Massachusetts Institute of Technology OpenCourseWare

8.03SC **Fall 2012**

Problem Set #2 Solutions

Problem 2.1: Driven oscillator with damping

a) An object of mass m is hung from a spring with spring constant 80 N/m. The resistive damping force on the object is given by $-bv$, where v is the velocity and $b = 4$ N m^{-1} sec. So the constants for the damped motion are $\gamma = \dfrac{b}{m} = 20\ s^{-1}$ and $\omega_0 = \sqrt{\dfrac{k}{m}} = 20\ s^{-1}$ Let the oscillations of the spring be along the x axis. The spring force and damping force acting on the mass are $F_{restoring} = -kx$ and $F_{damping} = -bv = -b\dot{x}$

Newton's 2nd law: $m\ddot{x} = F_{net} = F_{restoring} + F_{damping} = -kx - b\dot{x}$ so $\ddot{x} = -\dfrac{k}{m}x - \dfrac{b}{m}\dot{x}$

Hence the differential equation describing the motion of the mass is:

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0 \qquad \text{or} \qquad \frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_o^2 x = 0$$

The frequency and period of such damped oscillations are:

$$\omega^2 = \omega_0^2 - \frac{\gamma^2}{4} = 300 \quad \Rightarrow \quad \omega = 10\sqrt{3}\ s^{-1} = 17.3\ s^{-1} \qquad T_{period} = \frac{2\pi}{\omega} = \frac{2\pi}{10\sqrt{3}}\ s \ \approx\ 0.36\ s$$

b) The equation of motion for this harmonically driven damped oscillator is:

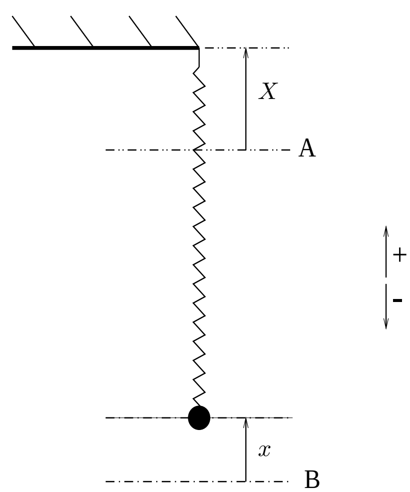
$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = \frac{F_0}{m}\sin(\omega t) \tag{1}$$

The amplitude of oscillations in the steady state is given by the formula:

$$A(\omega) = \frac{F_0/m}{[(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2]^{\frac{1}{2}}} = \frac{2/0.2}{[(20^2 - 30^2)^2 + (20\cdot 30)^2]^{\frac{1}{2}}} = 0.0128 \text{ m} = 1.28 \text{ cm}$$

substituting values for ω_o, F_0, ω and γ.

c) In the figure, A is the equilibrium level of the top end of the spring and B is the equilibrium level of the mass.



$X = X_0\cos(\omega t)$ is the harmonic displacement of the top end of the spring from its equilibrium position A, and x is the displacement of the spring from its equilibrium position. The spring force and the damping force acting on the mass are given by $F_{restoring} = +k(X - x)$ and $F_{damping} = -bv = -b\dot{x}$. For $X > x$ the spring force is in the + direction. Newton's 2nd law:

$$m\ddot{x} = F_{net} = F_{restoring} + F_{damping} = +k(X - x) - b\dot{x}$$
$$\ddot{x} = \frac{k}{m}(X - x) - \frac{b}{m}\dot{x}$$

So, $\dfrac{d^2x}{dt^2} + \gamma\dfrac{dx}{dt} + \omega_0^2 x = \dfrac{k}{m}X_0\cos(\omega t)$. This equation has the same form as Eq. 1. The harmonic displacement of the top end of the spring is equivalent to the application of a driving force with amplitude $F_0 = kX_0$.

