

## 8.03SC Physics III: Vibrations and Waves, Fall 2012

### Transcript – Lecture 6: Driven Coupled Oscillators

PROFESSOR: I will start today calculating for you the normal mode frequencies of a double pendulum. And then, I will drive that double pendulum. And then, we'll see very dramatic things that are changing.

So let us start here with the double pendulum. This is the equilibrium position if it hangs straight. Length is  $l$ , mass is  $m$ . This angle will be  $\theta_1$ . And this angle,  $\theta_2$ . I call this position  $x_1$ , and I call this position  $x_2$ .

I'm going to introduce a shorthand notation that  $\omega_0^2$  is  $g/l$ . I want to remind you that the sine of  $\theta_1$  is  $x_1$  divided by  $l$  and that the sine of  $\theta_2$  is going to be  $x_2$  minus  $x_1$  divided by  $l$ .

For small angle approximation, we do know the tensions. Here, we have a tension which I will call  $T_2$ . That is here, of course,  $mg$ . So it has also a tension  $T_2$  here. Action equals minus reaction. And then, there is here the tangent,  $T_1$ . And here is  $mg$ . Those are the only forces on these objects.

At small angles, I do know that  $T_1$  must be approximately  $2mg$  because it's carrying both loads, so to speak. And we know that  $T_2$  is very close to  $mg$ . And we're going to use that in our approximation.

So let me first start with the bottom one that has only two forces on it, so that may be the easiest,  $m\ddot{x}_2$ . And so the only force that is driving it back to equilibrium in the small angle approximation is, then, the horizontal component of the  $T_2$ . So that equals minus  $T_2$  times the sine of  $\theta_2$ .

And for that, I can write minus  $mg$  divided by  $l$  times  $\theta_2$   $x_2$  minus  $x_1$ . We divide out  $m$ . And we use our shorthand notation, so we get that  $\ddot{x}_2$ . And now we have here minus  $mg$  over  $lx_2$ . So that comes in. That becomes a plus. So I got plus  $\omega_0^2$  times  $x_2$ . I have here plus  $x_1$ . When that comes in, that becomes a minus. So I get minus  $\omega_0^2$   $x_1$ . And that equals 0.

So this is my first differential equation for the second object. So now, I'm going to my first object. So now, we get  $m\ddot{x}_1$ . And now, there is one force that is driving it back to equilibrium. That is the horizontal component of  $T_1$ . But the horizontal component of  $T_2$  is driving it away from equilibrium in the drawing that I have made here.

So you get minus  $T_1$  times the sine of  $\theta_1$  plus  $T_2$  times the sine of  $\theta_2$ . And so I'm going to substitute in there now the sines. And I'm going to substitute in there  $T_2$  equals  $2mg$  and  $T_2$  equals  $mg$ . So this becomes equal to minus  $2mg$  times the sine of  $\theta_1$ , which is  $x_1$  divided by  $l$ . And then, I get plus  $T_2$ , which is  $mg$  times the sine of  $x_2$  minus  $x_1$  divided by  $l$ .

And I'm going to divide by  $m$ . And I'm going to use my shorthand notation. And when I do that, I will come out here.  $x_1$  double dot is this one. Then, the  $m$  goes.  $g/l$  becomes  $\omega_0^2$ . But now, look closely. There is here a 2 times  $x_1$  and here there is a 1 times  $x_1$ . And both have a minus sign.

So when they come out, I get  $3\omega_0^2$ . So I get plus  $3\omega_0^2$  times  $x_1$ . And then, I have to bring  $x_2$  out. That becomes minus  $\omega_0^2$  times  $x_2$ . And that is a 0.

And so we have to solve now this differential equation coupled with  $x_1$  and  $x_2$  together with this one. My solution has to satisfy both. I want to go over that one to make sure that it is correct. It probably pays off because one sign wrong and you hang. The whole thing falls apart. You're dead in the water. So it pays off to think about it again.

So we have  $x_2$  double dot. I can live with that, plus  $\omega_0^2 x_2$ . That has the right smell for me. Minus  $\omega_0^2 x_1$ . I can live with that differential equation.

And I go to this one. Of course, the 3 is well known in a system like this that you get the 3. I have an  $x_1$  double dot plus  $3\omega_0^2 x_1$ . I think we are in good shape.

And so now, we're going to put in our trial solutions,  $x_1$  is  $C_1$  times cosine  $\omega t$  and  $x_2$  is  $C_2$  times cosine  $\omega t$ . And we're going to search for the frequencies. So this  $\omega$ , we have to solve for this  $\omega$ . This is not a given.

Since we're looking for normal mode solutions, these two  $\omega$ s must be the same-- can't write down  $\omega_1$  and  $\omega_2$ . And I don't have to worry about any phase angles because since we have no damping, either they're in phase or they're 180 degrees out of phase. And 180 degrees out of phase simply means a minus sign.

So that's the great thing about this. The signs will take care, then, of the possible phase angles. So now, we're going to substitute this solution in these two differential equations.

And I'm going to write it down in the form that I put the  $C_1$  to the left and the  $C_2$  to the right. And so I'm first going through this one that is my object number one. I take the second derivative. So I get  $C_1$  times minus  $\omega^2$  because the second derivative here gives me a minus  $\omega^2$ .

I ditch the cosine  $\omega t$  terms because I'm going to have a cosine  $\omega t$  everywhere. So I'm not going to write down the cosine  $\omega t$ . I have here plus  $3\omega_0^2$  times  $C_1$ . I have here minus  $\omega_0^2$  times  $C_2$ , and that is 0. Notice I put the  $C_1$ 's here and the  $C_2$  there.

I'm going to the other equation. I'm going to put this  $C_1$  on the left. So we get minus  $\omega^2$  times  $C_1$ . That's the term you have here. And then, we have plus  $C_2$  times  $\omega^2$  minus  $\omega_0^2$ . And this minus  $\omega^2$  comes from the second derivative of this one, second derivative of this one, minus  $\omega^2$ .

You get  $\omega^2$  comes out. And then, the cosine goes to a sine. That gives you the minus sign. So now, we have here two equations with three unknowns. That is typical for normal mode solutions of a system with two oscillations.

