8.03SC Physics III: Vibrations and Waves, Fall 2012
Transcript – Lecture 5: Coupled Oscillators

PROFESSOR: To avoid possible misunderstandings, my lectures start at 9:30 Eastern Standard Time, which is different from lobby seven time, as you may have noticed. The clock in lobby seven is seven minutes slow.

Today we're going to cover coupled oscillators, which is a big part in 8.03, though we leave out damping in order to avoid a major complication. Imagine that I have a pendulum-- length l, mass m-- and I have another pendulum, also mass m, length l. And I connect the spring between them, as you see there-- spring constant k.

Imagine now that time t equals 0, that I give this object-- which I call object number 1, and this is object number 2-- give it a certain position. I give it a certain velocity. So I have four choices. And I let the system go. What you're going to see is something extremely chaotic. And our task today is to predict what the position of this one is at any moment in time, and what the position of that one is at any moment in time.

And to show you how chaotic that motion is, I will just show you this. So I take this one and just displace it from equilibrium. I displace this one from equilibrium, and at time t equals zero, I will give the one in my left hand just a certain velocity.

And if you now look at the position of the individual objects, it would seem nearly impossible to come with an analytic solution which tells you what these motions are. You will see that the amplitudes build up, of certain ones, amplitude goes down. This one is hardly moving at all now. Now it's picking up again. And so our task today is to work on that.

What is by no means obvious-- but I will show that to you-- that any motion, no matter how you started off, is going to be the superposition of two normal mode solutions. Anyway you start it, it can always be written as the superposition of two normal mode solutions. What is a normal mode? A normal mode is, in this case, that both objects have exactly the same frequency-- that is fundamental to normal mode-- and that they are either in phase with each other or out of phase with each other, nothing in between. Because there's no damping, so it's either in phase, or it's out of phase. That is a normal mode.

In other words, if I call one of those frequencies, omega minus-- minus means it is the lowest frequency. There are two frequencies in this system because there are two objects. Omega minus-- I call that The lowest frequency. Then it would mean if they are in phase with each other, the two, that they come to a halt at the same moment in time-- that means in phase-- and in the same direction. So they come to a halt at the same moment in time in the same direction and they have the same frequency.

If, then, I have another frequency which is a higher-- there are two normal modes because we have two objects. If we have three objects, there are three normal modes. If we go to the higher normal mode, the higher frequency, they have the same frequency, but they are 180 degrees out of phase. So when one comes to a halt here, the other one comes to a whole there. That's what it means, 180 degrees out of phase.

I can excite, and I will excite, this system into its normal modes only. I can excite this normal mode alone and this normal mode alone if I choose the correct initial conditions. So for any randomly chosen initial condition, the motion of each object can be written as a linear combination of these two modes.

If you take my word for that for now-- but of course I will demonstrate to you and I will prove that to you-- it would mean that [INAUDIBLE] function of time can then be written as having some kind of an amplitude, x0-- I give it the minus sign because it's related to that normal mode frequency-- times the cosine of omega minus t plus phi minus plus some other amplitude, which I call x0 plus, which is going to be related to this frequency times the cosine of omega plus t plus phi plus.

Let's look at this. Let's try to see through this, what this means. It means that, if I know my initial conditions, that I can determine this amplitude. I can determine this phase. I can determine this amplitude. And I can determine this phase.

There have to be four adjustable constants, because I have the choice between the positions at t0 and the velocity. So there must be four. What you see is that omega minus and omega plus, which are these normal mode frequencies, are independent of the initial conditions.

So now I'm going to write down the position in time for object number two. And I know that in this mode, it must have the same frequency, so there must be here a cosine minus t plus exactly the same phase. Because I told you, normal mode means same frequency in phase or same frequency out of phase. And it is the next one which is going to be out of phase. But you will see that shortly. So this term must become omega plus t. For now I will say plus phi plus. But you're going to see how the out of phase comes in very shortly.

Notice this omega minus must be that omega minus, otherwise it wouldn't be a normal mode. This omega plus must be this omega plus, otherwise it wouldn't be a normal mode. These phis are the same to allow them to be in phase. These phis are the same even though they're out of phase, but you will see very shortly that I will get a minus sign here, which will take care of the 180 degrees.

I can, with this system, excite the lowest mode so that it only oscillates in that lowest mode. So this term is not there, and this term is not there. I can do that. I could always oscillate something in one of the normal modes, but if I set it off at random, then is going to be a superposition. Even though it is early for you, can you show me-- using your hands and your legs, whatever you want to-- when I set this off, just the right initial conditions, what will that lowest normal mode look like? How will these pendulums oscillate? Very good. Very good. So you have a good instinct. Very good.

Let's do that. Let's set them off in the same direction and let them go. Those are ideal initial conditions for this mode. And you will see that it is a normal mode, namely, they have the same frequency and they're in phase with each other. They come to a halt at the same moment in time.

So what you see already is that this value here must also be x0 minus. That's a must. You've just seen it. In this case, what it means is that you might as well remove the spring. The spring is not doing anything. The spring never gets any longer, never gets any shorter.

