# Massachusetts Institute of Technology OpenCourseWare

**8.03SC** **Fall 2012**

## Notes for Lecture #12: Dispersion, Phase Velocity, & Group Velocity

In all examples of propagating waves seen so far, the medium was *non-dispersive*, which means characterized by a single speed for all waves. A number of important media are non-dispersive. For example, sound in air is characterized by a single speed. However, in some cases, waves are subject to *dispersion*, and the speed of propagation may depend on the wavelength or frequency of the wave. In fact, a system of beads on a string does show dispersion. This system was analyzed previously purely in terms of normal modes or standing waves. More recent lectures have explored the correspondence between a standing waves and two oppositely-directed traveling waves. This correspondence is now explored for beads on a string.

The case of $N = 5$ beads of mass $M$ separated by $\ell$ on a string of total length $L$ is examined and we are reminded that $\omega_n = 2\omega_0 \sin\left(\frac{n\pi}{2(N+1)}\right)$, with $\omega_0 = \sqrt{T/M\ell}$ (**2:20**). Given that the sine has a maximum of 1, the maximum possible frequency is $\omega_{max} = 2\omega_0$. Note, however, that this requires $n = N + 1$ which is a trivial case of no motion since the equation at the top of page 2 in Lecture Notes #7 shows that all the amplitudes would be 0.

For standing waves, the boundary conditions require $k_n = n\pi/L$. Therefore, the speed $v = \omega_n/k_n$ cannot be the same for all frequencies, since the frequency goes up with the sin of $n$ while the wavenumber is directly proportional to $n$. It may not be immediately obvious why the same restriction on $k_n$ applies to traveling waves. Recall, however, the correspondence between a standing wave $y_n \propto \sin(k_n x)\cos(\omega_n t)$ and the two traveling waves $y \propto \sin(kx + \omega t)$ and $y \propto \sin(kx - \omega t)$. In this case, the speed of propagation, $v = \frac{\omega_n}{k_n} = \frac{2L\omega_0 \sin\frac{n\pi}{2(N+1)}}{n\pi}$, is lower for higher $\omega$ (i.e. higher $n$), since the sine curve flattens out. The degree of nonlinearity can be visualized using a plot of $\omega$ versus $k$ (called a *dispersion relation*) (**4:20**). On such a plot, the slope of the line from the origin to the point $(k, \omega)$ is the speed (phase velocity). There are many examples of dispersion in nature.

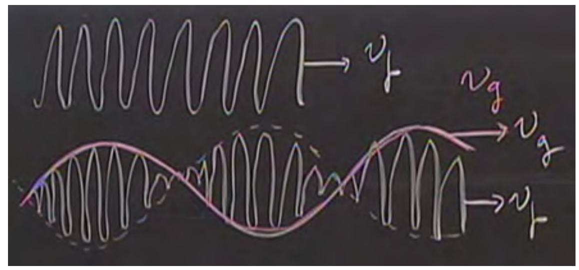
Adding two waves $y_1 = A\sin(k_1 x + \omega_1 t)$ and $y_2 = A\sin(k_2 x + \omega_2 t)$, with phase velocities $v_1 = \omega_1/k_1$ and $v_2 = \omega_2/k_2$ respectively (**8:20**), the sum is found using a trig identity:

$$y = y_1 + y_2 = 2A\sin\left(\frac{k_1 + k_2}{2}x - \frac{\omega_1 + \omega_2}{2}t\right)\cos\left(\frac{k_1 - k_2}{2}x - \frac{\omega_1 - \omega_2}{2}t\right)$$

Substituting $k$ for the average wavenumber $k = \frac{1}{2}(k_1 + k_2)$, the same for frequency $\omega = \frac{1}{2}(\omega_1 + \omega_2)$, and the differences $\Delta k = k_1 - k_2$ and $\Delta\omega = \omega_1 - \omega_2$, we can write

$$y = 2A\sin(kx - \omega t)\cos\left(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right) \ (\mathbf{11:10}).$$

If the frequencies are close together, then $k \approx k_1 \approx k_2$, $\Delta k$ is small, and the same applies for $\omega$. The equation itself is exact, it is the latter assumption that Prof. Lewin calls approximate. This is clearly very similar to the beat equation, with the sine term corresponding to a rapidly oscillating traveling wave at the phase velocity $v_p = \omega/k$, and the cosine term corresponding to a slowly oscillating wave with a new velocity called the *group velocity*, $v_g = \dfrac{\Delta\omega}{\Delta k}$, which will later be generalized to the derivative or slope $v_g = \dfrac{d\omega}{dk}$ (**12:15**).