We don't know  $C_1$ . We don't know  $C_2$ . And we do not know  $\omega$ . And so we remember from last time that you can always solve for the ratio  $C_1/C_2$ , and you can solve for  $\omega$ . You can only find  $C_1$  if you also know the initial conditions, which I have not given. Instead of solving it the high school way that I did last week, the simple way, the fast way, I'm going to do it now using Cramer's Rule.

And the reason why I want to do this once, even though in this case it really is not necessary, when you have three coupled oscillators or four, there's just no way that the high school method will do it for you. And you have to have a more general approach. I've sent this to you by email, all of you. And I also assume that you have worked on this a little bit since I requested that you would prepare for that.

So I'm first going to write down what  $D$  is, which is the determinant of these columns that you see. There are  $a$ ,  $b$ , and  $c$ . Of course, we only have here two columns, the  $C_1$ 's and we have the  $C_2$ 's. So let me write down here what  $D$  is.

So I'm going to get  $3\omega^2$ . That is the  $C_1$ . Then, I get here minus  $\omega^2$ . That's the  $C_2$ .

And then, I get here minus  $\omega^2$ . And here I get  $\omega^2$  minus  $\omega^2$ . And the determinant of this, that's  $D$ . And I presume you all know how to get the determinant of this very simple matrix. We'll do that together, of course.

So following Cramer's Rule, then, my  $C_1$  which is  $x$  there, that's the one I want to solve for. So my  $C_1$ , I now have to take this  $0$ -- this is also a  $0$ , by the way. I forgot to mention that this is a  $0$ . This equation is a  $0$ .

So this column now,  $0, 0$ , has to come first. So I get here  $0, 0$ . And then, the second column is the same as it was here, minus  $\omega^2$ . And then I get  $\omega^2$  minus  $\omega^2$ . And that's divided by  $D$ . So we now know that that is  $C_1$ .

And then, we go to  $C_2$ . So now the  $0, 0$  column shifts towards the right, goes here. And the first column is unchanged. So we have here  $3\omega^2$  minus  $\omega^2$  minus  $\omega^2$ .  $0, 0$ , divided by  $D$ .

You've got to admit that the upstairs here, the determinant of this matrix, is  $0$  because you have two  $0$ 's here. And it's also  $0$  for this one. But clearly, zero solutions for  $C_1$  and  $C_2$  are meaningless. They're not incorrect because you will get, in this differential equation, that  $0$  is  $0$ , which is rather obvious. So we don't want the solutions that  $C_1$  and  $C_2$  are  $0$ .

And the only way that we can avoid that is to demand that  $D$  becomes 0. Because now, you get 0 divided by 0. And that's a whole different story. That's not necessarily 0. And that, then, is the idea behind getting the solutions to the searched for normal mode frequencies.

So we must demand in this case that this becomes 0. Otherwise, you get trivial solutions which are of no interest. So when we make  $D$  equal 0, we do get that the determinant of that matrix becomes three  $\omega_0$  squared minus  $\omega$  squared times  $\omega_0$  squared minus  $\omega$  squared, and then minus the product of that becomes a plus-- that becomes a minus, not a plus, becomes a minus. Because minus minus is already plus, so I get minus  $\omega_0$  to the 4 equals 0.

And this equation is an equation in  $\omega$  to the 4. You can solve that. You can solve that for  $\omega$  squared. I will leave you with that solution. This is utterly trivial. And out comes, then, that  $\omega$  minus squared, which is the lowest frequency of the two, is 2 minus the square root of 2 times  $\omega_0$  squared. And  $\omega$  plus squared is 2 plus the square root of 2 times  $\omega_0$  squared.

So this step for you will take you no more than maybe half a minute. But be careful, because you can easily slip up of course. So we now have a solution that  $\omega$  minus is approximately, if I calculate the 2 minus square root of 2, it's approximately 0.77, 0.76  $\omega_0$ . Not intuitive at all, so it's lower than the resonance frequency of a single pendulum. And then, I can substitute this value for  $\omega$  either back into my equations, or I substitute it back in this if you want to.

You get 0 divided by 0, which is not going to be 0. And you're going to find, then, that  $C_2$  divided by  $C_1$  in that minus mode, in that lowest possible mode, you will find that it is 1 plus the square root of 2, which is approximately plus 2.4.

And the  $\omega$  plus solution gives you a frequency which is about 0.85  $\omega_0$ . That solution, you can put back into your differential equations here. Or not the differential equations, I mean into this one. Or you put it back into this if you want to. And you will find now that  $C_2/C_1$  for that plus mode-- I call this the plus mode, that's my shorthand notation-- is going to be minus 1 divided by 1 plus the square root of 2.

So it's going to be minus 1 divided by 2.4. So it's going to be roughly minus 0.41. So those are, then, the formal solutions for the normal modes. And if I gave you the initial conditions, then of course you could calculate  $C_1$ . And then, you know everything because you know the ratio  $C_2/C_1$ . But without the initial conditions, you cannot do that.

Each of these normal modes solutions satisfies both differential equations. So therefore, a linear superposition of the two normal mode solutions is a general solution. So when you start that system off at time  $t$  equals 0, you specify  $x_1$ . You specify the velocity of object one. You specify  $x_2$ , and you specify the velocity of that object. Then, it's going to oscillate in the superposition, the linear superposition of two normal modes solutions.

One you see here has  $\omega$  minus. And this will be the ratio of the amplitudes. And this is the  $\omega$  plus. And that'll be the ratio of the amplitudes. And that is non-negotiable. That's what the system will do.

Now, we're going to make a dramatic change. Now, we're going to drive this system. So now, this is the equilibrium position now of the whole system. And I am going to drive it back and forth, holding it in my hand like this. And I'm going to shake it. I call the position of my hand  $\eta$  equals  $\eta_0$  times the cosine  $\omega t$ .

So  $\eta_0$  is the amplitude of my motion of my hand. That is what I'm going to do. And so when I look now at this equation, at this figure, not the equation but at the figure, I call this now  $x_1$ . I call this now  $x_2$  because you should always call  $x_1$  and  $x_2$  the distance from equilibrium. And this is now equilibrium.