What do you think will the pendulums do when I excite the other mode, which has a higher frequency? Very good. Can I see some other hands or heads or legs? Good. That' this. Symmetry-- you see? There is a symmetry in this system, and that's why you people are so smart and see immediately the connection.

So let me make here a drawing of the omega minus. And then the amplitudes are the same. And here I make a drawing of the omega plus-- higher frequency. And the amplitudes are now the same, but 180 degrees out of phase. So it's easy to excite that. There we go. Notice they oscillate with the same frequency and they are coming to a halt at the same moment in time. But if one is here, the other is there. And so therefore, this one here must be minus x0 plus. That minus sign now gives me 180 degrees. So I could still keep that phi plus there.

It is utterly trivial, without any work, to calculate or to tell you what omega minus is. That's just the frequency of a single pendulum. So that's easy. So omega minus is simply the square root of g over l. There is no spring, right? Each one is doing its own thing and the spring is never pushing or pulling.

But we should be able to calculate omega plus. So let's do that. And I will only make a drawing of one of those pendulums. I don't need them both. OK, here's one of those pendulums. And let this separation, this distance, be x from equilibrium. And I call this angle theta. And so here is that spring. And the other one is on the other side.

Let's put in all the forces that we can think of. So there's gravity-- mg. There is tension-- t. But there's more. What else is there? Spring. How much longer is the spring than it wants to be? Two heads, exactly. In other words, there is a force here-- I call that the spring force, Fs-- which is bringing it also back to equilibrium. And that one is 2kx.

Now the tension is very close to mg. We have discussed that several times. I'm going to introduce a shorthand notation. Omega 0 squared is g divided by l, and omega s squared-- making reference to the spring-- is k over m.

And now we have to set up the differential equation. I have to apply Newton's Second Law and remind you that the magnitude of the spring force is 2kx. So, Newton's Second Law. m x dot equals-- now there are two forces that I have to take into account. First, the spring force, which is minus 2kx. And then I need the horizontal component of the tension-- remember, that's the only one that is important here-- which is t sine theta, with t as mg and sine theta as x divided by l. l Is the length of the pendulum. So we get minus t, which is mg, times the sine of theta, which is x divided by l. And so this is the differential equation.

I'm going to divide m out and I'm going to bring everything to one side. So I'm going to get x double dot plus 2 omega s squared times x plus omega 0 squared times x equals 0. And our task now is to solve this differential equation. And that, of course, you can do in three seconds, because you recognize this differential equation. Is x double dot plus something times x. And so the new frequency, which I will call omega plus-- that new frequency, omega plus is the square root of 2 omega s squared plus omega squared.

And what you see something that I already anticipated, which was consistent with your intuition. It is larger than omega 0. Because this is the effect of the spring. So omega 0 is the one, here. And omega plus now is the square root of 2 omega s squared plus omega 0 squared. Omega s squared is defined this way.

So if I now turn to my general solution-- for now you accept the fact that that is the general solution, the superposition of the normal modes-- then the key point is that independent of your initial conditions is omega minus. That was the square root of g over l. I never put in any initial conditions. Independent of the initial conditions is omega plus. That's this one. I never put any initial conditions. Independent of the initial conditions is this ratio, which is plus 1. And independent of the initial conditions is this ratio that is minus 1.

In other words, if I didn't tell you the initial conditions, which I haven't, I can predict that this ratio is plus 1. And I can predict that this ratio is minus 1. If I make this 3, and this is minus 3. If I make this 5, then this is minus 5. The ratio is minus 1. So the ratios are independent of initial condition. And the frequencies are independent of initial condition. If I tell you the initial conditions, then of course, you can also calculate the individual amplitudes.

Suppose now I start this system off in some way at time t equals 0, which I can choose. And I choose the following. At t equals 0, I make x1 C-- just some number C that we can choose, 3 centimeters, whatever you want to choose. But I may v1 0. So as I release it, no speed. And suppose I make x2 0 and I make v2 0.

So if you don't know what that means, of course, I'm going to demonstrate it. It means that this one here-- the other one I offset, that's all I do, and I let this one go. Pull my hands off, and then I want to know what's going to happen, right? Because I released this one with 0 speed. That's what you're seeing there. That is one of an infinite number of initial conditions that you may choose.

So this now has to be substituted into my general solution. And so you have to take the derivative of x1. So you have to do x1 dot and put that equal to 0. I believe that in your competent hands. Then you have to take x2 dot and you have to make that equal to 0. And I leave that into your competent hands. That's easy.

And you will find, then, that in this particular case, with this particular example, phi minus equals 0, and phi plus equals 0. It will not take you more than a few minutes to find that. I didn't want to waste your time on taking a derivative of such a simple function.

So if we take this right now, then I will substitute these results in the equation. So you did the hard work. You did the velocities. I will do the positions. So I'm going to substitute in that equation t equals 0. I know already that the phis are not there. And so then I get that C at time t equals 0 is going to be x0 minus. t is 0, so this is 1. t is 0, so this is 1 plus x0 plus. That's this initial condition.

