin\omega t \qquad \text{or} \qquad \ddot{I} + \gamma\dot{I} + \omega_0^2 I = \frac{-V_0\omega\sin\omega t}{L}$$

where $\gamma = R/L$, $\omega_0^2 = \frac{1}{LC}$, and the result is amazingly similar to the driven spring equation (**46:30**). There are similar limiting cases at high and low frequency as well as resonant behavior as found for the spring, with the resistance acting like damping. The resonant peak current is V_0/R at $\omega = \omega_0$. The current output for a driven RLC circuit with $R = 50\ \Omega$, $L = 50\ mH$, $C = 0.5\ \mu F$ ($\omega_0 = 6.3\times10^3\ \text{sec}^{-1}$ $Q = \omega_0/\gamma \approx 6.3$) as a function of driving frequency is demonstrated (**55:00**). This system has a decay time of 2 msec so the transient behavior is irrelevant. The current output for higher resistances is also shown.

The absorption of power by a system driven right at resonance, as well as the fact that going off resonance dramatically reduces the power absorption, is demonstrated using two matched tuning forks which have very high Q and therefore very narrow resonance curves (**1:01:30**).

In atomic physics (**1:06:00**), electrons have discrete orbits with discrete energy levels and can make transitions with a change of energy, which is carried in or out by electromagnetic radiation.

Energy of light is proportional to frequency (red is low energy, violet is high energy). The surface of the Sun radiates all frequencies as a continuous spectrum (black body). As this goes out through the atmosphere, resonant absorption by gas removes certain frequencies and these are observed as dark spectral lines where not as much light comes out. These lines allow identification of the elements present in the Sun and other stars. Helium (Greek for Sun is Helios) was first discovered on the Sun (**1:11:15**) since the corresponding line was previously unknown. Very-high-Q resonant absorption on the atomic scale (light absorbed by sodium) is demonstrated (**1:15:00**). The line in the yellow region at the edge of the red band appears dark because the absorbed light is re-emitted in all directions, so very little of it goes in the direction of the incident beam of light.

8.03SC Physics III: Vibrations and Waves
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These viewing notes were written by Prof. Martin Connors in collaboration with Dr. George S.F. Stephans.