If I don't shake at all, I have the pendulum here. And so it's hanging straight down. So this is now equilibrium. And this location here at a random moment in time is now  $\eta$ .

What has changed? Well, at first sight, very little has changed. The only thing that has changed now is that the sine of  $\theta_1$  is now  $x_1$  minus  $\eta$  divided by  $l$ . So I have here a minus  $\eta$ . That looks rather innocent. But the consequences will be unbelievable.

So I can leave everything on the blackboard the way I have it. All I have to do is to put instead of the sine  $\theta_1$   $x_1$  divided by  $l$ , I have to put in  $x_1$  minus  $\eta$  divided by  $l$ . See, nature was very kind to me. It already left some space there. You see that? Nature anticipated that I was going to do that.

So you get a minus  $\eta$  there. Well, if now you divide  $m$  out and you're going to use the shorthand notation, then leaving  $\eta$  on the right side, which is nice to do, you get minus minus  $\eta$ . So this  $0$  now becomes  $2\omega_0^2$  times  $\eta_0$  times the cosine of  $\omega t$ . It becomes  $2\omega_0^2$  times  $\eta$ . You see?

So it is no longer a  $0$ . And so when we now substitute in there these trial functions, they now have a completely different meaning.  $\omega$  is no longer negotiable.  $\omega$ , as set by me, is a driver. So we're not going to solve for  $\omega$ .  $\omega$  is a given.

That means, if  $\omega$  is a given, that we're going to end up not with two equations with three unknowns--  $C_1$ ,  $C_2$ , and  $\omega$ -- but we're going to end up with two equations with two unknowns, only  $C_1$  and  $C_2$  because  $\omega$  is a given.

And so in the steady state solution, you get a number for  $C_1$ . And you get a number for  $C_2$  in terms of  $\eta_0$ , of course. You will see how that works.

So when we put in these functions now, these  $\omega$ s are fixed, are no longer something that we search for. They're my  $\omega$ s. I can make them  $0$ . I can make them infinite. I can make them anything I want to. And that's what we want to study.

So if we want to go back now to these two equations, what changes here since we have put in this trial function, the only thing that disappears is this cosine  $\omega t$ . So we end up here with  $2\omega_0^2$  times  $\eta_0$ . And nothing else changes. But now, we're looking now at our solution for  $C_1$ . And we're going to look at our solution for  $C_2$  in steady state.

It's easy, right? We apply Cramer's Rule. And the only thing that changes is this one, which has to be replaced by this,  $2\omega_0^2 \eta_0$ .  $\eta_0$ , remember, is the amplitude of my hand.  $\eta_0$  is the displacement of my hand at any moment in time.  $\eta_0$  is the amplitude.

And so here, we get then also  $2\omega_0^2 \eta_0$ . And so now, we can solve for  $C_1$  and  $C_2$ . We get an answer-- not just a ratio only, we get an answer. We know exactly what  $C_1$  is going to be. And we know exactly what  $C_2$  is going to be because  $\omega$  is non-negotiable.  $\omega$  is now a known. When we solved this, we were searching for  $\omega$ . And out came these  $\omega$ s. That's not the case anymore. We know  $\omega$ . It's called  $\omega$ , and that's it.

So I can write down now  $C_1$ . So I take the determinant there of the upstairs. So that gives me  $2\omega_0^2 \eta_0^2 \omega_0^2 - \omega^2$ . That is this diagonal. And this one is 0.

And I have to divide that by  $D$ . Now, I could write  $D$  in that form. And I could do that. There's nothing wrong with that. But I can write it in a form which is a little bit more transparent.

We do know that  $\omega - \omega_0$  and  $\omega + \omega_0$  will make  $D = 0$ . So you can write this then in the following way. You can write here  $\omega^2 - \omega_0^2$  times  $\omega_0^2 - \omega^2$ . That must be the same as what I have there because you see that this one becomes  $\omega - \omega_0$ , then this goes to 0. And when this one becomes  $\omega + \omega_0$ , that is also 0. So it's a different way of writing. Gives you a little bit more insight.

It reminds you that the downstairs will go to 0 when you hit those resonance frequencies. And so I will also write down  $C_2$ , then. This one is 0. I get minus this one. So I get plus  $2\omega_0^2$  to the 4 times  $\eta_0$  divided by that same  $D$ . You can write this for it.

So let me check that, see whether I'm happy with that.  $2\omega_0^2 \eta_0^2 \omega_0^2 - \omega^2$ , right? Is  $\omega^2 - \omega_0^2$  squared? That looks good. This looks good. This looks good. And I have here-- I'm happy.

Our task now is not to look at these equations but to see through them. And they are by no means trivial, what they're going to do as a function of  $\omega$ . It's an extraordinarily complicated dependence on  $\omega$ .

And I have plotted for you these values of  $C_1$  and  $C_2$  for which you get an answer now. You also know  $C_1/C_2$  of course because you know  $C_1$  and you know  $C_2$ . And I'm going to show you this as a function of  $\omega$ . And then, we will try to digest that together.

And this plot, like the other plots that I will show you later today, will be on the 803 website. They will be part of lecture notes. So you have to click on Lecture Notes. And then, you will see these plots. This is the first one, which is the double pendulum.

What is plotted here horizontally is  $\omega/\omega_0$ . So that's the  $\omega_0$  that we mentioned there. And so you see the first resonance here is about at  $0.76\omega_0$ . And the second resonance here is at that  $1.85\omega_0$ .

And what we plot here is the amplitude,  $C$ , divided by  $\eta_0$ . Because obviously if you make  $\eta_0$  larger, you expect that has an effect on the  $C$ 's, of course. If you drive it to the larger amplitude, of course the object will also respond accordingly. And so therefore, we have it as a function of  $\eta_0$ . You see the  $\eta_0$  here?  $C_1$  is linearly proportional with  $\eta_0$ .  $C_2$  is linearly proportional to  $\eta_0$ . That's no surprise, of course. So we divided by  $\eta_0$ .