Then I go to the second initial condition that 0, this x0 minus, minus x0 zero plus the minus sign. And so what do I find? I have solved now the general solution. I will find that x0 minus is 1/2 C and x0 plus is also 1/2 C, which of course should not come as a surprise to you.

So let me write down now the general solution that we have for this specific initial condition. So we know everything. We know x0 1. We know x0 minus. We know x0 plus. We know phis. We know everything. So we're going to write it down here. So x1. It's going to be 1/2 C-- remember? We found it at 1/2 C-- times the cosine of omega t-- because phi is 0-- plus 1/2 c times-- this is a minus by the way, cosine omega minus t-- cosine omega plus t. That is x1. And x2 is 1/2 C times the cosine omega minus t minus 1/2 C times the cosine of omega plus t.

Take a deep breath. Substitute in there t equals 0. And you see immediately that x is C. And substitute t equals 0 in the second, and there you see indeed that x2 is 0. No surprise, because that's my initial condition.

Now I remember from my high school days that the cosine, alpha, plus the cosine of beta is twice the cosine of half the sum times the cosine of half the difference. So I can write down x1 as twice the cosine of half the sum times the cosine of half the difference. So that 2 that I get eats up these 1/2s. So I get C times the cosine omega minus plus omega plus divided by 2 times t times the cosine of omega minus minus omega plus divided by 2 times t.

I've just rewritten it in a different form. And x2 as a function of time-- I now have the cosine of alpha minus the cosine of beta. That is twice the sine, half the sum times the sine of half the difference. So now I get here the sine of omega minus plus omega plus divided by 2 times t times the sine of omega minus minus omega plus divided by 2 times t.

Notice when t is 0 that the sine of this one is zero-- consistent with our initial conditions, x2 is 0, remember? It's all there. Initial conditions are all there, just written in a different form.

Now, imagine now in your mind that these two frequencies are not too far apart. Then these equations have the smell of what?

[SNIFFS]

PROFESSOR: Beats. This here is the fast turn-- this one and that one. And this one is the slow one. And so if these two are close, then what you will see is something quite remarkable. At t equals 0, this one stands still. And this cosine term is going to be 1 because t is 1. This one is going to oscillate happily with this frequency.

But this cosine term is very gradually going to 0. And that cosine term gradually goes to 0, this one will stop oscillating, but this sine term becomes plus 1. And so the other one will start to oscillate. And then a little later in time, the cosine term will become minus 1, so it starts to oscillate. But when that happens, the sine term is 0 again. So it stops.

So you see a beautiful beat phenomenon. One, the first one will gradually come to a halt, and the other one will pick up. And then the other one will come to a halt, and then it transfers, in a way, energy. It is, of course, consistent with the conservation of mechanical energy.

And I want to demonstrate that. So we have this here. So I have to do is offset x1 over a distance C that we can choose. And then we release this one at 0 speed. And then we'll just watch it. And there you should see that strange phenomenon. You ready for this? This one I have to hold in place. And three, two, one, zero.

Look, this one is standing still. Look, this one is standing still. That's beating. And you see that the energy is transferred from one to the other. And that is a beat phenomenon that follows immediately from this.

What would happen if I moved the spring up? Suppose I move the spring here-- higher, say halfway. Can anyone, without looking too much at the blackboard, sort of use your intuition? What would happen? Would the same phenomena happen? Yeah? Yeah?

AUDIENCE: The spring have less effects.

PROFESSOR: The spring what?

AUDIENCE: It would have less effect.

PROFESSOR: Yeah, so what-- would the same phenomena happen?

AUDIENCE: Yeah, the same phenomena.

PROFESSOR: But what would happen with the beat period?

AUDIENCE: [INAUDIBLE]

PROFESSOR: Now you may look at the blackboard. Do you think-- these have a certain separation, right? You make that separation smaller, you make the effect of the spring smaller. So the omega minus becomes even closer to the omega plus. And so the beat period will increase. It will take longer for one to stop.

Let's try that. So we're going to move these up. I'll put it roughly halfway. OK, you ready for this? And so we're doing it again. And then you will see the same phenomenon, except it will take longer for the first one to come to a halt. So I have decreased the coupling. You see, it's still swinging happily. Now it's beginning to change its mind. You see? Now it's standing still. It took longer. Do we agree?

Now there's something mind-boggling, something that you really want to see. Suppose I bring this all the way to the top. That's interesting, because then omega minus is exactly the same as omega plus. Because look, this term goes away. What now will happen? It's almost this-- cutting out the spring. There's no spring anymore. What now will happen if I start one here?

Three, two, one, zero. What do you think? Yeah?

AUDIENCE: One will swing and one won't.

PROFESSOR: Ah. Brilliant. Brilliant. One will swing and the other won't. Isn't that shocking?

[LAUGHTER]

PROFESSOR: Isn't that shocking? You have two pendulums and they're not even connected anymore. And you start this one swinging, and the other one will never start to swing. Isn't that amazing? For $30,000 tuition, you learned something fantastic. Except that the beat period is infinitely long. So with a bit of patience-- so let's demonstrate that.