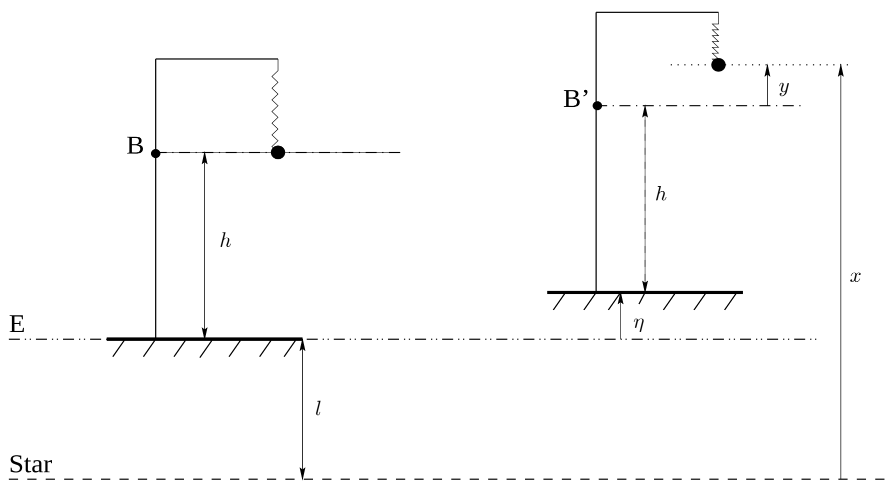
d) The amplitude of the mass in steady state is: $A(\omega) = \dfrac{kX_0/m}{[(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2]^{\frac{1}{2}}}$

Substituting values $\omega_o = 20$ s^{-1}, $k = 80$ N/m, $\omega = 0, 30, 300$ s^{-1} and $\gamma = 20$ s^{-1}.

$$A(\omega) \quad = \quad \frac{0.4/0.2}{[(20^2 - \omega^2)^2 + (20\omega)^2]^{\frac{1}{2}}} \quad A(0) = 0.5 \text{ cm} \quad A(30) = 0.256 \text{ cm} \quad A(300) = 0.00223 \text{ cm}$$

Problem 2.2: (French 4-6)[1] Seismograph

a) The displacement of mass M relative to the earth is y and η is the displacement of the earth's surface relative to the distant stars. Let x be the distance of mass M relative to the distant stars.



Left Figure: The horizontal dashed line through E is the equilibrium position of the earth relative to the star. The horizontal dashed line through B is the equilibrium position of the mass relative to the star. It is also the equilibrium position of the mass relative to the seismometer.

Right Figure: The dashed line through E, is the same as in the left figure. B is now a distance η farther away from the star than B (we indicate this with B'). The dashed line through B' is no longer the equilibrium position of the mass relative to the star, but it is the equilibrium position relative to the seismometer.

We can see from the figures that: $x = l + y + h + \eta$ or $\ddot{x} = \ddot{\eta} + \ddot{y}$. Newton's 2nd law only applies to an inertial reference frame. The acceleration of M is $\ddot{x}$. However, the spring force and the damping force depend on the displacement and velocity relative to the Earth (i.e. relative to B'). The amount by which the length of the spring changes is y in both reference frames (that of the star and that of the seismograph). Thus the magnitude of the spring force is ky. Since it is assumed that the air inside the closed box of the seismograph follows the motion of the Earth, the

[1]The notation "French" indicates where this problem is located in one of the textbooks used for 8.03 in 2004: French, A. P. *Vibrations and Waves. The M.I.T. Introductory Physics Series.* Cambridge, MA: Massachusetts Institute of Technology, 1971. ISBN-10: 0393099369; ISBN-13: 9780393099362.

damping force is $-b\dot{y}$. Notice, if the air does not follow the Earth then the damping force would be $-b(\dot{y}+\dot{\eta})$. Hence: $M\ddot{x} = -ky - b\dot{y} \quad 0 = \ddot{\eta} + \ddot{y} + \frac{k}{M}y + \frac{b}{M}\dot{y}$