When we plot it upstairs, it means that the amplitude is in phase with the driver. That's the way we have written it. And if it is below the zero line, it means that the amplitude of the object is out of phase with the driver. That's all it means. That's the sign convention.

So let us now look at this and try to digest it and use, to some degree, our intuition and see whether that agrees with our intuition. And let us start when  $\omega$  goes to 0. And don't look now at the solutions. If  $\omega$  is 0, I have a double pendulum in my hand. And I'm going to move it to the left. And 25 years from now, I'm going to go back. And I move it to the right again.

So the pendulum is always straight. And it is clear that  $C_1$  and  $C_2$  to both must be  $\eta_0$ . And you better believe it that if you substitute in there  $\omega_0$ , that's what you will find.

So  $C_1$  must be  $C_2$  and must be  $\eta_0$ . And it must even be plus  $\eta_0$ . It must be in phase with the driver. And you see that the ratio-- not the ratio, which is  $C$  divided by  $\eta_0$ . That is a ratio that is plotted at plus 1. And so both are in phase with the driver.

So that's a trivial result. It means that the pendulum which is here, 25 years from now is here. That's  $\omega_0$ , almost 0. So notice that we now know the ratio  $C_2/C_1$ , which is plus 1. The ratio  $C_1/C_2$  or  $C_2/C_1$  are now entirely dictated by this. Nothing to do with normal modes anymore, so don't be surprised that  $C_1/C_2$  is now plus 1.

Now, look what happens. I'm going to increase my  $\omega$ . And what you see is that the red curve, which is the second object, the lowest one of the two, is going to have a larger amplitude than the top one. You see, it already begins to grow very rapidly.

And when you reach resonance, the ratio is going to be plus 2.4, of course. That's obvious. Now at resonance, you get an infinite amplitude for each. That, of course, is nonsense. It has physically no meaning. So you should think of it as going to infinity. For one thing, if  $C_1$  became anything, that means if this point here ends up there in the hole, it's hard to argue that it's a small angle, right?

So in any case, the solution wouldn't even hold apart from the fact, of course, it has always some damping. And so we never go completely off scale, so to speak. However, you will see that when you drive the system, the double pendulum, and you approach resonance, that you will very quickly see that the ratio of the amplitude of the second one over the top one will grow, will become 1 and 1/2, will become 2. And then, in the limiting case, you will hit plus 2.4.

So as  $\omega$  goes up, you're going to see that  $C_2/C_1$  is going to be larger than one, right there, this one. And then ultimately, it will reach that 2.4. But that is that extreme case of resonance.

And when you look at the motion of the pendulum, if you drive it somewhere here, notice that they are both in phase with the driver. And  $C_2$  is larger than  $C_1$ . So what you will see is this is  $C_1$ . And then, this is  $C_2$ . And so  $C_2$  is larger than  $C_1$ . And they are in phase with the driver. And so when they return, you will see the pendulum like this. So that's the sweeping that you will see.

There's something truly bizarre. I go a little higher in frequency. I go over the resonance. And I see here a point whereby the frequency happens to be exactly the frequency of a single pendulum--  $\omega$  divided by  $\omega_0$  is 1-- and the top one refuses to move. But the bottom one does.

The bottom one here has an amplitude which is twice the amplitude of the driver. You look, you see a 2 here. And it is out of phase with the driver. That is unimaginable. It's unimaginable what you're going to see.

So when  $\omega$  is  $\omega_0$ ,  $C_1$  becomes 0.  $C_2$  becomes a minus 2  $\eta_0$ . And so this pendulum is going to look, then, as follows. I'll make a drawing here. I have a little bit more space.

So if my hand is here at  $\eta_0$ , then number 1 will stand still, won't do anything. But number two will have an amplitude which is twice  $\eta_0$ . But it's on the other side. So this is 2  $\eta_0$ .

And then, if you look half a period later, it will look like this. So this is  $\eta_0$ . And then, this is 2  $\eta_0$ . And this one doesn't move. That's what it tells you.

How on Earth can the lower pendulum oscillate if the upper one stands still? I want you to think about that. I'm still having sleepless nights about it. And so maybe you will have some tool. And if you have some clever ideas, come to see me.

But the logical consequence of what we did is that this one would stand still. And the other one will still oscillate and be driven by my hand. I have to keep moving this. Otherwise, this will go to pieces. I must be doing this all the time at that frequency, which is exactly the resonance frequency of a single pendulum. It is that  $\omega_0$ . I must keep doing this. And this one does nothing. And this one has double the swing of this and is out of phase.

If, then, you go even higher, then you will see that the two objects will go out of phase. The upper one will be in phase with the driver. And the lower one will be out of phase with the driver. And there, you hit that second resonance when things will get out of hand. And you get that ratio, minus 0.42, back of course.

I will want to demonstrate to you this situation here and this situation to see whether they make sense. And so now, I'm going to use a double pendulum and drive it with my own frequency, which I determine. I'm the boss. I determine  $\omega$ , not looking for normal mode solutions. I determine  $\omega$ .

And I'm going to first drive it with  $\omega_0$  with  $\omega$  is about 0. In other words, I'm going to drive it here. And what you're going to see is, of course, fantastic, absolutely fantastic. It's

hanging straight down now. And I'm moving it over a distance of one foot. So  $\eta_0$  is one foot. And  $C_1$  is one foot, and  $C_2$  is one foot. And they're in phase with the driver. Physics works.

If I go a little higher up here, so I go somewhere here approaching resonance, then you will see that  $C_2$  becomes larger than  $C_1$ . This is  $C_2$  and this is  $C_1$ . And you get to see a picture which is very much like this. And I'll try that.

So I'll drive it below resonance but not too far below. And there you see it. You see?  $C_1$  is smaller than  $C_2$ . No longer 1 plus 1. If this was 1, your  $C_1/C_2$  is 1. That's no longer the case now. You really see that  $C_2$  is getting ahead-- not in terms of phase ahead, but in terms of amplitude, very clear.