So let's demonstrate this. So there we go. Unbelievable. Physics works. And look at this equation. When omega minus is omega plus, this term is always 0. So you see x2 remains 0. Ha-ha-ha! That's what you see. And then omega minus is omega plus. This cosine term is always plus 1. Ha-ha-ha. That's what you see. Isn't it amazing, the power of physics?

So it is truly remarkable that we can describe, for any initial condition, the motion in terms of the linear superposition of the two normal modes. And so what originally looked like an impossibility when I started-- the first 30 seconds of my lecture was that demonstration-- when I started them off in a random way, it looked so chaotic that it looked almost unimaginable that we would be able to sort that out and be able to predict the motion of each one individually as a function of time.

I want you to appreciate that we have two coupled oscillators here. That gives you two normal modes. If you have three coupled oscillators-- we will get back to that-- then there are three normal modes. And if you have four, then you have four normal modes.

Now clearly, what we need is a general recipe. In this case, the problem was so beautifully symmetric. We could do omega minus in just a split second, omega plus. We could see exactly the motion. So in no time would you get that the ratio of the amplitude was plus 1 and was minus 1.

But let me tell you. If I break the symmetry, oh man, all hell breaks loose. For instance, if you change the ratio of the masses, you make one longer than the other, you have three beats connected with springs. If you break the symmetry, it is very, very hard work. And for that, we need a general recipe. And I'm going to give you the general recipe and then I'm going to apply that general recipe to this case. So everything that you have seen will then come out after a lot of algebra. But at least you know that the recipe is working.

So the first thing that you do-- that's number one-- you give each object a displacement from equilibrium. Even though you're free to choose the direction, I always-- a matter of discipline-- set them all off in the same direction. That's not a must, but that reduces the chance of mistakes. But you give them a displacement, and I always do them always in the same direction. That's number one.

Once you have done that, you write down Newton's Second Law for each, which means if you have two objects, you have three unknowns. What are the three unknowns? So you want to find one of the normal mode frequencies. You want to find this one, and independently you want to find that one, which has to come out of that recipe.

What are, in principle, the three unknowns? This ratio plus 1 holds here, but that doesn't hold in other cases. And so, in principle, you have as an uncertainty this amplitude and omega. So you always end up with two equations with three unknowns. So if I say two objects, then you have three unknowns and you have two equations.

But that looks like a problem. Number three-- you are going to put in the condition for normal modes, which means that x1 is C1 times cosine omega t. But x2 can be some other amplitude-- which we have to calculate-- also times the same omega t. I don't have to put a phase in there, because it's either in phase or out of phase. And out of phase means that you get minus signs.

And so you see immediately that if you substitute this in Newton's Second Law that you get two equations, three unknowns. C1 is unknown, C2 is unknown, and omega is unknown. So you substitute this in here, and then comes the problem. How are you going to deal with two equations with three unknowns?

Well, think about what I told you earlier. If you don't know the initial conditions, you can never find x0 minus, or you can never find C1. But the ratio is independent of the initial conditions. So that means the ratio C1 over C2 cannot be dependent on the initial conditions. In other words, you can always solve these equations in terms of C1 divided by C2 and omega. And I will show you that that works.

So you follow this recipe. And we will do that religiously together. We will find, then, in this case, two values for omega. We will find an omega minus, which has an associated value of C over C2. I will put a minus there. And we will find that omega plus, which has its own ratio associated with it-- in the case of our simple symmetric system, this would be plus 1 and this will be minus 1. But that is not the case when you break symmetry.

So my task is now to apply this recipe in its most general form to this system. It will be 16 minutes of grinding. We will go through each step. And out comes something that we already know. But at least you will see that I have made no assumptions, which I did there, about symmetry. Therefore, this is an ideal moment for you to have your five minute break so we can take a deep breath to get ready for this 16-minute marathon.

All right, the general recipe. Someone asked me during the intermission whether this omega is the same as that one. Yes, of course, otherwise it wouldn't be a normal mode. We are going to search for the normal modes. And in a normal mode, the frequencies must be the same of all of objects, whether w have five or six or two or three. But the ratios are different. That's a different issue. However, they are in phase or out of phase. So the ratios can be negative or positive, but you never have any phase angles although than the 180 degrees or 0 because we have no damping. We have taken the damping out. So it is essential that when you substitute that in Newton's Second Law that these omegas are the same, because they will then give you the normal modes.

OK, if you're ready. I'm going to offset the first one over a distance, x1, so they have mass m. They have length l, to remind you. And we are going to have omega 0 squared equals g over l, and omega s squared equals k over m.

And this one I offset over a distance, x2. Notice-- you don't have to do that-- I always offset them in the same direction. And then there is the spring that connects them. I will just make it very thin, this spring, otherwise the picture becomes a little bit too complicated. So let this angle be theta 1. And let this angle be theta 2. And so we have here mg. We have here mg. We have here the tension, which is roughly mg for small angles. And we have here the tension, which is roughly mg-- a little bit to big, the way I drew it, but that's not so important now.

And now there's another force that acts on both. And which force is that? That's the spring force. Now, do we agree what the magnitude of the spring force is? The magnitude, then we will argue about the direction. Nicole, magnitude. Look very closely. I offset one by x2, and I offset the other by x1.