$-\ddot{\eta} = \ddot{y} + \gamma\dot{y} + \omega_0^2 \quad$ or $\quad \frac{d^2y}{dt^2} + \gamma\frac{dy}{dt} + \omega_0^2 y = -\frac{d^2\eta}{dt^2}$ where $\gamma = \frac{b}{m}$ and $\omega_0^2 = \frac{k}{m}$.

b) Steady state solution for y when $\eta = C\cos(\omega t)$.

$$\eta = Ccos(\omega t) \quad \frac{d^2\eta}{dt^2} = -C\omega^2\cos(\omega t) \quad \frac{d^2y}{dt^2} + \gamma\frac{dy}{dt} + \omega_0^2 y = C\omega^2\cos(\omega t) \tag{2}$$

To solve the equation using the complex exponential method we reframe the above equation as follows $\frac{d^2z}{dt^2} + \gamma\frac{dz}{dt} + \omega_0^2 z = C\omega^2 e^{i\omega t}$. Let $z = Ae^{i(\omega t-\delta)}$ be the solution to the above equation. Now $y = Re(z)$. Substituting these in Eq. 2.

$$\begin{aligned}(-\omega^2 A + i\gamma\omega A + \omega_0^2 A)e^{i(\omega t-\delta)} &= C\omega^2 e^{i\omega t}\\ (\omega_0^2 - \omega^2)A + i\gamma\omega A &= C\omega^2 e^{i\delta}\end{aligned}$$

Equating the real and imaginary parts of the equation we get:

$$(\omega_0^2 - \omega^2)A = C\omega^2\cos\delta \quad \gamma\omega A = C\omega^2\sin\delta$$

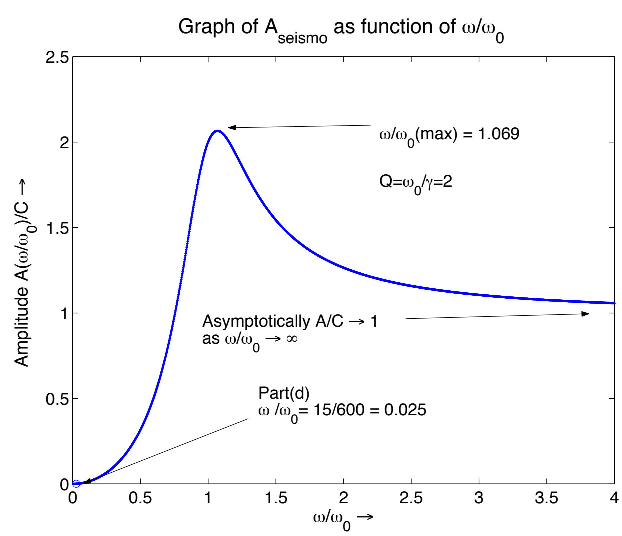
Therefore the steady state solution for y is $y = A\cos(\omega t - \delta)$ where

$$A(\omega) = \frac{C\omega^2}{[(\omega_0^2-\omega^2)^2 + (\gamma\omega)^2]^{\frac{1}{2}}} \quad \tan\ \delta(\omega) = \frac{\gamma\omega}{\omega_0^2 - \omega^2}$$

Behavior of $A(\omega)$ for various values of ω

$$\omega \to 0 \ \ A \to 0 \qquad \omega \to \omega_0 \ \ A \to QC \qquad \omega \to \infty \ \ A \to C$$

c) The graph of the amplitude A of the displacement y (in units of C) as a function ω is shown to the right. Note: Q=ω_0/γ is taken to be 2.



d) Period of the Seismograph T_s is 30 s and Q is 2.

$$T_s = 2\pi/\omega_0 = 30\text{ s} \quad \omega_0 = \frac{2\pi}{30} = \frac{\pi}{15}\text{ rad/s} \quad \gamma = \frac{\omega_0}{Q} = \frac{\pi/15}{2} = \frac{\pi}{30}\text{ rad/s}$$

Now the time period of oscillations of the earth's surface is 20 min and the amplitude of maximum acceleration is 10^{-9} m/s^2.