Now, I'm going to attempt to do the impossible. And the impossible is to try to hit this point, the point whereby the upper one stands still and whereby the lower one will have an amplitude which is twice that of my hand but out of phase with my hand. How on Earth can I ever drive this system with that frequency,  $\omega_0$ , which is the frequency of a single pendulum?

Well, maybe I can't. But I will try. And the way that I'm going to try this is the following. I know what the resonance frequency of a single pendulum is. That's this. I can feel it in my hands. I can feel it in my stomach. I can feel it in my brain. I feel it all over my body.

I can burn this frequency into my chips here. And then, I can close my eyes while you are looking and generate that frequency which was burned here and drive the system as a double pendulum. If I succeed, you will see the upper one stand still and the bottom one will have twice the amplitude of my hands. So the success of this depends exclusively on how accurately I can burn this frequency into my chips.

So you have to be quiet. So I'm going to count. One, two, three-- I'm burning now-- one, two, three, four, one, two, three, four. I'm closing my eyes. One, two, three, four. One, two, three, four. One, two, three, four. One, two, three, four. One-- you're not saying anything! Didn't you see this one standing still? Did you see it? Did you also see that the other one had twice the amplitude of my hands and out of phase with my hands? You didn't see that, right? Admit it. You didn't see it because you were not looking for it. Especially for you, I'll do it again.

So you really have to see number one that this one will practically standstill. Number two, that this one has double the amplitude of my hands but out of phase. The decay time of burning is only one minute. So I have to burn in again. One, two, three, four, five. One, two, three, four, five. One-- uh-oh. These things happen. You have to start all over with the burning.

There we go. One, two, three, four, five. One, two, three, four, five. One, two, three, four, five. One, two, three--- are you seeing it?

AUDIENCE: Yes!

PROFESSOR: All right.

[APPLAUSE]

PROFESSOR: This is an ideal moment for the break. I'm going to hand out the mini-quiz. And we will reconvene. I'll give you six minutes this time, so you can even stretch your legs. So I would like some help handing this out. And then, you bring it back. And I'll put some boxes out there.

So if you can help me handing it out here, you can start right away. Hand this out here. For those of you who have no seats, come forward and get some seats. Nicole, we are still friends, right? So why don't you hand that out. And why don't you hand this out here? You can also give it to people here, here, if you do that. You can start right away.

I'm now going to do something perhaps even more ambitious. And I'm going to now couple three oscillators-- not pendulums yet, but I'm going to couple three oscillators which I connect with four springs. I'm going to work on this. Three masses, equal masses, four springs, spring constant,  $k$ . And the spring constants are the same.

And I'm going to drive that system one, two, three, four. And this is the end. In other words, I have here a spring. And here's the first mass, second mass, third mass. And here, it is fixed. And I'm going to drive it here with a displacement,  $\eta$ , which is  $\eta_0 \cos \omega t$ .

So at a random moment in time, this is where my hands will be. So this is  $\eta$ . This is where the first mass will be. Remember, you always call the displacement  $x_1$  from its equilibrium, that is from its dotted line.

So here's the spring. This one is here. So I call this  $x_2$ . So here is the spring. And this one is here. So this is  $x_3$ . So here is the spring, and here is a spring. You may have noticed more than once now that I have a certain discipline that I always offset them in the same direction. Do you have to do that? No.

If you don't do it, your chance of a mistake on a sign slip is much larger than if you always set them off in the same direction. You'll see shortly why. So that is certainly something that is not a must, but it's a smart thing to do. I define this as my positive direction, but that, of course, is a complete free choice.

Now, let at this situation at this moment in time, let  $x_1$  be larger than  $\eta$ . Let  $x_2$  be larger than  $x_1$ . And let  $x_3$  be larger than  $x_2$ . And this assumption will have no consequences for what follows, at least for the differential equations.

If  $x_1$  is larger than  $\eta$ , that first spring is longer than it wants to be because I've assumed that  $x_1$  is larger than  $\eta$ . And so that means there will be a force in this direction because this spring is longer than it wants to be.

If  $x_2$  is larger than  $x_1$ , this spring is also longer than it wants to be. So it will contract. So there's a force in this direction. And so I can write down now the differential equation for my first object.

So that's going to be  $m\ddot{x}_1$ , that equals minus  $k$  times  $x_1$  minus  $\eta$  because that's the amount by which it is longer than it wants to be. So times  $x_1$  minus  $\eta$ , that is this force. And this force is now in the plus direction, is plus  $k$  times-- this spring here is longer than it wants to be by an amount  $x_2$  minus  $x_1$ . Not  $\omega$ , but  $x_1$ .

That's my differential equation for the first object. And this one is always correct, even if  $x_1$  is not larger than  $\eta$  because if  $x_1$  is not larger than  $\eta$ , then this force flips over. Well, this will also flip over. So that's why it's always kosher and advisable to make that assumption to start with because, again, it reduces the probability of making mistakes. That's all. There's nothing else to it, just reduce the chance of slipping up.

So let's now go to this object. If this spring is longer than it wants to be, it wants to contract. So this object will see a force to contract. But if this spring is longer than it wants to be because  $x_3$  is larger than  $x_2$ , it will experience a force to the right. So I can write down now the differential equation for object number two.

$m\ddot{x}_2$ , notice that the one that is here to the left is the same one that is here to the right, right? Action equals minus reaction. This pull is the same as this pull. So it is going to be this term which now has a minus sign. And you always see that in coupled oscillators that what was a plus here is going to come out here as a minus sign. You see, that comes out nicely because this spring is longer than it wants to be by an amount  $x_2$  minus  $x_1$ . And the force is in the minus direction.

And this one is now going to be plus  $k$  times  $x_3$  minus  $x_2$ . So now, I go to the next spring, to the next object. So this object here will experience a force to the left because this thing is longer than it wants to be. So it wants to contract. But this one is pushing. So therefore, the force due to this spring is now also in this direction because the end is fixed.

And so we get for the third object  $m\ddot{x}_3$  equals minus  $k$  times  $x_3$  minus  $x_2$ , which is this term, but it switches signs. And then in addition, I get minus  $k$  times  $x_3$ .