AUDIENCE: [INAUDIBLE]

PROFESSOR: I was asking you a question. You can't answer it with a question.

AUDIENCE: So it's going to be x2 minus x1?

PROFESSOR: Very good. The magnitude of that force is k times x2 minus x1. Non negotiable, right? Now, if x2 is larger than x1, do we agree that the spring force is in this direction? And do we agree, then, that this spring force must be in this direction? If x2 is larger than x1.

Well, that means I can leave everything the way it is now, as long as I give this a minus sign and this a plus sign, I'm OK, because if x2 is smaller than x1 it will automatically flip over. And my algebra is fine. And therefore in my head, I can just think of x2 being larger than x1, set up the differential equations, and I no longer have to worry about the fact that maybe x2, at certain moments in time, is not larger than x1.

So now I set up the differential equation mx1 double dot. So that's this one. Let's first do the pendulum. That is minus t times the sine of theta 1. And t is mg, so is minus mg times x1 divided by l. Do we agree there's that horizontal component here? The horizontal component of t is driving it back to equilibrium, minus sign. Fs, the spring force, is driving it away from equilibrium. So it's going to get plus k times x2 minus x1.

Look at this science. It is very important now, very important. And the next one, mx2 double dot, it has again the restoring force due to the pendulum. Which is minus mg x2 divided by l, because it's the sine of theta 2 now that it comes in. And now, it has a spring force, which now is a negative sign. So now we get minus k times x2 minus x1.

And when you have this done on your next exam, you take a deep breath and you go over each individual step to make absolutely sure that this is correct. If there's anything wrong with this, you're dead in the waters. The whole problem will fall apart. It may not even be a simple harmonic oscillator. You get something ridiculous. Signs are crucial. So therefore, look at this again. Take a pulse. This one, if there's a positive offset it is indeed in the negative direction-- the component off t sine theta. This one, if x2 is larger than x1, it's indeed in that direction. And you notice if x2 becomes smaller than x1, well, then this becomes automatically a negative.

So are you OK? So this is fine? This one has a restoring force due to the pendulum, which is t sine theta 2 negative sign, and the restoring force of the spring-- always opposite this one-- I'm happy. I am happy.

So now we're going to rearrange this a little bit. And we're going to introduce the omega 0 squared notation. So I'm going to divide m out. And so I'm going to get x1 double dot. And then I'm going to bring the x1s to the left. And so I get plus omega 0 squared times x1. But you have a minus kx1, and you divide by m. So you get plus omega s squared. And that's both times x1. You notice that? They both have an x1 term. And now the x2 comes out. The x2 has a plus sign on the right side, so it gets a minus sign. So I get minus omega s squared times x2. And that is 0.

And then we do the next one. We get x2 double dot. Ah, I get the same terms. I get an omega 0 squared. And I get an omega s squared times x2. And then I get this plus-- minus times minus is plus-- so I get minus omega s squared times x1 equals 0.

When you have reached this pointed in your exam, you take a deep breath and you go over each little term to make sure that you haven't accidentally slipped on a minus sign, because if you slip up on one minus sign, you're dead in the waters. It may not even become a simple harmonic oscillation. So let me do that. I agree with that. I agree with this. I agree with that. I agree with that. The differential equation is fine.

Notice that I already did point number one and that already I did even number two. I set up the differential equations, Newton's Second Law. So now comes number three. Number three means I'm going to substitute in there x1 is C1 cosine omega t and x2 is C2 cosine omega t. We go slowly. I want you to follow each step. And I'm going to work on here so that you could see it high.

So I'm going to substitute this in here so the x1 double dot will give me a minus omega squared-- right, because you get twice omega out-- minus omega squared times C1. And to hell with cosine omega t. Why to hell with cosine omega t? Because every term will have cosine omega t, so we might as well divide out right away. Right, every term that you're going to have in here, will have a cosine omega t in it, so I'll leave the cosine omega t already out now.

So I get minus omega squared C1. Then I get plus omega 0 squared plus omega s squared times C1. That is this term minus omega s squared times C2 equals 0. Because remember, x2 has the C. Yeah? Am I going too fast? Beautiful.

So I go to the second one. The second one is x2 double dot, so I also get a minus omega squared times C2. And then here, I get a C2, so I get plus omega 0 squared plus omega s squared times C2. And then I get minus omega squared times C1 equals 0.

And when you have reached this point in your exam, you take a deep breath and you make sure that all your signs are correct. If not, you're dead in the waters and the problem will fall apart. So let's do that. Minus omega squared C1-- I can live with that-- plus that term with the C1-- yes-- minus omega s squared C2. Even if you make a little step of the pen, and you change this into a 1 and this into a 2, it's all over. And then the next equation-- minus omega squared C2. Then I have this times C2 and then it's minus that times C1.

We're almost there, even though doesn't look that way, does it? Remember what I said. You cannot solve two equations with three unknowns. There's no way that you can solve for C1, C2, and for this omega, which is what you're really after. This is the omega you're after. But you can solve for C1 divided by C2, and omega. Knowing that already, I'm going to eliminate C1 over C2. And you'll see how I do that.