The name "group" comes from the fact that a packet of waves would travel at this speed. In a non-dispersive medium, it does not matter if one takes the ratio of differences or just the ratio, so both speeds are the same. This is not true in a dispersive medium. Spatially, the sine has a wavelength $\lambda = 2\pi/k$ , while the cosine has the much bigger value $\lambda = 4\pi/\Delta k$. The shorter sine waves move with the phase velocity while the envelope given by the cosine moves with the group velocity. The overall effect is like moving beats, and if the medium is dispersive, the envelope will move at a different speed than the individual waves, either faster or slower as determined by the dispersion relation ($\omega$-$k$ diagram). On that diagram, a straight line indicates a non-dispersive medium, with the phase and group velocities equal (**16:30**). In the case of a downward curving dispersion relation (as was found for beads on a string), the slope at large $k$ is smaller than at small $k$, so the group velocity is smaller than the phase velocity. For an upward curving dispersion relation, the opposite is the case. It is even possible to have a region on the $\omega$-$k$ diagram with negative slope, so that the phase velocity is in the opposite direction from the group velocity in some regimes of $k$ (**19:20**). A demo is shown using two overlapping transparencies of bars with those on one transparency being 5% larger and 5% farther apart, creating a beat pattern. The overall pattern moves 20 times faster than an individual sheet in such a case (**24:00**).

We now return to strings, where the solution of the wave equation $\dfrac{1}{v^2}\dfrac{\partial^2 y}{\partial t^2} = \dfrac{\partial^2 y}{\partial x^2}$ requires $v = \sqrt{T/\mu}$, so that the velocity of propagation does not depend on frequency but rather only on the tension and mass density of the string (**27:30**). Thus the string is a non-dispersive medium with $\omega = vk$ or $\omega^2 = v^2k^2$. However, some approximations, mainly neglecting stiffness (resistance to bending), were made in this derivation. More realistically, the equation should be $\omega^2 = v^2k^2 + \alpha k^4$, which bends upward. A piano string, for which $\omega^2 = \frac{T}{\mu}k^2 + \alpha k^4$, with $\alpha = 10^{-2}$, $T = 250\ N$, $L = 1$ m, and $\mu = 10$ g/m= $10^{-2}$ kg/m is one example (**32:50**). The square of the speed is $T/\mu = 2.5\times 10^4\ (\text{m/s})^2$, and the tenth harmonic is considered. The fundamental would consist of one-half wave in the string length $L$, so here instead $\lambda_{10} = 2L/10$=0.2 m. Thus $k_{10} = 2\pi/\lambda_{10} = 10\pi\ \text{m}^{-1}$. With these numbers, we can solve that $\omega^2 = 2.5 \times 10^7 + 10^4$, slightly higher than it would be for a non-stiff wire. This gives $\omega \approx 5000$ rad/s, or $f \approx 800$ Hz (**35:20**). What is important is that the extra term results in a 0.2% increase (1/6 Hz) of the tenth harmonic above what it would be without dispersion. An even stiffer wire would have an even larger increase. The tenth harmonic is slightly higher than ten times the fundamental, which is called a sharpening of the frequency.

For a computer demo, a "toy model" dispersion relation $\omega^2 = v^2k^2(1+\alpha k^2)$ is used, which although superficially resembling that of a stiff wire, is actually quite different (**36:30**). The initial condition is six waves, separated by 2.5% in wavelength, with their maxima initially aligned at the origin. First, the non-dispersive case ($\alpha$=0) is examined, and the pulse envelope moves along at the same speed as all six waves and nothing changes in the shape (**41:00**). All of the waves making the beat pattern move at the same speed. If, instead, the phase velocities are not all the same, something different happens. The component waves change how they line up, and the envelope pattern changes (in this example the group velocity is larger than the phase velocities) (**43:30**), but it does so slowly since the dispersive term is small ($\alpha$=0.001). In the next demo, $\alpha$ is negative and the group velocity ends up negative, and this case is more dramatically dispersive (**45:30**).

If we think of this in Fourier space, a non-dispersive case has a shape coming back to its original form since all the components (harmonics) repeat in one full period. If there is dispersion, this is no longer the case, so the pulse shape changes (**48:40**). In traveling waves, the sharp features come from high frequencies and these will be dispersed more quickly if the medium is dispersive so that a sharp pulse will smooth out. (Not mentioned in the lecture, you can imagine that this is not a desired feature in a digital communication system based on transmitting sharp square pulses). This effect is demonstrated, first without dispersion, where the original shape is preserved. With dispersion ($\alpha$=0.01), the pulse shape degrades very quickly (**52:00**).