When you reach this point on an exam, you pause, take a deep breath, and you go over every term and every sign. If you slip up on one sign, one casual mistake that you just even though you know it you casually write here for instance a 1 instead of a 3, it's all over. You're dead in the water. The problem will fall apart. And it may not even oscillate in a simple harmonic way.

So therefore, let's look at it.  $m\ddot{x}_1$ --  $x_1$  is larger than  $\eta$ . Therefore, force is in this direction. I love it. The other one is in this direction, perfect,  $x_2$  minus  $x_1$ . That same force here is going to pull on the second one. So if this is correct, this is also correct. This one is driving it away from equilibrium,  $x_3$  minus  $x_2$ , got to be right. This term shows up here with a minus sign, can't go wrong there. And since this spring is always shorter here, if it's pushed to the right, I am happy with my differential equations.

So now, you're going to substitute in here  $x_1$  is  $C_1 \cos \omega t$ , trial functions.  $x_2$  is  $C_2 \cos \omega t$ . And  $x_3$  is  $C_3 \cos \omega t$ . Are we looking for  $\omega$ ? Oh, no, oh, no.

Omega is given by me. I am telling you what omega is. You're not going to negotiate that with me.

We are only solving for  $C_1$ ,  $C_2$ , and  $C_3$ . And in the steady state, you will be able to do that because omega is nonnegotiable. You're going to get three equations with three unknowns---  $C_1$ ,  $C_2$ ,  $C_3$ . You don't have to settle to only calculate the ratios of the amplitude. No, you're going to get a real answer for  $C_1$ , for  $C_2$ , and for  $C_3$  which, of course, will depend on  $\eta_0$ -- sure, if you know  $\eta_0$ .

Do we worry about phase angles here? No, because there's no damping. And if there's no damping, either the objects are in phase or they're out of phase because it is the damping that gives these phase angles in between. And 180 degrees out of phase is a minus sign. So we have the power to introduce 180 phase changes and 0 phase. For that, we have plus and minus signs.

Now, you are going to do some grinding. I did all the grinding on every detail of the double pendulum. Now, you're going to do the grinding. However, I want to make sure that if you go through that effort to make the grinding that you indeed end up with the right solution. So in that sense, I'm going to help you a little bit by giving you the  $D$ , which is the  $D$  that we have here.

But you have to bring me to the  $D$ . And so the  $D$  is going to be-- so we have to divide by  $m$ . You have to also--  $\omega^2$ ,  $k/m$ , shorthand notation. Some of you may want to call it  $\omega_0^2$ . That's fine because there's only one-- only springs, there are no pendulums and springs. But I still call it an  $s$  to remind you that it is the resonance frequency or a single spring.

Then, my  $D$  becomes minus  $\omega^2$ . That's always the result of that second derivative, remember? You always get that minus  $\omega^2$  out. Then you get plus  $2\omega^2$ . Then, you get in the second column minus  $\omega^2$ . And in the third column, you get a 0.

No surprise that you get a 0 in the third column because the first differential equation has no connection with  $x_3$  at all. And so you never see anything in the third column that will be a 0. But if you look at the second differential equation, that has an  $x_1$ , an  $x_2$ , and an  $x_3$  in it. So now, you don't see 0's. So what you're going to see is minus  $\omega^2$ . That's going to be the  $C_1$  term.

And then, you get here minus  $\omega^2$ . I think it is plus  $2\omega^2$ . And your last column is going to be minus  $\omega^2$ .

Now, the third differentiable equation, there's no  $x_1$ . Therefore, that is a 0 here. And then, you get minus  $\omega^2$ . And then here, you get minus  $\omega^2$  plus  $2\omega^2$ . And you have to take the determinant of this matrix. That is  $D$ .

Let me check it to make sure that I didn't slip up with a minus sign so when you get home that you don't wonder, why didn't you get that result? And I think that looks good to me. Remember, all those  $\omega^2$ 's always come from those second derivatives because you have to take

the second derivative of cosine  $\omega t$ . That always brings out the minus  $\omega$  squared. So no surprise that this is a minus, that this is a minus, and that that is a minus.

So now, we want to know what  $C_1$  is. And the first column will reflect this  $\eta$  because the right side now is not going to be 0, remember, like the double pendulum? So you're going to get here in the first column, you're going to get  $\omega^2 \eta$ . And there, you're going to get a 0 and a 0.

And then, this column comes here. And this column comes here. That is the determinant of the upstairs. And you divide it by  $D$ .

That, then, is  $C_1$ . And of course, I will only go one step further to go to  $C_2$ . But that becomes a little boring now.  $C_2$ , then, you get that's this one. And then, the second column according to Cramer's Rule is going to be  $\omega^2, \eta, 0, 0$ . And then, the third one is going to be this. And then, you go on, and we divide it by  $D$ , of course. And then, you can write down  $C_3$ . You're on your own. I'll help you.

So when you do this, you could, if you wanted to, first solve for what we call the resonance frequencies. The resonance frequencies are the ones which are the normal mode frequencies. That's a resonance. And so you may want to put in  $D = 0$ . So you make that determinant equal 0, which gives you, then, the three resonance frequencies, which earlier we would have called normal modes in case we are not driving.

And so for those of you who worked this out, the lowest frequency, resonance frequency which was a normal mode, is  $2 \pm \sqrt{2} \omega$ . The one that follows is  $2 \omega$ . And the one that follows then, which I will call  $\omega_{++}$ , is going to be  $2 + \sqrt{2} \omega$ . None of these are, of course, intuitive. But none of the resonance frequencies of our coupled double pendulum were intuitive.

There's no way that you could even look at this and say oh yeah, of course. Excuse me? What did I do wrong?

AUDIENCE:  $\omega$ , or--

PROFESSOR: Yes, thank you very much. I have an  $\omega$  there. You deserve extra credit. I've called that  $\omega$ . If you want to change all the  $\omega$ 's and  $\omega$ 's, that's fine. But you cannot have one  $\omega$  and the other  $\omega$ . Yes, so this is the square of the frequency or a single spring oscillating mass  $m$ . Thank you very much for pointing that out.