C1 divided by C2. I go to the first equation. This is not so hard, what I'm doing. I put, in my mind, this on the right side. So I get a plus omega s squared times C2. And then I divide C1 one by C2. And so what I get, then, is the following. I get omega s squared-- upstairs, which is this one. And all this comes downstairs-- minus omega squared plus omega 0 squared plus omega s squared. Do we agree? That's the first equation. I've simply written it as C1 divided by C2. I can always do that, right? Bring the C1s to one side, the C2s to one side, and divide them. That's this.

And I'm going to the same with the next equation. I bring this one to the right side, then I have C1s there and C2s here. And now this becomes upstairs. So we get minus omega squared plus omega 0 squared plus omega s squared. And I get downstairs omega s squared. And now I have eliminated C1 and C2, because if I solve this equation, I get my solution for omega.

It is one equation. It is one unknown that's omega. You've got to admit. And there'd better be two solutions. There'd better an omega minus and there'd better be an omega plus coming out of this. Untouched by young human hands, I must find two solutions.

Let's first make sure that this is correct. And the answer is yes. So what do I do now? Well, I'm going to simplify find it one step further, which is now very easy. I multiply this by this and this with this, so we get omega s to the power of 4 is minus omega squared plus omega 0 squared plus omega s squared squared. One equation with one unknown-- omega. It's the only unknown. Take the square root, left and right. So I get minus omega squared plus omega 0 squared plus omega s squared equals plus or minus omega s squared. Do not forget the plus or minus, because the square root of omega s to the fourth is plus or minus omega s squared.

And now you're going to see the light of the day. You won't believe this. I bring omega squared to the other side, and I get omega 0 squared plus omega s squared minus or plus omega s squared. And the simplicity is overpowering. I feel it over my whole body. It is unbelievable. Whether you have a minus sign here, I find that omega is omega 0. That's my omega minus. That's the one. That's my omega minus.

When there is a plus sign here, I find exactly what we predicted before. I find that omega plus is the square root of omega 0 squared plus 2 omega s squared. Because the plus sign made this 2. Isn't that elegant? Isn't that beautiful?

So they just pop out. And I predicted that it will be two solutions. You have them both. So by substituting this in there, automatically come out two solutions. But what now we see 1 over C2? We haven't solved for that yet. But I promise, you can always solve it in terms of omega and in terms of C1 over C2.

Any volunteers? Any volunteers look at the blackboard. Somewhere hidden in those equations-- I said we're going to eliminate C1 over C2. You think C1 over C2 likes to be eliminated? Let's call it back. And look at this equation. Substitute in here-- for omega squared, substitute in omega minus. Then you get an answer for C1 over C2. What do you think that answer is? What do you think the answer is?

AUDIENCE: 1.

PROFESSOR: Plus 1. Not even 1, plus 1!

[LAUGHTER]

PROFESSOR: So you get C1 over C2 is plus 1. If you take this omega 0 and you put it in here you get omega 0 squared with the minus sign. With the plus sign you have omega s squared divided by omega s squared and it's plus 1. And now you take the second solution and you put it in here. What do you think you're going to find?

AUDIENCE: Minus 1.

PROFESSOR: Minus 1. C1 over C2 is now minus 1. And so what you have seen now is that the general recipe comes up with the frequencies, comes up with the ratios, of the amplitudes-- not with the individual amplitudes because you don't know the initial condition-- comes up with the ratio of the amplitudes. And now you can write any solution, provided that you know the initial conditions. If you know the initial conditions, then you can also find C1. And therefore, since you know the ratio, you automatically have C2. And you can find this C1. I'll give this a minus sign and this a plus sign, and then you know the ratio in that mode.

So now you may think erroneously that life is easy. Oh, that's far from the truth. Suppose you have something as simple as this, l, m, l, m. It's called a double pendulum. There's no symmetry. I want to test your intuition. There are going to be two modes, two normal modes, a lower one and a higher one, because there are two objects.

In the lowest frequency-- use your hands and your legs-- what will this pendulum look like? You're are all doing sort of the right thing. Would it look like this? Or would it look like this? In other words, let me make the difference a little larger. Would it look like this or would it look like this? Slightly exaggerated. Who's for this one? OK, that means that the ratio, C2 over C1 is going to be plus 2, right? Who is for this one? That's the way it is, believe it or not. C2 over C1 is going to be 1 plus the square root of 2. And it will take you 30 minutes of grinding to find that. And I'm not exaggerating when I say 30 minutes. You have to go through the whole procedure. And then you'll find that this ratio is 2.4. And I will demonstrate it.

The highest mode must be something like this, right? Because they must be out of phase. Any idea of the ratio? C1 over C2-- we'll call this C1 and call this C2. Any one of you, any intuition? Minus, yeah? Boy, you're good. You're good. It is minus 2.4. This one will be much further away than this one.

Do any of you want to make a guess what omega minus is and anyone want to make a guess what omega plus is? No way on earth that you or I or anyone else can look at this and say, oh, yes, of course, omega minus is this. It's a long road, 30 minutes of calculating these frequencies. And out of that, then pops the ratios.