The phase velocity is $v_p = \dfrac{\omega}{k}$ and the group velocity is $v_g = \dfrac{d\omega}{dk}$, both of which depend on the shape of the dispersion relation. Although, ironically, water waves are too complicated to consider in this course on waves, their $\omega - k$ relation is well known and is a good example of dispersion. The formula is $\omega^2 = gk + (S/\rho)k^3$ for deep water waves (shallow water effects are different, and the depth criterion is relative to the wavelength), with $g$ the gravitational acceleration, $S$ the surface tension, and $\rho$ the density (**54:30**). For fresh water, $S = 0.072$ N/m and $\rho = 1000$ km/m$^3$. If the wavelength is greater than 1 cm, the surface tension is unimportant, and for wavelength 1 m, $gk \approx 62$ and the last term is an insignificant 0.02 (both in SI units) (**56:00**). It turns out that the phase velocity is about 1.25 m/s and the frequency in Hz is also about 1.25, which seem like reasonable numbers to anybody having seen such waves. These numbers are found using $\omega \approx \sqrt{gk}$ so that the phase velocity $v_p = \omega/k = \sqrt{g/k}$. This is proportional to the square root of the wavelength so that longer waves go faster (recall what you may know about tsunamis). The group velocity comes from the derivative $v_g = \frac{d\omega}{dk} = \frac{d}{dk}(gk)^{\frac{1}{2}} = \frac{1}{2}(gk)^{-1/2} g = \frac{1}{2}\sqrt{g/k}$ which is exactly half of the phase velocity (**58:50**). If you carefully observe the ripple going out from a rock thrown into a deep pond, you can see this, the individual waves of the ripple move through the ripple envelope going outward!

In the opposite case of short wavelengths, where surface tension does play the dominant role, the dispersion relation becomes $\omega^2 = \frac{s}{\rho}k^3$ and $v_p = \frac{\omega}{k} = k\sqrt{\frac{s}{\rho}}$ which is proportional to the inverse of the wavelength, so that short waves move faster. In contrast, shallow water waves are non-dispersive, which may be somewhat intuitive since these are essentially pressure waves (**1:00:50**). Sound is, of course, pressure waves in air, and it is fortunate that it is non-dispersive, since otherwise we could not communicate or enjoy music.

Electromagnetic waves, to be discussed in more detail in future lectures, are non-dispersive and move at a speed in vacuum of $c = 3\times10^8$ m/s. From this, one can calculate wavelengths which range from radio with $f$ of the order of $10^6$ Hz and $\lambda \approx$300 m through radar with $f \approx 10^{10}$ Hz and $\lambda \approx$3 cm up to "visible" light and beyond with even higher frequencies and shorter wavelengths (**1:04:15**). In matter, the speed of light is slower depending on the electric and magnetic permeabilities which are properties of the material. Of these, $\kappa_m$ is almost always 1 in materials capable of transmitting light, but $\kappa_e$ varies widely and depends on frequency. In water, $\kappa_e$ varies from about 78 over the range 0 to $10^{10}$ Hz, but drops to about 1.77 for visible light. The speed of light is given by $c/\sqrt{\kappa_m\kappa_e}$ and so varies from about $c/9$ to $c/1.33$ over the large span of frequencies (**1:07:30**). Even within the range of visible light at frequencies near $10^{15}$ Hz, the speed changes with frequency, so that light in materials is dispersed.

The passage of electromagnetic radiation in waveguides (for example, a metal pipe) is another example where strong frequency dependencies appear. Passage of radar waves at 10 GHz ($10^{10}$ Hz) or 3 cm wavelength between two plates with separation $a$ is considered (**1:10:00**). If $a$ is less than half the wavelength, waves cannot pass. The dispersion relation is $\omega_n = c\sqrt{(n\pi/a)^2 + k_z^2}$, where the first term is reminiscent of equations we had for strings or sound in boxes. The lowest frequency possible is with $n = 1$. Given this dispersion relationship, the frequency must exceed a certain minimum or *cutoff* value $\omega_c = c\pi/a$ for propagation to occur (**1:13:45**). For waves with fixed frequency, this actually places a condition on $a$, that it must be greater than 1.5 cm for these 3 cm waves. The cutoff is very dramatic in the demo, with just a squeeze below 1.5 cm needed to completely stop the waves (**1:16:30**). With this dispersion relation, the phase velocity ($v_{pz} = \omega/k_z$) is greater than the speed of light at all points, and in fact goes to infinity at $k_z = 0$. The group velocity ($v_{gz} = d\omega/dk_z$) goes to 0 at the cutoff.

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