Massachusetts Institute of Technology OpenCourseWare

8.03SC **Fall 2012**

Formulas, Physical Constants, and Trigonometry Identities

Formulas

General differential equation for oscillators $\ddot{x} + \gamma\dot{x} + \omega_0^2 x = f_0\,\cos(\omega t)$ has solutions:

$$x(t) = A\,\mathrm{e}^{-\frac{\gamma t}{2}}\,\cos\left(\sqrt{\omega_0^2 - \frac{\gamma^2}{4}}\;t + \alpha\right) + x_{ss}(t) \qquad \omega_0 > \frac{\gamma}{2}$$

$$x(t) = (A + B\,t)\;\mathrm{e}^{-\frac{\gamma t}{2}} + x_{ss}(t) \qquad \omega_0 = \frac{\gamma}{2}$$

$$x(t) = A\,\mathrm{e}^{-\Gamma_1 t} + B\,\mathrm{e}^{-\Gamma_2 t} + x_{ss}(t) \qquad \omega_0 < \frac{\gamma}{2}$$

where $\Gamma_{\frac{1}{2}} = \frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2}$ and the steady-state solution is $x_{ss}(t) = A(\omega)\,\cos(\omega t - \delta(\omega))$

$$A(\omega) = \frac{f_0}{\left[(\omega_0^2 - \omega^2)^2 + \gamma^2\,\omega^2\right]^{1/2}} \qquad \tan\delta(\omega) = \frac{\gamma\,\omega}{\omega_0^2 - \omega^2}$$

Non-dispersive wave equation $\frac{\partial^2}{\partial x^2}y(x,t) = \frac{1}{v^2}\,\frac{\partial^2}{\partial t^2}y(x,t)$ where $v = \sqrt{T/\mu}$ (string) or $\sqrt{\kappa/\rho}$ (gas).

Kinetic, potential energy and power:

$$\frac{dK}{dx} = \frac{1}{2}\,\mu\left(\frac{\partial y}{\partial t}\right)^2 \qquad \frac{dU}{dx} = \frac{1}{2}\,T\left(\frac{\partial y}{\partial x}\right)^2$$

$$P(t) = -T\left(\frac{\partial y}{\partial t}\right)\left(\frac{\partial y}{\partial x}\right)$$

Reflection and transmission coefficients: $R = \frac{v_2 - v_1}{v_2 + v_1}$ $\qquad T = \frac{2v_2}{v_2 + v_1}$

Fourier series for a function $f(\theta) = f(\theta + 2\pi)$ $\quad f(\theta) = \sum_{m=1}^{\infty}\left[\frac{A_0}{2} + A_m\,\cos(m\,\theta) + B_m\,\sin(m\,\theta)\right]$

$$A_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\,\cos(m\theta)\,d\theta \quad m = 0, 1, 2, \ldots \qquad B_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\,\sin(m\theta)\,d\theta \quad m = 1, 2, 3, \ldots$$

Dispersion: $v_{phase} = \frac{\omega}{k}$ and $v_{group} = \frac{\mathrm{d}\omega}{\mathrm{d}k}$

Maxwell's equations:

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{B} = \mu_0\,\epsilon_0\,\frac{\partial\vec{E}}{\partial t} + \mu_0\,\vec{J}$$

EM force, flux, energy, intensity:

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) \qquad \vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B}$$

$$U_E = \frac{\epsilon_0}{2}\left|\vec{E}\right|^2 \qquad U_M = \frac{1}{2\mu_0}\left|\vec{B}\right|^2$$

Dipole approximation:

$$\vec{E}_{rad}(\vec{r},t) = \frac{-q\vec{a}_\perp(t-r/c)}{4\pi\epsilon_0 c^2 r} \qquad \text{Volt/m}$$

$$\vec{B}_{rad}(\vec{r},t) = \frac{1}{c}\hat{r} \times \vec{E}_{rad}(t) \qquad \text{Tesla}$$

$$\vec{S}_{rad}(\vec{r},t) = \frac{1}{\mu_0}\vec{E}_{rad} \times \vec{B}_{rad} \qquad \text{Watt/m}^2$$

$$P = \frac{q^2 a^2}{6\pi\epsilon_0 c^3} \qquad \text{Watt}$$

Boundary conditions at the surface of a perfect conductor:

$$E_{//} = 0 \qquad \left|B_{//}\right| = \mu_0\,|J_S|$$

$$E_\perp = \frac{\rho_S}{\epsilon_0} \qquad B_\perp = 0$$

Transmission lines:

$$\frac{\partial V}{\partial z} = -L_0\frac{\partial I}{\partial t} \qquad v_p = \frac{1}{\sqrt{L_0 C_0}} \qquad \frac{V_r}{V_i} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$\frac{\partial I}{\partial z} = -C_0\frac{\partial V}{\partial t} \qquad Z_0 = \sqrt{\frac{L_0}{C_0}} \qquad \frac{I_r}{I_i} = \frac{Z_0 - Z_L}{Z_L + Z_0}$$