And now I'm going to demonstrate to you just this case. And the way I'm going to do that, even though we have not discussed driven coupled oscillators, in order to set it off in these normal modes-- a normal mode is also a resonance frequency. We call it, often, natural frequency. It's what the system likes to do if you leave it alone. And it's easy for me, when I just tease it a little bit, to excite it in that resonance mode. So therefore, I am going to drive it, but only very briefly, and then I will leave it alone. And I will get it into the state that you will see that it's really the normal mode. There's one and only one frequency. They come to a halt at the same moment in time. They're in phase. But you will clearly see that this distance more than doubled this one. And then I will go do this one.

So we'll try that All right. Double pendulum. Oh my goodness. This is a double pendulum. Yeah. All right. So, I have to drive it a little-- but it's really only very little-- at resonance. And then I will stop driving it. This is the mode. Do you see that the bottom one is further away than twice the top one? This is a normal mode. Can you see that it's not a straight line? Now, it's hard to say that it is 1 plus the square root of 2, of course, from your seats. But it is. That's what it is.

I'll now excite the second normal mode, which is a resonance. That's it. And I can almost stop swinging it. Do you notice that the upper one has a much larger amplitude than the lower one? And the ratio is 1 plus the square root of 2. Do you notice they are out of phase? It's the minus sign. They're out of phase. But in the other mode, they were in phase.

And so all of that will follow from the recipe. If I give this problem on an exam, you would have reasons to kill me, because it would take you too long. I would certainly wear a bulletproof vest on campus. That is easy because you have symmetry. This doesn't have that symmetry.

AUDIENCE: You've got C2 over C1 there. Should that not be C1 over C2?

PROFESSOR: Yeah, C2 over C1. This is C2. I haven't specified what I call 1 and 2. Is that the problem? I call this 1, and I call this 2. So C2 C1 is plus 2.4. Yeah? No, it's good that you asked that. And here, I changed the sequence, because I knew that this one was larger than this one. So here we have C1 over C2 is minus 2.4. The minus sign means out of phase. Yeah? Good that you asked that. Very good.

Well, I'm slowly going to turn you into experts. And now we're going to try something else to see how good your intuition is. I want you to understand, though, that your intuition is no better, no worse, than my own. In other words, there is no way that I could have done any better than you did. When I was in high school, or whatever it was-- maybe college-- when I first saw this, also my first impression was, what the hell, a straight line. But if you give it a little thought, you will come to the conclusion it cannot be. And then, ultimately comes out this way. So don't feel bad if your intuition lets you down. I'm as bad as you are when it comes to that.

So now we're going to evaluate the system, which will take you an hour and a half to work out with the general recipe. And that is four springs and three cars. One car, two car, three car, and all springs have spring constant k, nicely symmetric, you will think. And all objects have mass m. I want to see whether we have any intuition, without being too quantitative, for the three normal mode solutions. There must be an omega minus where all three are in phase. Then there must be omega plus, which is more complicated. And then that must be an omega plus plus, which is the highest of all three.

In the highest possible frequency, every object-- two adjacent objects are always out of phase. So if you have 20 objects in the highest mode, number one is out of phase with number two, number two is out of phase with number three, number three is out of phase with number four, and so on.

Let's first look at the omega minus. I won't even dare ask you whether you have any idea what omega minus is. You don't have that. I don't have that. Where would the object to be if, for instance, I displaced the first one over a distance which I just normalized to be plus 1? I call that plus 1.

Any idea where this one will be? Any idea where that one will be? What you think the second one will be? Plus 1. What will the third one be? Plus 1. Wrong. Why doesn't it have to be wrong? Suppose this one is at plus 1. And suppose this one is also at plus 1. Then this spring is no longer than it wants to be. And this spring is no longer than it wants to be. So there's no force on this object. So that's not possible. That object cannot go anywhere.

Any suggestions? Which of those is wrong? Is this correct, that it's plus 1? How about the middle one? Where do you want it? A little further away or a little less?

AUDIENCE: A little less.

AUDIENCE: Minus 1.

PROFESSOR: Have to be in phase. Don't use the word minus sign. If you use the word minus sign, man, how can you do that? They have to be in phase. It must be on this side. Do we make the distance larger or smaller?

AUDIENCE: Smaller.

PROFESSOR: It's larger. Shall I tell you what it is? Shall I tell you what it is? Do you want to know? Tommy, you really want to know, right? I knew it. Is that obvious? No.

Now, the next one. Ah, the next one is interesting. I will again put this one at plus 1, because that is my starting point. I always put that at plus 1. Now what? This one you can guess. You really can. Think now about the symmetry of the system. Yeah?

AUDIENCE: The one in the middle stays still, and the other one's opposite.

PROFESSOR: Very good. Very good. The middle one stays still, and this one comes to minus 1. Now the third one. Now remember, in the highest mode, neighbors are always out of phase with each other, no matter how many you have. So it's already certain that this one must be on this side, and it's already certain that this one must be on that side. Any volunteers? Would you like to try that again? Are you plus 1? Are you minus 1? Shall we put this one at plus 1? Oh sorry, minus 1? And this one at plus 1? Or shall we change the minus 1 here?