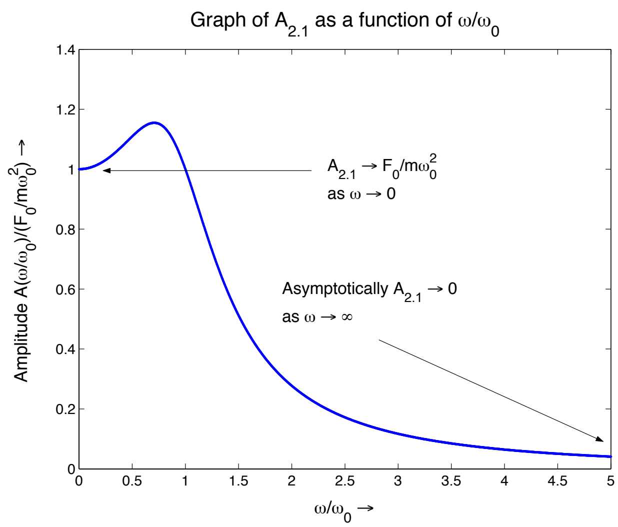
$$\omega = \frac{2\pi}{T_s} = \frac{2\pi}{1200} = \frac{\pi}{600} \text{ rad/s} \quad a_{max} = C\omega^2 = 10^{-9} \text{ m/s}^2 \quad C = \frac{a_{max}}{\omega^2} = 3.6 \times 10^{-5} \text{ m}$$

Substituting values for ω, ω_0, γ and C in the equation for amplitude A we get:

$$A(\omega) = \frac{C\omega^2}{[(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2]^{\frac{1}{2}}} \quad A = 2.28 \times 10^{-8} \text{ m}$$

Notice that C (amplitude of the Earth's oscillations) is about 1600 times larger than A. It seems to us that this is a very poorly designed seismometer. Values of A of the order of 2.3×10^{-8} m must be observable for this tremor to be detected. If the frequency of the oscillations $\omega \gg \omega_0$ the value of $A \to C$ (see the figure for Part (c) above). The amplitude of the earthquake oscillations can then directly be read off the seismometer.

e) Problem 2.2 and 2.1 are very different. In the figure below, we show the amplitude $A(\omega)$ versus ω for Problem 2.1 [Note: $\omega_0 = 20$ Hz, $\gamma = 20$ Hz, and $F_0/m = 2$ N/kg]. Compare this to the plot in Problem 2.2 as shown in Part (c) above. This difference is best demonstrated by comparing their amplitudes at very low (near zero) and very high frequencies. Let the amplitude in Problem 2.1 be $A_{2.1}$ and the amplitude of the seismometer be A_{seismo}.



$$\omega \to 0 \Rightarrow A_{2.1} \to \frac{F_0}{m\omega_0^2} \quad A_{seismo} \to 0$$

$$\omega \to \infty \Rightarrow A_{2.1} \to 0 \quad A_{seismo} \to C$$

$$\omega \to \omega_0 \Rightarrow A_{2.1} \to \frac{F_0}{\gamma m \omega_0} \quad A_{seismo} \to QC$$

As you can see, there is a major difference between harmonically displacing the top end of the spring and harmonic oscillations of the earth.

Problem 2.3: (French 4-10) Power dissipation

a) Let dW be the work done against the damping force in time dt. Now the work done is the dot product of the force and the distance over which it is applied, $dW = F_{anti-damping}dx = bv \cdot dx$. Hence, the instantaneous rate of doing work against the damping force is:

$$P = \frac{Work\ Done}{Time\ Taken} = \frac{dW}{dt} = bv\frac{dx}{dt} = bv^2 \tag{3}$$

b) The equation of motion is of the form $x = A\cos(\omega t - \delta)$, hence the mean power dissipated can be calculated from part (a) as shown below:

$$\bar{P} = (b\dot{x}^2)|_T = [b(-A\omega\sin(\omega t - \delta))^2]|_T = bA^2\omega^2[\sin^2(\omega t - \delta)]|_T$$

$$\bar{P} = bA^2\omega^2\frac{1}{2} \quad (\text{ Since } \sin^2(\omega t - \delta)|_T = \frac{1}{2}\ ) \text{ or } P = \frac{bA^2\omega^2}{2}$$

c) The value of A for any arbitrary frequency is given by the expression shown below

$$\begin{aligned}
A(\omega) &= \frac{F_0/m}{[(\omega_0^2-\omega^2)^2+(\gamma\omega)^2]^{\frac{1}{2}}} \quad \Rightarrow \quad A(\omega)^2 = \frac{F_0^2/m^2}{(\omega_0^2-\omega^2)^2+(\gamma\omega)^2} \\
\bar{P}(\omega) &= \frac{b\omega^2}{2}\frac{F_0^2/m^2}{\omega^2\omega_0^2[(\frac{\omega_0}{\omega}-\frac{\omega}{\omega_0})^2+(\frac{\gamma}{\omega_0})^2]} = \frac{bF_0^2}{2\omega_0^2 m^2}\frac{1}{(\frac{\omega_0}{\omega}-\frac{\omega}{\omega_0})^2+(\frac{\gamma}{\omega_0})^2} \\
&= \frac{F_0^2\gamma}{2k}\frac{1}{(\frac{\omega_0}{\omega}-\frac{\omega}{\omega_0})^2+(\frac{\gamma}{\omega_0})^2} = \frac{F_0^2\omega_0}{2kQ}\frac{1}{(\frac{\omega_0}{\omega}-\frac{\omega}{\omega_0})^2+\frac{1}{Q^2}}
\end{aligned}$$

This result for the mean power dissipation is the same as given in Eq. (4-23) on page 98 in French.

Problem 2.4: Transient behavior

a) The period of free oscillations can be measured off the graph to be T_0 approximately 4 sec, hence $\nu_0 = \frac{1}{T_0} \approx \frac{1}{4} \approx 0.25$ Hz $\quad \omega_0 \approx 0.5\pi$ rad/s

b) The homogenous solution is given by $x(t) = x(t=0)e^{-\gamma t/2}\cos(\omega t+\phi)$. To determine we use the envelope of the exponential decay. A couple of points on the exponential decay envelop are measured to be $x(t \approx 0.7) \approx -10.5$ m and $x(t \approx 3.0s) \approx +6.2$ m. Now

$$\left|\frac{x(t=3.0\ s)}{x(t=0.7\ s)}\right| = e^{-2.3\gamma/2} = \frac{6.2}{10.5} \quad \Rightarrow \quad \gamma = \frac{2}{2.3}\ln\left(\frac{10.5}{6.2}\right) \approx 0.46 \text{ rad/s}$$