AUDIENCE: [INAUDIBLE]

PROFESSOR: Ah, boy, you also deserve extra credit. I really set you up with this, didn't I? I wanted two people to get extra credit. You deserve one. Come see me later. And who was the other one? You did one, thank you. Yeah, there's a square here. Oh, there are more people than one who claim this one. All right, thank you very much.

OK, so now for any given value of  $\omega$ , for any given value of  $\omega$  you find three answers. You get  $C_1$ . You get  $C_2$ . And you get  $C_3$  in the steady state solution.

Of course, you get infinite amplitude, which is physically meaningless. So you've got to stay away in your solutions from the infinities. But you'll see if you don't come too close to infinities that the results you get are quite accurate for high  $q$  systems. And so now I will show you the three amplitudes for this system, which will also be put on the web this afternoon. And so that is the three car system which we have set up there.

The only difference with the previous one is that we don't plot here  $\omega$  divided by  $\omega_0$  but the squares. This plot was provided to me by Professor Wyslouch who lectured 803 a year ago. I was a recitation instructor at the time. And it was very kind of him that he gave me this plot. He even added this at my request, which is very nice.

Car 1 is the first car, green. And then, the car 2 is the red one, is the second car. And this is supposed to be blue. You don't see it. But in any case, if you think it's black, that's fine. That's then the black line. So horizontally, it is the ratio of the frequency squared. So you see that the second resonance is indeed at a 2 here. You notice that 2 that I have there on the blackboard.

If we plot it above the 0, it means that it is in phase with the driver. If we plot it below the 0, it means it's out of phase with the driver. If now you look at this, then already at  $\omega_0$  you see something that is by no means intuitive.

Notice that  $C_1$ ,  $C_2$ , and  $C_3$  are all substantially lower than  $\eta_0$  because this is in units of  $\eta_0$ . And they're not even the same. They're all three different. Would I have anticipated that? No, I wouldn't.

Maybe two would be the same for me, but not all three different. And that's the case. They're all three different. When we approach resonance, things go out of hand. All three are in phase. And when you just cross over the first resonance, they are all three again in phase but out of phase with the driver. That's not so surprising all by itself.

But now, look at this ridiculous point. If I drive that system with the resonance frequency of the individual spring with one mass on it-- because remember, if this squared is 1, then  $\omega$  divided by  $\omega_0$  is also 1, but that's exactly the square root of  $k/m$ -- then this one will stand still. And these two have roughly the same amplitude. You could eyeball it here. It looks like it almost crosses over there. And it's about minus  $\eta_0$  because it's about minus 1.

Let me write that on the blackboard. That's a quite bizarre situation. So at that one frequency, so  $\omega$  equals the square root of  $k/m$ , I get  $C_1$  equals 0. And my  $C_2$  is about  $C_3$ , maybe even exactly  $C_3$ . I never checked that. And that is roughly minus  $\eta_0$ . You can just see that there.

So it means that this car will stand still. It's closest to the driver. These two are in phase with each other, have the same amplitude. But they're out of phase with the driver. Crazy! Got to be wrong, right? How on Earth can this one stand still, and these happily go hand in hand and go--? It can't be right. But I will demonstrate to you that it is right at least to a high degree of accuracy.

I'm going to concentrate on two points in that graph. The first point that I want to concentrate on is this one, the hardest one of the two. I'm going to drive this system at a frequency that we think is almost there. I'm going to turn it on right now because it will take three to four minutes for the transience to die out. And then in the meantime, I'll explain what you're going to see.

I'm driving it now with that frequency. So don't look at it now. It looks chaotic. It's going into a ridiculous transient mode. This one is also oscillating. It's not supposed to oscillate. Look there. It's oscillating, and it looks chaotic. The amplitudes are changing. Just let it cook.

Now, what are you going to see? If the transients die out which can really take three minutes, you will see that this is my  $\eta_0$ . This is  $2\eta_0$ . You see this arrow? That is  $2\eta_0$ . That's the driver. So you can calibrate the amplitude  $\eta_0$ .

You will see, then, that these two cars have the same displacement but out of phase. So when this one goes to the right for me, they go to the left for me and vice versa. They have an amplitude which is very closely the same. And this one is going to stand still.

What you're going to see is not something truly spectacular. But it's extremely subtle. You have never seen it before and I don't think this has ever been demonstrated in any lecture hall. It's extremely subtle, first of all, to have the patience to wait four minutes for this to happen, number one. And then, number two, to make such a daring prediction that this one will almost come to a halt and that the other will have an amplitude which is the same as the driver but 180 degrees out of phase.

So be patient. And it will pay off. Don't fall asleep now. Let's look at this middle one to see whether its amplitude is becoming constant. As long as the amplitudes are not constant, there is still transience. Well, these two are already going hand in hand. You see that? And I would say very much with the same amplitude.

And look at this. Hey, [INAUDIBLE]. They are already out of phase with each other. You see that? Now this one, oh boy. That amplitude is nowhere nearly as large as this one. And yet, when we started it, it was even larger than this one because of the transient phenomenon.

Look at this one. It's almost standing still already. Can we find that precise frequency? No. We have to set a dial somehow. We do the best we can. I think it's fantastic. It's growing my mind.

I can't believe it. Physics is working. This one is practically standing still. If I had to estimate this amplitude, I would say maybe  $1/10$  of  $\eta_0$ . And these, you can mark, you see the marks here. This one is really  $\eta_0$ . For those of you who see these marks, that is the same as this. It's really  $\eta_0$ . And boy, they go hand in hand. Aren't you thrilled?

[APPLAUSE]

PROFESSOR: Now, I'm going to try to make you see this point. I'm going to start it. And then, I'll explain to you what is so special about that point. Because again, we have to be patient. And we have to wait for the transience to die out.

I'm going to drive it very close to resonance right below to resonance. You get a huge amplitude of C1 and C3. But look what C2 is going to do. C2 is going to have an amplitude which you and I may have thought is 0, namely the outer ones do this and the middle one stands still. No, no, look at 1 and 3, by the way. They go nuts already. You see? That is this. This is 1 and this is 3.