Don't feel bad. I wouldn't have been able to do it either. Shall I tell you what it is? Now, I can demonstrate this. And the way I'm going to demonstrate this is with the air track that I have here. See, the beauty with this air track is that I can place these cars wherever I want and they're not going to move, because I don't have any air. Then I turn on the air, and they're immediately free to go. In other words, if I offset this one in the positive direction over a distance 1-- we've just called this positive, I apologize for that, because I call this positive. And the other one has to be offset by 1 in the same direction, which I just did. But this one has to be offset by the square root of 2, which I just did.

And now, turn on the air, and I just have exactly the right initial conditions for this mode. So it is going to oscillate in the superposition of three normal modes, but there is only one. And that's the one you're going to see. Because I chose the initial conditions. You ready for this? Now, admit it. This is fantastic.

[LAUGHTER]

PROFESSOR: They are all in phase. That was a criterion for a normal mode. They're in phase. They come to a halt at the same moment in time. They all have the same frequency. And the amplitude here is the square root of two times larger. That is not so obvious. But you see, this is the criterion for normal modes-- same frequency and in phase.

This one-- OK, I'm going to put this one at 0. I'm going to put this one at plus 1. And I'm going to put this one at minus 1. Now, this one is offset by 1 in this direction. This one is offset by 1 in this direction, and this one is not offset. I turn on the air. What do you think will happen? That's right. It's going to oscillate in the superposition of three normal modes, but there is only one. That's the omega plus. Impressive, isn't it?

Next one. This one. So we're first going to put this at 0, this at 0, and we're going to put this at 0. And now what we have to do-- this has to be put over plus 1. The other one has to be put minus the square root of 2, and we have measured that here. And we've come so close that this spring is not going to like that. And now this one is going to be, again, plus 1 in the same direction as this one. That's the highest frequency. And there we go. You ready? Three, two, one, zero. One and only one frequency, and every adjacent one is out of phase with the other. These are out of phase, but these are in phase, because these are out of phase.

Now, I had one student calculate this for me. And I'll be frank with you, I even paid him for that because I was too lazy to do it myself. I put a job on the market, and he came within a few days. There were a few people-- one said, an A plus for 8.03, and he was going to do that in no time at all. 20 hours later, he had it right.

[LAUGHTER]

PROFESSOR: Cost me a fortune.

[LAUGHTER]

PROFESSOR: Now, I'm going to test you once more to show you that these things are not so straightforward-- not to make you feel bad at all, on the contrary. My intuition is no better. We're gonna make a triple pendulum. This is a triple pendulum. Omega minus-- what do you think it will look like? My favorite student is here. Do you think it will be a straight line?

AUDIENCE: I don't think so.

PROFESSOR: I don't think so either. I think what you're going to see-- without making any predictions about amplitudes-- exaggerated, you're going to see this. That's what I think. Can I predict omega minus? No way. No idea what it is. How about omega plus?

This is not going to work. It's a whole different system. Two of them are probably going to be in phase and one out of phase. Which two will be in phase? Will you see this? Now these two are in phase and this is out of phase. Or will you see this? Now these two are in phase and this one is out. Who is in favor of this? Who is in favor of that? I wanna see the hands more. Who is in favor of this? Who is in favor of that? Yea. Well, we'll see.

[LAUGHTER]

PROFESSOR: How about the highest mode? That's the easiest one, because now they must go like this. Now, of course, the ratio C1 over C2, C1 over C3, that's a different story. That is your 20 hours of grinding if you get paid. If you don't get paid, you can probably do it in two hours.

So I'm going to demonstrate this to you. I'm going to find these three modes. Again, I do that by a little bit of driving because of their resonances. And then I want you to eyeball these ratios-- C1 over C2, and C1 over C3. No problem. No problem. There we go.

All right, so now, this is the knot where the lengths are the same. So to get this one into the lowest mode is easy for me. Just give it a swing and then let it go, and then we'll just do nothing. And then I will stop even driving it. That's it. And now it's nicely in that mode. You see the bottom one. Notice that the lines are never in the same direction? You see the offset of the angles? Can you all see that? So our first picture on the blackboard is quite accurate.

Now the second one-- that's harder for me to find. Let me first make sure that they are not moving. OK, so I'm going to try to search for it. I got it. This is it. So the first solution is the correct one. The upper two are in phase. And the bottom one-- look at the large amplitude of the bottom one. That's not so intuitive. The bottom one is out of phase with the upper two. You see it? Normal mode. Why is it a normal mode? Because they all have the same frequency and they're in phase or they're out of phase.

Now, the highest frequency. Now, the highest frequency-- let me see if I can get that one, Got it. That's it. Now look how small the amplitude of the bottom one is. And that can all be calculated. Oh, I'll do it once more. They're beginning to rotate a little bit. There we go. That's the third one. So now you know everything there is to be known about coupled oscillators. But the most important thing that you probably have learned is that if you really want to get the general solution for a system, it's hard work. See you next Tuesday. Have a good weekend.