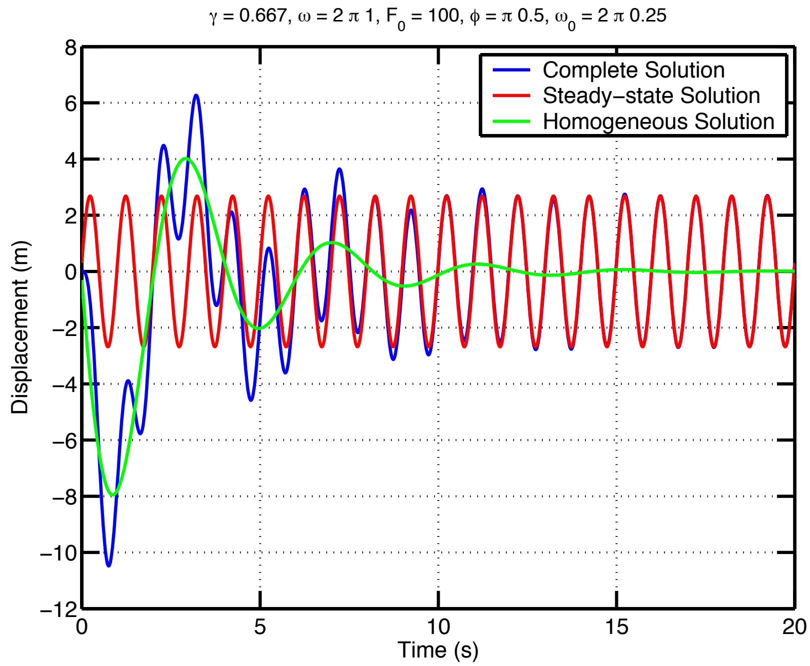
Hence the damping coefficient $b = m\gamma \approx 0.46$ Ns/m. A much better way would be to subtract the steady state solution from the total curve and then to derive γ from the decay of the remaining curve. That would certainly give a more accurate value. However, we did not do that here. This may explain why our value for γ is more than 20% off the value that was used to generate the curve; see part (e).

c) The frequency of the driving force can be measured quite accurately with the period of the steady state solution. The period appears to be 5 cycles in the last 5 seconds or $\nu \approx 1$ Hz. Hence, $T = \frac{1}{\nu} \approx 1\ s$ and $\omega = 2\pi\nu \approx 2\pi$ rad/s. The frequency of the driving force is ~ 1 Hz. It is four times larger than the frequency of the free oscillations.

d) The amplitude of the steady state response of the oscillator can be measured quite accurately $A(\omega) = \frac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2+(\gamma\omega)^2}} = 2.5$ m from which we can find the amplitude of the driving force:

$$\begin{aligned}
F_0 &= mA(\omega)\sqrt{(\omega_0^2-\omega^2)^2+(\gamma\omega)^2} = 1 \times 2.5\sqrt{16\pi^4(0.25^2-1^2)^2+4\pi^2(0.46^2\times 1^2)} \\
&\approx 93\ N \simeq 90 \text{ N}
\end{aligned}$$

e) We find the initial phase of the driving force by extending the steady-state part back to $t=0$. The zero crossings in the steady state solution are at $t = 10, 15$ and 20 sec. Thus it also crosses zero at t=0 sec, and the steady state displacement starts off in the positive direction. We also know that when $\omega \gg \omega_0$, the displacement $x(t)$ is π radians out of phase with the driving force $F(t) = F_0 \cos(\omega t + \phi)$. Thus at $t = 0$ the driving force must be zero and must increase in the negative direction. Thus $F(t = 0) = F_0 \cos\phi = 0$ and $F(t = +\epsilon) < 0 \Rightarrow \phi \approx \frac{\pi}{2}$ radians. The figure shows a graph of the transient (green), the steady state (red) and the composite (blue). At the top of the plot we list the input parameters that we used in preparing this problem. Compare them with the approximate values that



we derived from the blue plot. We were dead on in the case of ω, ω_0 and ϕ. We were within 10% of F_0, but our estimate of γ was off by more than 20%.

Problem 2.5: Driven RLC circuit

a) Potential differences across the resistor and the capacitor are as follows: $V_R = IR = R\frac{dq}{dt}$ and $V_C = \frac{q}{C}$. Faraday's Law states $\oint \vec{E} \cdot \vec{dl} = -\frac{d\phi_B}{dt}$. The inductor has no ohmic resistance. Thus the $\int \vec{E} \cdot \vec{dl}$ in going through the wire of the inductor from one end to the other end is zero. The closed loop integral going into the direction of the current then becomes

$$IR + V_C - V_0 \cos(\omega t) = -L\frac{dI}{dt}$$

Notice: Kirchhoff's "voltage" rule does *NOT* hold as the E field here is non-conservative. Substituting $I = dq/dt$ and $dI/dt = d^2q/dt^2$, the differential equation for charge on the capacitor:

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{L}\frac{q}{C} = \frac{V_0}{L}\cos(\omega t) \qquad \Rightarrow \qquad \frac{d^2q}{dt^2} + \gamma\frac{dq}{dt} + \omega_0^2 q = \frac{V_0}{L}\cos(\omega t) \tag{4}$$

$$\frac{R}{L} = \gamma \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

We differentiate the above equation to find the equation for current $\frac{d^2I}{dt^2} + \gamma\frac{dI}{dt} + \omega_0^2 I = -\frac{V_0}{L}\omega\sin(\omega t)$

b) To solve for $q(\omega, t)$, we use the fact that Eq. 4 has the same form as Eq. 1 in Problem 2.1(b). Hence the solution is:

$$q(\omega, t) = q_0(\omega)\cos(\omega t - \delta) \qquad q_0(\omega) = \frac{V_0/L}{[(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2]^{\frac{1}{2}}} \qquad \tan\delta(\omega) = \frac{\gamma\omega}{\omega_0^2 - \omega^2}$$

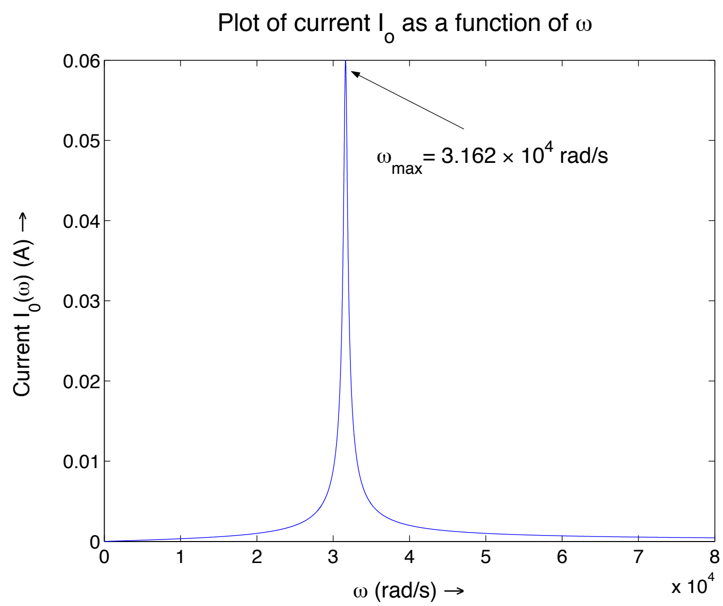
c) We calculate $I(\omega,t)$ by differentiating our results from above for $q(\omega,t)$:

$$\begin{aligned} I(\omega,t) &= -\omega q_0(\omega)\sin(\omega t-\delta) = -I_0(\omega)\sin(\omega t-\delta) \\ |I_0(\omega)| &= \frac{\omega V_0/L}{[(\omega_0^2-\omega^2)^2+(\gamma\omega)^2]^{\frac{1}{2}}} \qquad \tan\delta(\omega) = \frac{\gamma\omega}{\omega_0^2-\omega^2} \end{aligned} \tag{5}$$

The equation for $I_0(\omega)$ is often written in the form $I_0 = \dfrac{V_0}{\sqrt{R^2+(X_L-X_c)^2}}$ where $X_C = 1/\omega C$ and $X_L = \omega L$ are the capacitive and inductive reactances, respectively.

At resonance, $\delta = \pi/2$. That means that the driving voltage is IN PHASE with the current. Because $\cos\omega t = -\sin(\omega t - \pi/2)$. As mentioned in lectures, at resonance the circuit behaves as if there is no C and no L. Thus Ohm's Law is at work which dictates that the voltage and the current are in phase. Consequently, for low values of ω when $\delta = 0$, the current is leading the voltage by a phase angle $\pi/2$ which corresponds to a quarter of a period (the capacitor rules!), and for very high ω, the current is lagging the driving voltage by a quarter of a period (the self-inductor rules!).

d) Substituting values for V_0 , R, L , and C [$V_0 = 3$ V, $R = 50\ \Omega$, $L = 100$ mH, and $C = 0.01\ \mu$F]; the plot for current I as a function of ω is shown in the figure to the right.