Now you see the transient phenomenon. Now, it's picking up again. So we have to be patient. But there's something very special about this C2. Why is this C2 not 0, which is what you would expect? You would expect that the outer two go like this and the middle one would just stand still.

AUDIENCE: [INAUDIBLE]

PROFESSOR: That's one answer, yes, of course. But isn't it so that this must be 0? You don't believe that this upstairs is going to be 0? Just say you don't believe it.

AUDIENCE: No.

PROFESSOR: It is 0! Why now, even though the upstairs is 0, why do we get an amplitude which is half  $\eta_0$  but with a minus sign? Because we get 0 divided by 0 because right at that frequency here, that is a resonance frequency. So D is 0. So you get an amazing coincidence that the upstairs is 0, the downstairs is 0. But that's what we have 1801 for. The ratio is not 0. The ratio is minus  $1/2 \eta_0$ . And that's what you see, it's minus  $1/2 \eta_0$ .

Now look. 1 and 3 are happy. They oscillate happily out of phase with each other. Number 1 has to be in phase with the driver. It's doing that.

Number 3, out of phase with the driver, it's doing that. And look at the middle one. The middle one, for those of you who can see, the arrows are given  $\eta_0$  to  $\eta_0$ , its right half of it. Just walk up to it and you can see. It's amazing, this.

We hit that right on the nose. The reason why this is easy is look, my  $\omega$  doesn't have to be precisely here. Even if it's a little bit to the left, I'm still OK. That amplitude of number 2 is not changing very much. You see that? This is so nicely horizontal. So this, for me, was a piece of cake. This was hard. Piece of cake. So you see there?

Now, I'm going to move to the triple pendulum. In other words, now you've seen the structure of my talk. We first did the double pendulum. The double pendulum was-- I worked out all the way for you. Then, we did the three cars with the four springs. I set it up for you so that you can't go wrong anymore. I gave you the final solution. And I demonstrated that indeed it is working the way we calculated.

Now, I'm going to simply show you the results of a triple pendulum. And the plots that I'm going to show you will be on the web, but no calculations at all. So that triple pendulum, or that-- let me put these down so that we have not so much shadow on this board.

Triple pendulum, this is a triple pendulum. And we're going to drive it here,  $\eta$  equals  $\eta_0$  times the cosine of  $\omega t$ . The top one is going to be green. The middle one is going to red. And the bottom one is going to be blue even though it may look black there, but I'll make it blue. That is the color code that we have on our plot, which will be put on the web this afternoon.

So here it comes. Horizontally, we plot again  $\omega$  divided by  $\omega_0$  where  $\omega_0$  is the square root of  $g/l$ , the resonance frequency of a single pendulum. And vertically, we do the same thing. We plot  $C$  divided by  $\eta_0$ . Everything has the same meaning, namely plus 1 means in phase with the driver and the same amplitude as  $\eta_0$ . And minus is out of phase with the driver.

Now, if you take this pendulum and you move it with 0 frequency, you don't have to know any differential equations that it will be here. And 20 years from now, it will be there. And this separation is, then,  $\eta_0$ . And so you expect that  $C_1$ ,  $C_2$ , and  $C_3$  will all three be  $\eta_0$ . And they will be plus  $\eta_0$ -- not minus, but plus, in phase with the driver. And look, that is what you see there. So that is by no means a surprise.

You go closer to resonance. And you see that the bottom one, which is the blue one, black here, is picking up an amplitude which is larger than the middle one and the red one. And the red one, larger than the top one.

So you're going then into the domain where you're going to see something like this. So this one,  $C_1$ ,  $C_2$ ,  $C_3$ . And then finally, when you hit resonance, I do not know what the ratios are. But everything gets out of hand anyhow.

And now, there comes a point which boggles the mind right here. The top one is not moving, stands still. And there's nothing really so special about that frequency. That frequency for which the top one stands still, so  $C_1$  goes to 0, that  $\omega$  if I try to eyeball it, I would say it's about 0.75, 0.77. 0.77 times  $\omega_0$ .

And somehow at that frequency, the top one will not move. The other two will move. They have an amplitude which is not stunning. It's nothing to write your mother about. But it is about  $\eta_0$ . It is a minus sign, so it's out of phase with the driver. And this one is a little more. Bizarre.

Could I demonstrate this? No way on Earth can I generate 0.77 times  $\omega_0$ . I can burn into my chip  $\omega_0$ , as I did. But I cannot do 0.77. There's no way. So forget it. I have to disappoint you.

But now comes the good news. Look at this point here. That is a point where the middle one will stand still. And that is truly amazing. The middle one stands still. The top one has an amplitude of about 0.7  $\eta_0$ . I just eyeball that. And the bottom one has an amplitude of about minus 1.5  $\eta_0$ . And let me try to make a drawing of that.

So this now is at  $\omega$  equals  $\omega_0$ . So my hand is here, which is  $\eta_0$ . The top one has an amplitude which is about 0.7  $\eta_0$ , this one. So it is roughly here. In phase, so this is connected. So this is roughly 0.7  $\eta_0$ .

The next one stands still, believe it or not. It's here. And then, the next one is out of phase with me and has roughly an amplitude of 1 and 1/2 times this. So that's 1 and 1/2. So I call that minus 1.5 times  $\epsilon_0$ .

And then half a period later, my hand is here. And this object is here. And this one stands still. And this one is here.

Truly, these are almost not believable. And then, there's another frequency whereby the top one would stand still. I will attempt-- I will make the daring attempt to aim for this solution. And the reason why I can try that is because the frequency is  $\omega_0$ , which I can burn into my chips. And that's the last attempt I will make today.

So this is the triple pendulum. So I have to go through the exercise of learning again the period. And then, as I close my eyes and drive it with this frequency, the idea then is that you would see the top one move in phase with my hand, the one below that will stand still. And then, the one below that will have an even larger amplitude.

So one, two, three, four, five, six, seven, one, two, three, four, five, one, two, three, four, five, one-- you see it?

AUDIENCE: Yes.

PROFESSOR: Three. You see it? Did you see that one stand still? Isn't it amazing that physics works? OK, see you Thursday.

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