Boundary conditions at the surface of a perfect dielectric:

$$E_{//}^{(1)} = E_{//}^{(2)} \qquad \frac{B_{//}^{(1)}}{\mu_1} = \frac{B_{//}^{(2)}}{\mu_2}$$

$$\kappa_{e1}\,E_\perp^{(1)} - \kappa_{e2}\,E_\perp^{(2)} = \frac{\rho_s}{\epsilon_0} \qquad B_\perp^{(1)} = B_\perp^{(2)}$$

Fresnel equations: $n_1 \sin\theta_1 = n_2 \sin\theta_2$

$$r_\parallel = E_{0r\parallel}/E_{0i\parallel} = \frac{n_1\cos\theta_2 - n_2\cos\theta_1}{n_1\cos\theta_2 + n_2\cos\theta_1} = -\frac{\tan(\theta_1-\theta_2)}{\tan(\theta_1+\theta_2)}$$

$$r_\perp = E_{0r\perp}/E_{0i\perp} = \frac{n_1\cos\theta_1 - n_2\cos\theta_2}{n_1\cos\theta_1 + n_2\cos\theta_2} = -\frac{\sin(\theta_1-\theta_2)}{\sin(\theta_1+\theta_2)}$$

$$t_\parallel = E_{0t\parallel}/E_{0i\parallel} = \frac{2n_1\cos\theta_1}{n_1\cos\theta_2 + n_2\cos\theta_1} = \frac{2\sin\theta_2\cos\theta_1}{\sin(\theta_1+\theta_2)\cos(\theta_1-\theta_2)}$$

$$t_\perp = E_{0t\perp}/E_{0i\perp} = \frac{2n_1\cos\theta_1}{n_1\cos\theta_1 + n_2\cos\theta_2} = \frac{2\sin\theta_2\cos\theta_1}{\sin(\theta_1+\theta_2)}$$

Special case of normal incidence ($\theta_1 = \theta_2 = 0$)

$$r_{\parallel,\perp} = \frac{E_r}{E_i} = \frac{n_1 - n_2}{n_1 + n_2} \qquad t_{\parallel,\perp} = \frac{E_t}{E_i} = \frac{2\,n_1}{n_1 + n_2}$$

Doppler Effect:

$$\frac{\lambda'}{\lambda} = \frac{1-\beta\,\cos\theta}{\sqrt{1-\beta^2}} \qquad \text{for EM waves}$$

$$\frac{f'}{f} = \frac{v_s + v_r\,\cos\theta_r}{v_s - v_t\,\cos\theta_t} \qquad \text{for sound waves}$$

N source interference and diffraction:

Interference $$I = I_0\left[\frac{\sin(N\delta/2)}{\sin(\delta/2)}\right]^2 \qquad \delta = \frac{2\pi}{\lambda}\,d\,\sin\theta$$

Diffraction $$I = I_0\left[\frac{\sin\beta}{\beta}\right]^2 \qquad \beta = \frac{\pi}{\lambda}\,D\,\sin\theta$$

Diffraction gratings: $$I = I_0\left[\frac{\sin(N\delta/2)}{\sin(\delta/2)}\right]^2\left[\frac{\sin\beta}{\beta}\right]^2$$

Physical Constants

Speed of light	c	3×10^{8}	m s^{-1}
Vacuum permeability	μ_0	1.26×10^{-6}	$\left(\mathrm{V\ m^{-1}}\right)/\mathrm{A}$
Vacuum permittivity	ϵ_0	8.85×10^{-12}	$\mathrm{C}/\left(\mathrm{V\,m^{-1}}\right)$
Electron rest mass	m	9.1×10^{-31}	kg
Elementary charge	e	1.6×10^{-19}	C
Gravitational constant	G	6.7×10^{-11}	$\mathrm{N\,m^2/kg^2}$

Trigonometric Identities

$$\sin(a+b) = \sin a\cos b + \cos a\sin b \qquad \cos(a+b) = \cos a\cos b - \sin a\sin b$$

$$\sin a + \sin b = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right) \qquad \sin a - \sin b = 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$

$$\cos a + \cos b = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right) \qquad \cos a - \cos b = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$

$$\sin(\theta\pm\tfrac{\pi}{2}) = \pm\cos\theta \qquad \cos(\theta\pm\tfrac{\pi}{2}) = \mp\sin\theta \qquad \sin(\theta\pm\pi) = -\sin\theta \qquad \cos(\theta\pm\pi) = -\cos\theta$$

Complex exponentials: $$\mathrm{e}^{j\theta} = \cos\theta + j\,\sin\theta \qquad \sin\theta = \frac{\mathrm{e}^{j\theta}-\mathrm{e}^{-j\theta}}{2\,j} \qquad \cos\theta = \frac{\mathrm{e}^{j\theta}+\mathrm{e}^{-j\theta}}{2}$$