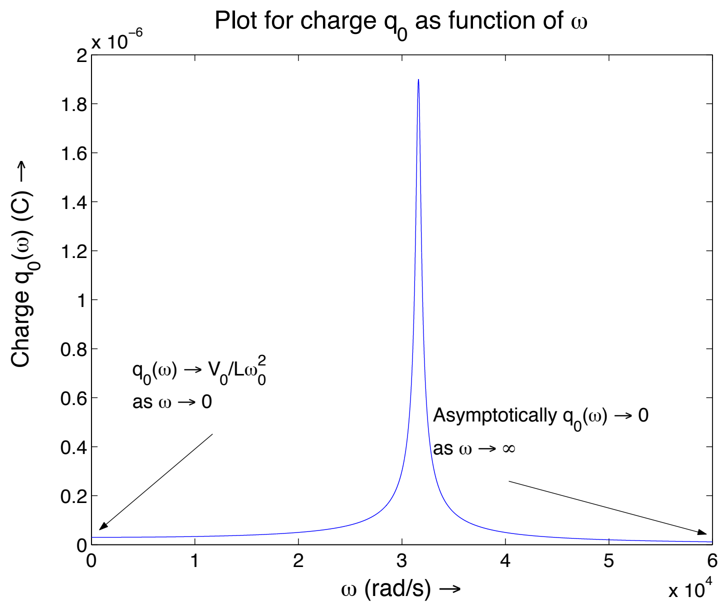
e) We can see from the equation for I_0 in part (c) that for the current I_0 through the circuit to be maximum, $Z = \sqrt{R^2+(X_L-X_c)^2}$ has to be minimized.

$$\begin{aligned} X_L &= X_C \Rightarrow \omega L = \frac{1}{\omega C} \\ \omega_{Imax} &= \frac{1}{\sqrt{LC}} = \omega_0 \\ \omega_0 &= 3.162\times 10^4 \text{ rad/s} \end{aligned}$$

At frequency $\omega_{Imax} = 3.162\times10^4$ rad/s, the current through the circuit is maximum.

The quality factor Q for the system is $\gamma = \dfrac{R}{L} = 500$ rad/s $Q = \dfrac{\omega_0}{\gamma} = \dfrac{L}{R}\dfrac{1}{\sqrt{LC}} = 63.2$. The high value of $Q = 63.2$ explains the sharp peak around $\omega = 3.162\times10^4$ rad/s.

f) The plot for the charge amplitude q_0 as a function of ω is shown in the figure to the right.



g) To find the ω at which q_0 is maximum, we differentiate its value from the equation in part (b) with respect to ω and equate it to 0.

$$\frac{d[q_0(\omega)]}{d\omega} = \frac{d}{d\omega}\left[\frac{V_0/L}{[(\omega_0^2-\omega^2)^2+(\gamma\omega)^2]^{\frac{1}{2}}}\right]$$

$$0 = -\frac{V_0}{2L}\frac{4\omega(\omega^2-\omega_0^2)+2\omega\gamma^2}{[(\omega_0^2-\omega^2)^2+(\gamma\omega)^2]^{\frac{3}{2}}}$$

$$4\omega(\omega^2-\omega_0^2) = -2\omega\gamma^2 \qquad \omega_{qmax} = \sqrt{\frac{1}{2}(2\omega_0^2-\gamma^2)} = 3.160\times10^4 \text{ rad/s.}$$

At frequency $\omega_{qmax} = 3.160 \times 10^4$ rad/s, the charge on the capacitor is maximum.

The frequency ω_{qmax} at which charge q_0 on the capacitor is a maximum, is only slightly lower than the frequency ω_{Imax} at which the current I_0 through the circuit is maximized. The difference is very small as Q is very high (~ 63).